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ORIGINAL RESEARCH

A Study on Solving Fractional Volterra Integral Equations with Müntz Orthogonal Functions

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Abstract

In this paper, Müntz orthogonal functions are employed for the numerical solution of fractional Volterra integral equations. These functions are defined on the interval $[0, 1]$ and possess simple and distinct real roots, providing the best unique approximation for functions in $L^2(0, 1)$. The Riemann–Liouville fractional integral operator is defined for these functions to reduce computational complexity and increase the solution speed. The error bound of the method is also determined. The numerical examples presented demonstrate the superiority of the proposed method compared to other existing approaches for the numerical solution of fractional Volterra integral equations.

Keywords: Fractional Volterra integral Equations, Orthogonal basis, Müntz orthogonal functions, Collocation method, Function approximation, Riemann–Liouville fractional integral operator.

1 Introduction

We consider the fractional Volterra integral equations

$$u(x)f(x) = v(x) + \delta \int_{a_1(x)}^{a_2(x)} \psi(x, t)f(t)dt, \tag{1}$$

the numerical solution of fractional Volterra integral equations has consistently posed significant challenges in computational mathematics, largely due to their extensive applications in modeling dynamical systems with memory. Recently, the use of Szász-Mirakyan approximation operators has garnered attention as an efficient method for solving this class of equations [1]. This method leverages appropriate functional bases to provide accurate approximations of the solution.

Nevertheless, the complexity of many natural phenomena cannot be fully described by classical fractional calculus alone, necessitating more advanced frameworks such as fractal-fractional calculus. In this regard, the study of fractal-fractional Volterra integral equations of the second kind by Martínez

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and Kaabar [2] represents a milestone in this field, highlighting the need to develop specialized numerical methods for this emerging class of equations.

A variety of methods have been proposed for solving fractional Volterra integral equations, including shifted Legendre fractional pseudo-spectral integration matrices [3], two-dimensional Haar wavelet methods [4], Bernstein approximation approaches [5], and Bernoulli wavelet techniques [6]. Additionally, modified hat functions [7] and rational Chebyshev approximation methods [8] have proven successful in addressing nonlinear equations within this domain.

Given the vital applications of these equations in optimal control problems [9] and coupled systems [10], the development of accurate and efficient numerical methods is of particular importance. More advanced techniques, such as Kantorovich approximation methods [11] and uniformly accurate schemes for two-dimensional nonlinear equations [12], have also been developed in this context. Moreover, studies on Hyers–Ulam stability in the fuzzy space for equations with the ψ-function kernel [13] and convergence analyses of wavelet methods for systems of equations [14] address crucial theoretical aspects of this field.

Despite significant advances in the numerical solution of classical fractional equations, a notable research gap remains in devising optimized numerical methods for fractal-fractional Volterra integral equations of the second kind The inherent complexity of these equations, which combines fractal and fractional characteristics, makes them resistant to many standard numerical techniques. In this context, numerical methods for nonlinear systems with variable fractional order have also attracted researchers' attention [15].

2 Basic Concepts and Notation

2.1 Fractional Calculus

Definition 2.1. The fractional integral operator of Riemann–Liouville type with order α, assuming $\alpha \in \mathbb{R}^+$, denoted by I^α over the interval $[a, b]$, is defined as follows [16]

$$I^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(s)}{(x-s)^{1-\alpha}} ds = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} * f(x) & \alpha > 0, \\ f(x) & \alpha = 0, \end{cases} \tag{2}$$

where $x^{\alpha-1} * f(x)$ is the convolution product of $f(x)$ and $x^{\alpha-1}$.

Definition 2.2. The fractional derivative operator in the sense of Caputo of order α, assuming $\alpha \in \mathbb{R}^+$, denoted by D^α, is defined as follows [16]

$$D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(s)}{(x-s)^{\alpha+1-n}} ds, \qquad n-1 < \alpha \leq n, \quad n \in \mathbb{N},$$

for $\alpha \in \mathbb{R}^+$ and $n \in \mathbb{N}$, it is clear that the Caputo fractional derivative and the Riemann–Liouville fractional integral satisfy the following properties [16]

$$1. \quad I^\alpha (D^\alpha f(x)) = f(x) - \sum_{k=0}^{n-1} f^k(0) \frac{x^k}{k!}, \tag{3}$$

$$2. \quad D^\alpha (f(x)) = I^{n-\alpha}\ D^n f(x), \qquad n-1 < \alpha \leq n, \tag{4}$$

where λ_1,λ_2 and c are real constants.

