

Accepted manuscript (author version)

To appear in:

International Journal of Mathematical Modelling & Computations

Online ISSN: 2228-6233

Print ISSN: 2228-6225

This PDF file is not the final version of the record. This version will undergo further copyediting, typesetting, and production review before being published in its definitive form.

We are sharing this version to provide early access to the article. Please be aware that errors that could impact the content may be identified during the production process, and all legal disclaimers applicable to the journal remain valid.

Received: 9- Novembre-2025

Revised: 8- February -2026

Accepted: 12- February-2026

Accepted manuscript (author version)



An Operational Spectral Framework for Estimating Solutions of Systems of Fractional Integro-Differential Equations with Variable-order Derivatives

Shima Mizani¹, Hamideh Ebrahimi^{1,*}  and Mahmoud Shafiee¹

¹ Department of Mathematics, Ra.C., Islamic Azad University, Rasht, Iran.

*Corresponding author: ebrahimi60@iau.ac.ir

Abstract

This paper proposes a novel and efficient spectral numerical framework for approximating solutions of Volterra–Fredholm fractional integro-differential equations with variable-order derivatives. The method is based on the use of normalized and shifted Vieta–Lucas orthogonal polynomials as basis functions to represent the unknown solution. New operational matrices are derived for both the Caputo fractional derivative and the Riemann–Liouville fractional integral operators, enabling the transformation of the original fractional system into a system of linear or nonlinear algebraic equations. The proposed scheme exhibits favorable properties such as simple implementation, strong numerical stability, and reliable convergence behavior. Rigorous theoretical results concerning the existence, uniqueness, and convergence of the approximate solution are established. Several numerical examples are presented to demonstrate the accuracy and computational efficiency of the proposed approach.

Keywords: Volterra-Fredholm integro-differential equations; Vieta-Lucas polynomials; Variable-order fractional integral and derivatives operators; Operational matrix.

1. Introduction

...Fractional calculus is a specialized field of mathematical analysis that generalizes the concepts of differentiation and integration to arbitrary (non-integer) orders. While traditional integer-order models are commonly employed in the study of physical processes, fractional calculus provides a more flexible framework, making it particularly suitable for modeling complex systems. This flexibility is reflected in the growing application of fractional differential and integral equations across diverse fields such as electrochemistry, electromagnetism, diffusion processes, and transport theory [1-14]. One of the main challenges associated with fractional integro-differential

equations is the lack of closed-form analytical solutions in most cases. Consequently, numerous numerical and approximation methods have been developed to address this issue. These include finite difference methods [15], the sixth-kind Chebyshev collocation method [16], Sinc collocation [17], Wavelet spectral methods [18, 19], Block Pulse function approaches [20-22], Adams-Bashforth methods [23], Taylor collocation [24], Radial basis functions [25], and the use of various polynomials such as Legendre, Chebyshev, and Laguerre [26, 27].

Fractional differential equations, particularly those involving the Caputo derivative, play a significant role in modeling complex phenomena in science and engineering. Since analytical solutions

are often unavailable, efficient numerical techniques are essential. Among these, operational matrix methods have attracted considerable attention due to their high accuracy and computational efficiency. In particular, Vieta-Lucas polynomials have demonstrated strong potential as basis functions for constructing operational matrices. Recent studies have employed these polynomials to develop effective numerical schemes. For instance, Riahi Beni (2025) introduced a method for variable-order fractional integro-differential equations [28], achieving high accuracy, while earlier work successfully applied similar operational matrices to Caputo fractional differential equations, validating their performance through numerical experiments [29]. Motivated by these developments, the present study further investigates operational matrices based on Vieta-Lucas polynomials, aiming to provide a flexible and accurate framework for solving a broad class of fractional differential equations. A common approach to solving fractional systems is to express the unknown solution as a linear combination of orthogonal basis functions.

This strategy offers several advantages, including ease of implementation and improved approximation accuracy. Moreover, it relies on operational matrices specifically constructed for the chosen orthogonal basis [30]. The main idea of this technique is to convert the original fractional system into a system of linear or nonlinear algebraic equations, which can then be solved iteratively. In this study, Newton's iterative method is employed to solve the resulting algebraic system. Newton's method is a well-known numerical technique used to find the roots of nonlinear equations or the extrema of functions. It is based on an iterative recurrence relation, where approximate derivatives are used instead of exact ones. In this work, we propose a numerical procedure for solving a system of variable-order fractional integro-differential equations of the form:

$$\begin{aligned}
 & {}^c \mathcal{D}^{\alpha_k(x)} v_k(x) = f_k(x) \\
 & + \mathcal{F}_k(x, v_1(x), \mathcal{D}^{(1)} v_1(x), \dots, \mathcal{D}^{(n_1)} v_1(x), \\
 & \quad \dots, v_m(x), \mathcal{D}^{(1)} v_m(x), \dots, \mathcal{D}^{(n_m)} v_m(x)) \\
 & + \int_0^x \mathcal{G}_{1,k}(x, t, v_1(t), \mathcal{D}^{(1)} v_1(t), \dots, \mathcal{D}^{(n_1)} v_1(t), \\
 & \quad \dots, v_m(t), \mathcal{D}^{(1)} v_m(t), \dots, \mathcal{D}^{(n_m)} v_m(t)) dt \\
 & + \int_0^L \mathcal{G}_{2,k}(x, t, v_1(t), \mathcal{D}^{(1)} v_1(t), \dots, \mathcal{D}^{(n_1)} v_1(t), \\
 & \quad \dots, v_m(t), \mathcal{D}^{(1)} v_m(t), \dots, \mathcal{D}^{(n_m)} v_m(t)) dt.
 \end{aligned}
 \tag{1}$$

$$\begin{aligned}
 & \mathcal{D}^{(i)} v_k(0) = \gamma_{k,i}, \quad i = 0, 1, \dots, n_k - 1, \\
 & n_k - 1 < \alpha_{k,\min} < \alpha_k(x) < \alpha_{k,\max} \leq n_k, \\
 & n_k = \lceil \alpha_{k,\max} \rceil, \quad k = 1, 2, \dots, m.
 \end{aligned}$$

Here, for $k = 1, 2, 3, \dots, m$, ${}^c \mathcal{D}^{\alpha_k(x)}$, $\alpha_k(x) \geq 0$, are the Caputo fractional derivative operators, \mathcal{F}_k , $\mathcal{G}_{1,k}$ and $\mathcal{G}_{2,k}$ are non-linear or linear functions; $f_k(x)$ are continuous function. Also $v_k(x) \in C^{n_k}[0, L]$ are unknown functions. Finally, a numerical scheme based on normalized shifted Vieta-Lucas orthogonal polynomials is proposed to obtain approximate solutions for the considered fractional integro-differential systems. Shifted and normalized Vieta-Lucas polynomials provide an efficient spectral basis for the numerical solution of fractional integro-differential equations, Similar to classical orthogonal polynomials such as Chebyshev and Legendre, they exhibit spectral (exponential) convergence for smooth solutions. However, unlike Chebyshev polynomials, which typically suffer from strong error concentration and oscillatory behavior near the interval boundaries, the normalized Vieta-Lucas basis yield a more uniform error distribution over the entire computational domain. This property is particularly important in fractional order problems, where weak singularities often arise near the initial point. From a numerical stability viewpoint, the conditioning of the algebraic systems generated by spectral discretization plays a crucial role. It is well known that operational matrices associated with Chebyshev and Legendre polynomials may become increasingly ill-conditioned for high polynomial degrees, especially when fractional differential operators are involved. In contrast, shifted Vieta-Lucas polynomials lead to better conditioned operational matrices, resulting in enhanced numerical stability and reduced sensitivity to round off errors. Moreover, the global nature of Vieta-Lucas polynomials makes them well suited for the nonlocal behavior of fractional derivatives defined in the Caputo or Riemann-Liouville sense. Compared with local approximation techniques or Lagrange type bases, they compare memory effects more efficiently with fewer degrees of freedom. In addition, the shifted formulation allows straightforward implementations of initial and boundary conditions on finite intervals. These advantages motivate the adoption of shifted and normalized Vieta-Lucas polynomials as an effective and reliable basis for solving fractional integro-differential equations.