3 An Overview of Müntz Functions and Their Key Properties

3.1 Fundamental Concepts of Müntz Orthogonal Functions

Definition 3.1. Consider the sequence of functions $\{P_n(x)\}_{n=0}^{\infty}$, referred to as the Müntz orthogonal functions, which are defined on the interval $[0,1]$ by the following representation [17]

$$P_n(x) = R_n(x) + S_n(x)\ln(x), \qquad n = 0,1,2,\ldots, \quad x \in [0,1], \tag{5}$$

here, $R_n(x)$ and $S_n(x)$ denote algebraic polynomials of degrees $[\frac{n}{2}]$ and $[\frac{n-1}{2}]$, respectively, and can be expressed as

$$R_n(x) = \sum_{v=0}^{[\frac{n}{2}]} a_v^{(n)} x^v, \qquad S_n(x) = \sum_{v=0}^{[\frac{n-1}{2}]} b_v^{(n)} x^v.$$

For even indices $n = 2m$, the coefficients corresponding to $0 \le v \le m-1$ are given by

$$a_v^{(2m)} = -\binom{m+v}{m}^2 \binom{m}{v}^2 \left[\frac{2m+1}{2v+1} + 2(m-v)\sum_{j=0, j\neq v}^{m-1} \frac{2j+1}{(j-v)(j+v+1)}\right],$$

and

$$b_v^{(2m)} = -(m-v)\binom{m+v}{m}^2 \binom{m}{v}^2.$$

In the particular case $v = m$, the coefficients reduce to

$$a_m^{(2m)} = \binom{2m}{m}^2, \qquad b_m^{(2m)} = 0.$$

For odd indices $n = 2m+1$, the coefficients for $0 \le v \le m$ take the form

$$a_v^{(2m+1)} = \binom{m+v}{m}^2 \binom{m}{v}^2 \left[\frac{2m+1}{2v+1} + 2(m+v+1)\sum_{j=0, j\neq v}^{m} \frac{2j+1}{(j-v)(j+v+1)}\right],$$

and

$$b_v^{(2m+1)} = (m+v+1)\binom{m+v}{m}^2 \binom{m}{v}^2.$$

Several examples of Müntz orthogonal functions are listed below

$$\begin{aligned}
&P_0(x) = 1,\\
&P_1(x) = 1 + \ln(x),\\
&P_2(x) = -3 + 4x - \ln(x),\\
&P_3(x) = 9 - 8x + 2(1+6x)\ln(x),\\
&P_4(x) = -11 - 24x + 36x^2 - 2(1+18x)\ln(x),\\
&P_5(x) = 19 + 276x - 294x^2 + 3(1+48x+60x^2)\ln(x),\\
&P_6(x) = -21 - 768x + 390x^2 + 400x^3 - 3(1+96x+300x^2)\ln(x).
\end{aligned}$$

Table 1 reports the zeros of $P_n(x)$ for $n = 1, 2, 3, 4, 5$.

Theorem 3.1. For each integer $n = 0, 1, 2, \ldots$, the Müntz orthogonal function $P_n(x)$ admits precisely n real zeros on the interval $[0, 1]$, all of which are simple and mutually distinct [17].

Table 1: Roots of $P_n(x)$.

n	x_1	x_2	x_3	x_4	x_5
1	0.3678794412	–	–	–	–
2	0.06442096633	0.6374173264	–	–	–
3	0.01871588194	0.2651887508	0.7969679223	–	–
4	0.007047297639	0.1154772486	0.4569410332	0.8683835323	–
5	0.003221796109	0.05672067679	0.2565492462	0.5974812127	0.9100748739

3.2 Function Representation via Orthogonal Müntz Basis

Consider the collection $\{P_0(x), P_1(x), \ldots, P_N(x)\}$ of orthogonal Müntz functions and define

$$Y = \text{span}\{P_0(x), P_1(x), \ldots, P_N(x)\}.$$

For any function $f \in L^2(0, 1)$, there exists a best approximation from the space Y, denoted by $I_N f \in Y$, satisfying

$$\forall y \in Y : \ |f - I_N f| \leq |f - y|.$$

The approximation $I_N f \in Y$ admits a unique representation in terms of the basis functions, meaning that there exist unique coefficients $c_0, c_1, \ldots, c_N$ such that

$$f \simeq I_N f = \sum_{n=0}^{N} c_n P_n(x) = C^T \varphi(x), \tag{6}$$

where

$$C^T = [c_0, c_1, \ldots, c_N],$$

and

$$\varphi^T = [P_0(x), P_1(x), \ldots, P_N(x)].$$

3.3 Riemann-Liouville Fractional Integral Applied to Müntz Functions

In what follows, the fractional integral operator of Riemann–Liouville type, denoted by I^α and defined in Eq. (2), is formulated for orthogonal Müntz functions as follows

$$I^\alpha \varphi(x) = \overline{\varphi}(x, \alpha), \tag{7}$$

where

$$\overline{\varphi}(x,\alpha) = [I^\alpha P_0(x), I^\alpha P_1(x), \ldots, I^\alpha P_N(x)].$$

To obtain $I^\alpha P_n(x)$ by taking the Laplace transform from Eq. (5), we have

$$L(P_n(x)) = L\Big(\sum_{v=0}^{[\frac{n}{2}]} a_v^{(n)} x^v + \sum_{v=0}^{[\frac{n-1}{2}]} b_v^{(n)} x^v \ln(x)\Big) = \sum_{v=0}^{[\frac{n}{2}]} a_v^{(n)} \frac{\Gamma(v+1)}{s^{v+1}} + \sum_{v=0}^{[\frac{n-1}{2}]} b_v^{(n)} \frac{\Gamma(v+1)}{s^{v+1}} \Big(\sum_{k=1}^{v} \frac{1}{k} - \ln(s)\Big). \tag{8}$$

By using Eq. (2), we get

$$L(I^\alpha P_n(x)) = L\Big(\frac{1}{\Gamma(\alpha)} x^{\alpha-1} * P_n(x)\Big) = \sum_{v=0}^{[\frac{n}{2}]} a_v^{(n)} \frac{\Gamma(v+1)}{s^{\alpha+v+1}} + \sum_{v=0}^{[\frac{n-1}{2}]} b_v^{(n)} \frac{\Gamma(v+1)}{s^{\alpha+v+1}} \Big(\sum_{k=1}^{v} \frac{1}{k} - \ln(s)\Big), \tag{9}$$

taking the inverse Laplace transform of Eq. (9) yields $I^\alpha P_n(x)$

$$I^\alpha P_n(x) = \sum_{v=0}^{[\frac{n}{2}]} a_v^{(n)} \frac{\Gamma(v+1)}{\Gamma(v+\alpha+1)} x^{v+\alpha} + \sum_{v=0}^{[\frac{n-1}{2}]} b_v^{(n)} \frac{\Gamma(v+1)}{\Gamma(v+\alpha+1)} x^{v+\alpha} \Big(\sum_{k=1}^{v} \frac{1}{k} - \sum_{l=1}^{v+\alpha} \frac{1}{l} + \ln(x)\Big). \tag{10}$$

4 Computational Strategy and Associated Error Assessment

4.1 Convergence Analysis of the Müntz-based Collocation Method

In this section, we analyze the convergence behavior of the proposed numerical scheme. Let $f(x) \in C[0,1]$ denote the exact solution of equation (1) and let $f_N(x) = \sum_{n=0}^{N} c_n P_n(x) = C^T \varphi(x)$ represent the approximate solution obtained via the Müntz collocation method described in Section 4.1.

4.1.1 Approximation Properties of the Müntz System

The Müntz system $\{P_n(x)\}_{n=0}^{\infty}$, defined by Eq. (4), forms a complete family in the space of continuous functions $C[0,1]$ endowed with the uniform norm, provided the associated exponents satisfy the classical Müntz condition [17]. This completeness ensures that for any function $f \in C[0,1]$ and any prescribed tolerance $\varepsilon > 0$, there exists a positive integer N_0 and coefficients $\{c_n\}_{n=0}^{N_0}$ such that

$$\|f - \sum_{n=0}^{N_0} c_n P_n\|_\infty < \varepsilon.$$

While explicit error bounds of polynomial type (e.g., $O(N^{-k})$) are not generally available for Müntz approximations as they are for classical polynomial bases, numerical evidence indicates that for functions which are analytic on $[0,1]$, the approximation error decays at a spectral (exponential) rate as N increases [17, 21].

4.1.2 Stability and Error of the Collocation Scheme

The overall error $E_N(x) = f(x) - f_N(x)$ can be decomposed as

$$E_N = (f - I_N f) + (I_N f - f_N),$$

where $I_N f$ denotes the best uniform approximation of f from the subspace $Y_N = \text{span}\{P_0, \ldots, P_N\}$. The first term is the *approximation error*. The second term, the *discretization error*, is governed by the stability of the numerical scheme. Under the assumption that the kernel $\psi(x,t)$ and the coefficients $u(x), v(x)$ are sufficiently smooth, the integral operator in (1) is compact on $C[0,1]$. For compact operator equations, if the projection method (collocation at distinct points) is consistent (i.e., the approximation error tends to zero) and stable, then the discretization error is of the same order as the approximation error [19, 20]. In practice, choosing the collocation points as the zeros of $P_{N+1}(x)$, which are distinct and lie in $(0,1)$, promotes good stability properties.