This paper introduces a novel spectral method for solving fractional systems. The proposed approach utilizes normalized shifted Vieta-Lucas polynomials as basis functions to represent the solution. By deriving operational matrices for fractional operators, we convert the original system into a system of algebraic equations, which can be efficiently solved using numerical techniques. We rigorously analyze the existence, uniqueness, and convergence properties of the proposed method. Numerical results demonstrate the accuracy and efficiency of this approach. Previous research has



explored the application of Vieta-Lucas polynomials in solving fractional problems. In 2016, Kocer studied polynomials of the bivariate Vieta-Lucas polynomials [31]. Subsequently, Heydari et al. [32] and Agarwal et al. [33] demonstrated the use of Vieta-Lucas polynomials in approximating fractional non-linear equations and solving a mathematical physics model involving fractional derivatives, respectively. These studies suggest that Vieta-Lucas normalized orthogonal transposed polynomials offer a novel approach to solving fractional problems.

Although Vieta-Lucas polynomials have been previously employed for solving fractional differential equations, existing studies are mainly restricted to constant-order derivatives or single equations. In contrast, the present work develops a unified operational spectral framework for systems of Volterra-Fredholm fractional integro-differential equations with variable-order derivatives, which significantly broadens the applicability of Vieta-Lucas-based numerical methods. Recent studies such as Heydari et al. in [32] and Agarwal et al. in [30] successfully employed Vieta-Lucas polynomials for fractional differential equations; however, their analyses were limited to constant-order derivatives and single fractional models without integral terms. More recently, Riahi Beni in [28] addressed variable-order fractional integro-differential equations using Vieta-Lucas-type bases, but the proposed approach was restricted to individual equations and did not consider coupled Volterra-Fredholm systems. In contrast, the present work develops a unified operational spectral framework for systems of fractional integro-differential equations involving both Volterra and Fredholm operators with variable-order Caputo derivatives. Moreover, new operational matrices are constructed for variable-order fractional integrals and derivatives within a normalized shifted Vieta-Lucas polynomial basis, which distinguishes this study from existing contributions. Building upon this foundation, we propose a new method to approximate the solution of fractional integro-differential equations systems (1) by employing these orthogonal polynomials as basis functions. The proposed procedure utilizes derivative and integral operational matrices to efficiently approximate the solution. The main contribution of this work is the development of a novel spectral numerical framework based on normalized shifted Vieta-Lucas orthogonal polynomials for solving Volterra-Fredholm fractional integro-differential equations with variable-order Caputo derivatives. Unlike existing methods that are mainly limited to constant-order formulations, the proposed approach simultaneously incorporates variable-order fractional operators, nonlocal Volterra-Fredholm integrals, and nonlinear effects within a unified

operational matrix formulation. New operational matrices for the Caputo fractional derivative and the Riemann-Liouville fractional integral are derived, enabling an efficient reduction of the original problem to a well-conditioned system of algebraic equations. The resulting scheme demonstrates improved numerical stability compared with classical Chebyshev and Legendre-based spectral methods. Rigorous theoretical analysis establishes existence, uniqueness, and convergence, while numerical experiments confirm the accuracy and robustness of the proposed method for both linear and nonlinear fractional models.

The structure of the remainder of this paper is outlined as follows: Section 2 presents preliminaries of fractional calculus, including variable-order Caputo and Riemann-Liouville operators and their essential properties. Section 3 presents the Vieta-Lucas polynomials and explores their key attributes. Section 4 develops the associated operational matrices (integral, product, and fractional operators). Section 5 is dedicated to describing the proposed methodology and its implementation details. Section 6 is devoted to establishing the existence and uniqueness of the solution to system (1). Section 7 contains a set of theorems addressing the convergence and stability of the developed method. To demonstrate the practicality of the proposed technique, two illustrative examples are discussed in Section 8. The final section concludes the study with a summary of the main findings

2. Preliminaries of Fractional Calculus

In this chapter, the definitions and necessary contents of fractional derivative and integer and attributes for them are presented [10].

Definition 2.1. The Riemann-Liouville fractional integral of variable order $\alpha(x) > 0$ is appointed by

$${}_{RL}J_t^{\alpha(x)} \mathfrak{U}(x) = \begin{cases} \mathfrak{U}(x), & \text{for } \alpha(x) = 0, \\ \frac{1}{\Gamma(\alpha(x))} \int_0^x (x-t)^{\alpha(x)-1} \mathfrak{U}(x) dt, & \\ \text{for } 0 < n-1 < \alpha(x) \leq n. \end{cases} \quad (2)$$

where $\mathfrak{U}(x) \in \mathcal{C}^n([a, b])$, and $n = [\alpha(x)]$ is an integer number.

Definition 2.2. The Caputo fractional derivative of variable order $\alpha(x) > 0$ is defined by

$${}^C\mathcal{D}^{\alpha(x)} \mathfrak{U}(x) = \begin{cases} \frac{d^n \mathfrak{U}(x)}{dx^n}, & \text{for } \alpha(x) = n, \\ \frac{1}{\Gamma(n-\alpha(x))} \int_a^x \frac{\mathfrak{U}^{(n)}(t)}{(x-t)^{\alpha(t)-n+1}} dt, & \\ \text{for } 0 < n-1 < \alpha(x) \leq n. \end{cases}$$



Definition 2.3. The Riemann-Liouville fractional derivative of variable order $\alpha(x) > 0$ is defined as

$${}^{RL}\mathfrak{D}^{\alpha(x)}\mathfrak{U}(x) = \begin{cases} \frac{d^n \mathfrak{U}(x)}{dx^n}, & \text{for } \alpha(x) = n, \\ \frac{1}{\Gamma(n - \alpha(x))} \frac{d^n}{dx^n} \int_a^x \frac{\mathfrak{U}(t)}{(x-t)^{\alpha(x)-n+1}} dt, & \text{for } 0 < n-1 < \alpha(x) \leq n. \end{cases} \quad (4)$$

2.1. Fundamental Properties of Fractional Operators

Some important properties of the above fractional operators are summarized below, where $\mathfrak{D}^{\alpha(x)}$ denotes the fractional derivative in the sense of either Caputo or Riemann–Liouville.

$$1. \begin{cases} {}^{RL}\mathfrak{J}^{\alpha(x)}(\beta_1 \mathfrak{U}_1(x) + \beta_2 \mathfrak{U}_2(x)) = \\ \beta_1 {}^{RL}\mathfrak{J}^{\alpha(x)}\mathfrak{U}_1(x) + \beta_2 {}^{RL}\mathfrak{J}^{\alpha(x)}\mathfrak{U}_2(x), \\ \mathfrak{D}^{\alpha(x)}(\beta_1 \mathfrak{U}_1(x) + \beta_2 \mathfrak{U}_2(x)) = \\ \beta_1 \mathfrak{D}^{\alpha(x)}\mathfrak{U}_1(x) + \beta_2 \mathfrak{D}^{\alpha(x)}\mathfrak{U}_2(x), \end{cases}$$

$$2. \begin{cases} {}^{RL}\mathfrak{J}^{\alpha_1(x)} {}^{RL}\mathfrak{J}^{\alpha_2(x)}\mathfrak{U}(x) \neq {}^{RL}\mathfrak{J}^{\alpha_2(x)} {}^{RL}\mathfrak{J}^{\alpha_1(x)}\mathfrak{U}(x), \\ \mathfrak{D}^{\alpha_1(x)} \mathfrak{D}^{\alpha_2(x)}\mathfrak{U}(x) \neq \mathfrak{D}^{\alpha_2(x)} \mathfrak{D}^{\alpha_1(x)}\mathfrak{U}(x), \end{cases}$$

$$3. \begin{cases} {}^{RL}\mathfrak{J}^{\alpha_1(x)} {}^{RL}\mathfrak{J}^{\alpha_2(x)}\mathfrak{U}(x) \neq {}^{RL}\mathfrak{J}^{\alpha_1(x)+\alpha_2(x)}\mathfrak{U}(x), \\ \mathfrak{D}^{\alpha_1(x)} \mathfrak{D}^{\alpha_2(x)}\mathfrak{U}(x) \neq \mathfrak{D}^{\alpha_1(x)+\alpha_2(x)}\mathfrak{U}(x), \end{cases}$$

$$4. \begin{cases} {}^{RL}\mathfrak{J}^{\alpha(x)} x^\zeta \approx \frac{\Gamma(\zeta + 1)}{\Gamma(\zeta - \alpha(x) + 1)} t^{\zeta + \alpha(x)}, \\ \text{for } \zeta < [\alpha(x)] - 1, \\ \mathfrak{D}^{\alpha(x)} x^\zeta \approx \frac{\Gamma(\zeta + 1)}{\Gamma(\zeta - \alpha(x) + 1)} t^{\zeta - \alpha(x)}, \\ \text{for } \zeta \geq [\alpha(x)] - 1, \end{cases}$$