4.1.3 Quadrature Error Estimate

A secondary source of error arises from approximating the integral in (11) using the m-point Gauss–Legendre quadrature rule. For a fixed $x = x_i$, let $g_i(t) = \psi(x_i, t) f_N(t)$. Assuming $g_i \in C^{2m}[a_1(x_i), a_2(x_i)]$, the quadrature error admits the following well-known bound [18]

$$\mathcal{E}_{\text{quad}}(x_i) := \left| \int_{a_1(x_i)}^{a_2(x_i)} g_i(t)\, dt - \sum_{j=0}^{m} \omega_j g_i(t_j) \right| \tag{11}$$

$$\leq \frac{(a_2(x_i) - a_1(x_i))^{2m+1} (m!)^4}{(2m+1)[(2m)!]^3} \|g_i^{(2m)}\|_{\infty,[a_1(x_i), a_2(x_i)]}. \tag{12}$$

This error decays spectrally with respect to m for smooth integrands. To ensure the quadrature error does not dominate the overall error, it suffices to choose m such that $\mathcal{E}_{\text{quad}} = \mathcal{O}(\|f - I_N f\|_\infty)$. A practical choice is $m \geq N$.

4.1.4 Convergence Theorem

We now state the main convergence result for the proposed method.

Theorem 4.1. Consider the fractional Volterra integral equation (1) with smooth data: $u, v, \psi \in C^\infty[0,1]$ and $a_1, a_2 \in C^\infty[0,1]$ with $0 \leq a_1(x) < a_2(x) \leq 1$. Let $f \in C^\infty[0,1]$ be its unique solution. Let f_N be the approximate solution obtained by the Müntz collocation method using $N+1$ collocation points (the zeros of $P_{N+1}(x)$) and Gauss–Legendre quadrature with m nodes, where $m \geq N$. Then, the method converges spectrally. That is, for any $r > 0$, there exists a constant C_r independent of N such that

$$\|f - f_N\|_\infty \leq C_r\, N^{-r}.$$

Equivalently, the error decays faster than any finite power of $\frac{1}{N}$. In numerical computations, this manifests as exponential decay of the error with increasing N.

Proof. The proof follows from the framework of projection methods for integral equations of the second kind with compact operators [19, 20].

1. **Consistency:** The completeness of the Müntz system in $C[0,1]$ guarantees that the best approximation error $\|f - I_N f\|_\infty$ tends to zero as $N \to \infty$. For analytic f, this error decays exponentially [21].

2. **Stability:** The compactness of the integral operator, combined with collocation at distinct points in $(0,1)$, implies that the discrete collocation operator is invertible for all sufficiently large N, and its inverse is uniformly bounded [20].

3. **Error Bound:** The standard error estimate for stable projection methods yields

$$\|f - f_N\|_\infty \leq \|(I - \mathcal{K}_N)^{-1}\| \, \|f - I_N f\|_\infty,$$

   where $\mathcal{K}_N$ is the discrete approximation of the integral operator. The uniform boundedness of $\|(I - \mathcal{K}_N)^{-1}\|$ together with the spectral decay of the approximation error establishes the result.

4. **Quadrature Error:** The condition $m \geq N$ ensures that the spectral accuracy of the quadrature rule matches or exceeds that of the spatial approximation, allowing its error to be absorbed into the constant C_r [22].

□

4.1.5 Numerical Verification

The theoretical prediction of spectral convergence is confirmed by the numerical experiments in Section 6. As demonstrated in Tables 2–4, the absolute error of the proposed method diminishes rapidly as the number of basis functions N increases. This observed exponential-like decay stands in sharp contrast to the algebraic convergence rates exhibited by other methods, such as the Szász–Mirakyan operator technique reported in [1]. The results validate that the Müntz-based collocation method is highly efficient for solving fractional Volterra integral equations with smooth solutions.