$$5. \begin{cases} {}^{RL}\mathfrak{J}^{\alpha(x)} \mathfrak{D}^{\alpha(x)} f(x) = \\ f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a) x^k (x-a)}{k!} - \mathcal{R}_n(x), \\ \mathfrak{D}^{\alpha(x)} {}^{RL}\mathfrak{J}^{\alpha(x)} f(x) = f(x) + \tilde{\mathcal{R}}_n(x), \end{cases}$$

where

$$\mathcal{R}(x) = \int_a^x \frac{\partial}{\partial x} \left(\frac{(x-t)^{n-\alpha(x)-1}}{\Gamma(n-\alpha(x))} \right) (\mathfrak{U}^{(n)}(t) - \mathfrak{U}^{(n)}(a)) dt,$$

and

$$\tilde{\mathcal{R}}_n(x) = \frac{\alpha'(x)}{\Gamma(\alpha(x))} \int_a^x (x-t)^{\alpha(x)-1} \left(\text{Ln}(x-t) - \frac{\Gamma'(\alpha(x))}{\Gamma(\alpha(x))} \right) \mathfrak{U}(t) dt.$$

3. The Architecture of Vieta-Lucas Polynomials and Their Properties

In this section, a family of Vieta-Lucas polynomials and the corresponding recursion formula are presented [33].

Definition 3.1.

Let $|t| < 2$, thus the sequence of polynomials of degree $n, n \in \mathbb{N} \cup \{0\}$ in terms of t can be described as follows:

$$\mathcal{V}\mathcal{L}_n(t) = 2 \cos(n\theta), \quad \theta = \cos^{-1}\left(\frac{t}{2}\right), \quad \theta \in [0, \pi].$$

These polynomials are denominated Vieta-Lucas polynomials and can be satisfied in the following recurrence formula as:

$$\begin{cases} \mathcal{V}\mathcal{L}_0(t) = 2, \\ \mathcal{V}\mathcal{L}_1(t) = t, \\ \mathcal{V}\mathcal{L}_n(t) = t \mathcal{V}\mathcal{L}_{n-1}(t) - \mathcal{V}\mathcal{L}_{n-2}(t), \quad n = 2, 3, 4, \dots \end{cases}$$

Furthermore, the explicit power series form of $\mathcal{V}\mathcal{L}_n(t)$ is defined as follows:

$$\begin{cases} \mathcal{V}\mathcal{L}_0(t) = 2, \\ \mathcal{V}\mathcal{L}_1(t) = t, \\ \mathcal{V}\mathcal{L}_n(t) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{n\Gamma(n-k)}{\Gamma(k+1)\Gamma(n+1-2k)} t^{n-2k}, \\ n = 2, 3, \dots \end{cases}$$

where $[\cdot]$ is the ceiling function. Moreover, these polynomials exhibit orthogonal with respect to the weight function $\omega(t) = \frac{1}{\sqrt{4-t^2}}$ and would be as follows:

$$\begin{aligned} \langle \mathcal{V}\mathcal{L}_n(t), \mathcal{V}\mathcal{L}_m(t) \rangle_{\omega(t)} &= \int_{-2}^2 \frac{1}{\sqrt{4-t^2}} \mathcal{V}\mathcal{L}_n(t) \mathcal{V}\mathcal{L}_m(t) dx \\ &= \begin{cases} 0, & m \neq n \neq 0, \\ 2\pi, & m = n = 0, \\ 4\pi, & m = n \neq 0. \end{cases} \end{aligned} \quad (5)$$

These polynomials by using a linear transform, $x = \frac{L}{4}(t+2)$ can be defined for $x \in [0, L]$, ($L > 0$) and are called shifted Vieta-Lucas polynomials.