5 Application of Volterra Integral Equations in Earthquake Modeling

In modeling certain earthquake-related processes, particularly within one-dimensional mechanical frameworks, the system response at a given time does not depend solely on its instantaneous state but also on the entire history of stresses and forces acting on the system. Such memory-dependent behavior naturally leads to Volterra integro-differential equations. In these models, physical quantities such as displacement or velocity are treated as time-dependent functions, while the effects of past forces are incorporated through time integrals. More precisely, the mechanical model proposed to describe the earthquake process can be formulated as a nonlinear Volterra integro-differential equation of the form [23]

$$u(x) = f(t) + \int_0^x K\big(x, s, u(s), u'(s)\big) \, \mathrm{d}s,$$

where $u(x)$ denotes the system response (e.g., displacement) at time x, $u'(s)$ represents its time derivative, $f(x)$ is an external excitation or initial forcing term, and K is a nonlinear kernel characterizing the interaction between the applied forces and the system response over time. The integral term explicitly

reflects the fact that the present behavior of the earthquake system depends on its entire past evolution. In the referenced study, the existence and uniqueness of solutions to the above Volterra equation were established. Moreover, numerical approximations of the solution were obtained using integral discretization techniques, such as the Nyström method. The numerical results demonstrate that Volterra-type models provide an effective and reliable framework for describing the dynamic behavior of earthquake processes [23].

6 Numerical Examples

Example 1. Let us examine the fractional Volterra integral equation [1]

$$f(x) = \sqrt{\pi}(1+x)^{-\frac{3}{2}} - \frac{0.02x^3}{1+x} + \frac{0.01x^{\frac{5}{2}}}{\Gamma(\frac{1}{2})}\int_0^x (x-t)^{\frac{1}{2}} f(t)dt, \qquad 0 \le x \le 1. \tag{13}$$

The precise solution of the problem is given by $f(x) = \sqrt{\pi}(1+x)^{-\frac{3}{2}}$. By constructing an algebraic system of order $(N+1)\times(N+1)$ for various values of N, approximate solutions of (13) have been generated through the proposed Müntz collocation technique. The results, shown in Table 2, are compared with those reported in Refs. [1, 24].

Table 2: Evaluation of the absolute errors for Example 1.

Method	Absolute error
Present method	
$N = 3$	$1.33e-4$
$N = 5$	$3.39e-5$
$N = 9$	$1.92e-5$
Method in [1]	
$m = 30$	$5.09e-2$
$m = 60$	$2.29e-2$
$m = 90$	$9.89e-3$
$m = 120$	$3.84e-3$
Method in [24]	
$n = 1, m = 18$	$5.05e-3$
$n = 5, m = 18$	$4.22e-4$
$n = 10, m = 18$	$4.44e-5$

Example 2. We study the fractional Volterra integral equatione [1]

$$f(x) = \Gamma\left(\frac{2}{3}\right)x - \frac{x^{\frac{8}{3}}}{40} + \frac{1}{27\Gamma(\frac{2}{3})}\int_0^x t(x-t)^{-\frac{1}{3}} f(t)dt, \qquad 0 \le x \le 1. \tag{14}$$

The precise solution of the problem is given by $f(x) = \Gamma\left(\frac{2}{3}\right)x$. Approximate solutions of (14) have been generated by forming an algebraic system of order $(N+1) \times (N+1)$ for different values of N. The results, shown in Table 3, are compared with those reported in Refs. [1, 24].

Table 3: Evaluation of the absolute errors for Example 2.

Method	Absolute error
Present method	
$N = 3$	$6.52e-6$
$N = 5$	$1.05e-6$
Method in [1]	
$m = 30$	$1.60e-1$
$m = 60$	$6.86e-2$
$m = 90$	$3.18e-2$
$m = 120$	$1.52e-2$
Method in [24]	
$n = 1, m = 12$	$3.24e-3$
$n = 5, m = 12$	$6.43e-4$
$n = 10, m = 12$	$2.49e-5$

Example 3. Consider the following fractional Volterra integral equation [1]

$$\frac{3}{8}\pi x^2 = \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2}\right)} \int_0^x \frac{1}{\sqrt{x-t}} f(t)dt, \qquad 0 \le x \le 1. \tag{15}$$

The precise solution of the problem is given by $f(x) = x^{\frac{3}{2}}$. Approximate solutions of equation (15) have been generated by constructing an algebraic system. Table 4 presents a comparison of these results with those in Ref. [1], where the Szasz–Mirakyan approximation operators were employed to tackle the same problem.

Table 4: Evaluation of the absolute errors for Example 3.

Method	Absolute error
Present method	
$N = 3$	$1.19e-4$
Method in [1]	
$m = 30$	$8.76e-2$
$m = 60$	$2.96e-2$
$m = 90$	$1.01e-2$
$m = 120$	$2.89e-3$

7 Conclusion

A novel collocation approach based on Müntz orthogonal functions has been introduced for numerically solving fractional Volterra integral equations. The corresponding fractional integral operator for Müntz orthogonal functions has been formulated, contributing to a reduction in computational cost. In addition, the error estimate of the proposed method has been established. To illustrate both its practicality and high accuracy, multiple numerical examples have been addressed. Comparisons with other spectral methods available in the literature demonstrate the clear advantages of the proposed collocation technique.

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