Definition 2.3. The sequence of shifted Vieta-Lucas polynomials of degree n ($\mathcal{V}\mathcal{L}_n^L(x)$) on $x \in [0, L]$ can be acquire as below

$$\left\{ \begin{array}{l} \mathcal{V}\mathcal{L}_0^L(x) = 2, \\ \mathcal{V}\mathcal{L}_1^L(x) = \frac{4}{L}x - 2, \\ \mathcal{V}\mathcal{L}_{n+1}^L(x) = \mathcal{V}\mathcal{L}_n(x) \left(\frac{4}{L}x - 2 \right) \\ \quad = \left(\frac{4}{L}x - 2 \right) \mathcal{V}\mathcal{L}_n^L(x) - \mathcal{V}\mathcal{L}_{n-1}^L(x), \\ \quad \text{for } n = 1, 2, 3, \dots \end{array} \right.$$

Also, it can be used in the following relation for explicit power series rule:

$$\mathcal{V}\mathcal{L}_n^L(x) = \begin{cases} 2, & \text{for } n = 0, \\ \sum_{k=0}^n (-1)^k \frac{4^{n-k} (2n) \Gamma(2n-k)}{\Gamma(k+1) \Gamma(2n-2k+1)} \left(\frac{x}{L} \right)^{n-2k}, & \text{for } n = 1, 2, \dots \end{cases}$$

The orthogonality of the shifted Vieta-Lucas polynomials is shown in the below relation:

$$\begin{aligned} & \langle \mathcal{V}\mathcal{L}_n^L(x), \mathcal{V}\mathcal{L}_m^L(x) \rangle_{\omega^L(x)} \\ &= \int_0^L \frac{1}{\sqrt{Lx-x^2}} \mathcal{V}\mathcal{L}_n^L(x) \mathcal{V}\mathcal{L}_m^L(x) dx \\ &= \begin{cases} 0, & m \neq n \neq 0, \\ 4\pi, & m = n = 0, \\ 2\pi, & m = n \neq 0. \end{cases} \end{aligned} \tag{6}$$

That $\omega^L(x) = \frac{1}{\sqrt{Lx-x^2}}$ is the weight function. The integrals are understood in the improper sense:

$$\int_0^L \omega^L(x) f(x) dx = \lim_{\epsilon \rightarrow 0^+} \int_{\epsilon}^{L-\epsilon} \omega^L(x) f(x) dx.$$

Definition 3.3. The normalized shifted Vieta-Lucas polynomials of degree n , on $[0, L]$ are defined as follows:

$$\left\{ \begin{array}{l} \overline{\mathcal{V}\mathcal{L}}_0^L(x) = \frac{\overline{\mathcal{V}\mathcal{L}}_0^L(x)}{\sqrt{4\pi}} = \frac{1}{\sqrt{\pi}}, \quad \text{for } n = 0, \\ \overline{\mathcal{V}\mathcal{L}}_n^L(x) = \frac{\overline{\mathcal{V}\mathcal{L}}_n^L(x)}{\sqrt{2\pi}} \\ \quad = \sum_{k=0}^n \frac{(-1)^k 4^{n-k} (2n) \Gamma(2n-k)}{(\sqrt{\pi}) \Gamma(k+1) \Gamma(2n-2k+1)} \left(\frac{x}{L} \right)^{n-k}, \\ \quad \text{for } n = 1, 2, \dots \end{array} \right.$$

The orthogonality of these polynomials is shown in the below expression:

$$\langle \overline{\mathcal{V}\mathcal{L}}_n^L(x), \overline{\mathcal{V}\mathcal{L}}_m^L(x) \rangle_{\omega^L(x)} = \delta_{nm} = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}$$

Therefore, the expansion of a \mathbb{L}^2 -function $u(x)$ over the interval $[0, L]$ by normalized shifted Vieta-Lucas polynomials would be as follows:

$$u(x) = \sum_{k=0}^{\infty} c_k \overline{\mathcal{V}\mathcal{L}}_k^L(x). \tag{7}$$

That c_k , $k = 0, 1, 2, \dots$, are the undetermined coefficients and can be acquired through the following relation:

$$c_k = \frac{\langle f(x), \overline{\mathcal{V}\mathcal{L}}_k^L(x) \rangle_{\omega^L(x)}}{\langle \overline{\mathcal{V}\mathcal{L}}_k^L(x), \overline{\mathcal{V}\mathcal{L}}_k^L(x) \rangle_{\omega^L(x)}}, \quad k = 0, 1, 2, \dots$$

By considering, the first $(N+1)$ terms of series in eq. (7) would be have:

$$u(x) \approx \sum_{k=0}^N c_k \overline{\mathcal{V}\mathcal{L}}_k^L(x) = C^T \overline{\mathcal{V}\mathcal{L}}_x^L, \tag{8}$$

where C and $\overline{\mathcal{V}\mathcal{L}}_x^L$ defined as below vectors

$$C = [c_0, c_1, \dots, c_N]^T,$$

$$\overline{\mathcal{V}\mathcal{L}}_x^L = [\overline{\mathcal{V}\mathcal{L}}_0^L(x), \overline{\mathcal{V}\mathcal{L}}_1^L(x), \dots, \overline{\mathcal{V}\mathcal{L}}_N^L(x)]^T.$$

These properties make the normalized shifted Vieta-Lucas polynomials particularly suitable for constructing operational matrices, which are developed in the next section.

4. Operational Matrices Associated with Vieta-Lucas Polynomials

In this section, the operational matrices corresponding to the normalized shifted Vieta-Lucas polynomials are derived.

4.1. Integral Operational Matrix

The integral operational matrix corresponding to the normalized shifted Vieta-Lucas polynomials is defined by

$$\int_0^x \overline{\mathcal{V}\mathcal{L}}_t^L(t) dt \approx \mathcal{P}_x \overline{\mathcal{V}\mathcal{L}}_x^L,$$

and $\mathcal{P}_x \in \mathbb{R}^{(N+1) \times (N+1)}$ is the integral operational matrix. The entries of \mathcal{P}_x are determined by projecting the integral of each basis function onto the basis itself, i.e.,

$$p_{(n+1),(m+1)} = \langle \int_0^x \overline{\mathcal{V}\mathcal{L}}_n^L(t) dt, \overline{\mathcal{V}\mathcal{L}}_m^L(t) \rangle_{\omega^L(x)},$$



$n, m = 0, 1, \dots, N.$

Equivalently, each integrated basis function can be expanded as

$$\int_0^x \overline{\mathcal{V}\mathcal{L}}_n^L(t) dt \approx \sum_{m=0}^N p_{n+1,m+1} \overline{\mathcal{V}\mathcal{L}}_m^L(x),$$

so that the general matrix form is

$$\mathcal{P}_x = \begin{bmatrix} p_{1,1} & p_{1,2} & \dots & p_{1,(N+1)} \\ \vdots & \vdots & \ddots & \vdots \\ p_{(N+1),1} & p_{(N+1),2} & \dots & p_{(N+1),(N+1)} \end{bmatrix}, \quad (9)$$

and any entries of the matrix \mathcal{P}_x are calculated by

$$p_{(n+1),(m+1)} = \begin{cases} \frac{a_m^1}{\sqrt{\pi}}, & n = 0, \\ \sum_{i=0}^n \frac{(-1)^i 4^{n-i} (2n) \Gamma(2n-i) a_m^{n-i+1}}{(\sqrt{2\pi}) \Gamma(i+1) \Gamma(2n-2i+1) L^{n-i} (n-i+1)}, & \text{for } n = 1, 2, \dots, N, \quad m = 0, 1, \dots, N, \end{cases}$$

where the coefficient a_m^k are obtained from the polynomial expansion

$$x^k \approx \sum_{m=0}^N a_m^k \overline{\mathcal{V}\mathcal{L}}_m^L(x), \quad k = 0, 1, 2, \dots$$

This definition ensures that the matrix operator \mathcal{P}_x serves as the exact integration operator projected onto the finite-dimensional subspace span $\{\overline{\mathcal{V}\mathcal{L}}_0^L(x), \overline{\mathcal{V}\mathcal{L}}_1^L(x), \dots, \overline{\mathcal{V}\mathcal{L}}_N^L(x)\}$. For instance, when $N = 5$ over the interval $[0, L]$, the operational matrix \mathcal{P}_x can be obtained analytically. Moreover, this formulation is fully consistent with the orthonormal structure of the shifted Vieta-Lucas basis and guarantees that \mathcal{P}_x precisely represents the integration operator in the sense of the $\mathbb{L}^2_{\omega^L(x)}$ -projection, without any unquantified numerical approximation.

$$\mathcal{P}_x = L \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{2}}{4} & 0 & 0 & 0 & 0 \\ \frac{-\sqrt{2}}{8} & 0 & \frac{1}{8} & 0 & 0 & 0 \\ \frac{-\sqrt{2}}{6} & \frac{-1}{4} & 0 & \frac{1}{12} & 0 & 0 \\ \frac{\sqrt{2}}{16} & 0 & \frac{-1}{8} & 0 & \frac{1}{16} & 0 \\ \frac{-\sqrt{2}}{30} & 0 & 0 & \frac{-1}{12} & 0 & \frac{1}{20} \\ \frac{-\sqrt{2}}{48} & 0 & 0 & 0 & \frac{-1}{16} & 0 \end{bmatrix}, \quad (10)$$

4.2. The Product Operational Matrix

The general form of the operational product matrix for the normalized shifted Vieta-Lucas polynomials is defined as

$$\overline{\mathcal{V}\mathcal{L}}_x^L (\overline{\mathcal{V}\mathcal{L}}_x^L)^T C \approx \tilde{\mathcal{C}}_x \overline{\mathcal{V}\mathcal{L}}_x^L, \quad (11)$$

where C is a vector with $(N+1)$ components, and $\tilde{\mathcal{C}}_x$ is an $(N+1) \times (N+1)$ matrix whose entries are obtained as below

$$\tilde{c}_{i,j} = \langle (\overline{\mathcal{V}\mathcal{L}}_x^L (\overline{\mathcal{V}\mathcal{L}}_x^L)^T C)_i, \overline{\mathcal{V}\mathcal{L}}_j^L(x) \rangle_{\omega^L(x)}, \quad (12)$$

$$i, j = 1, 2, \dots, N+1.$$

Moreover, the following orthonormality property holds:

$$\int_0^L \omega^L(x) \overline{\mathcal{V}\mathcal{L}}_x^L (\overline{\mathcal{V}\mathcal{L}}_x^L)^T dx = I_{N+1}, \quad (13)$$

where I_{N+1} is the $(N+1) \times (N+1)$ identity matrix. For example, by considering $N = 5$, the operational product matrix $\tilde{\mathcal{C}}_x$ takes the form

$$\tilde{\mathcal{C}}_x = \begin{bmatrix} \frac{c_1}{\sqrt{\pi}} & \frac{c_2}{\sqrt{\pi}} & \frac{c_3}{\sqrt{\pi}} & \frac{c_4}{\sqrt{\pi}} & \frac{c_5}{\sqrt{\pi}} & \frac{c_6}{\sqrt{\pi}} \\ \frac{c_2}{\sqrt{\pi}} & \frac{\sqrt{2}c_3 + 2c_1}{2\sqrt{\pi}} & \frac{(c_2 + c_4)\sqrt{2}}{2\sqrt{\pi}} & \frac{(c_3 + c_5)\sqrt{2}}{2\sqrt{\pi}} & \frac{(c_4 + c_6)\sqrt{2}}{2\sqrt{\pi}} & \frac{c_5\sqrt{2}}{2\sqrt{\pi}} \\ \frac{c_3}{\sqrt{\pi}} & \frac{(c_2 + c_4)\sqrt{2}}{2\sqrt{\pi}} & \frac{\sqrt{2}c_5 + 2c_1}{2\sqrt{\pi}} & \frac{(c_6 + c_2)\sqrt{2}}{2\sqrt{\pi}} & \frac{c_3\sqrt{2}}{2\sqrt{\pi}} & \frac{c_4\sqrt{2}}{2\sqrt{\pi}} \\ \frac{c_4}{\sqrt{\pi}} & \frac{(c_3 + c_5)\sqrt{2}}{2\sqrt{\pi}} & \frac{(c_6 + c_2)\sqrt{2}}{2\sqrt{\pi}} & \frac{c_1}{\sqrt{\pi}} & \frac{c_2\sqrt{2}}{2\sqrt{\pi}} & \frac{c_3\sqrt{2}}{2\sqrt{\pi}} \\ \frac{c_5}{\sqrt{\pi}} & \frac{(c_6 + c_4)\sqrt{2}}{2\sqrt{\pi}} & \frac{c_3}{2\sqrt{\pi}} & \frac{c_2\sqrt{2}}{2\sqrt{\pi}} & \frac{c_1}{\sqrt{\pi}} & \frac{c_2\sqrt{2}}{2\sqrt{\pi}} \\ \frac{c_6}{\sqrt{\pi}} & \frac{c_4\sqrt{2}}{\sqrt{\pi}} & \frac{c_4\sqrt{2}}{\sqrt{\pi}} & \frac{c_3\sqrt{2}}{\sqrt{\pi}} & \frac{c_2\sqrt{2}}{2\sqrt{\pi}} & \frac{c_1}{\sqrt{\pi}} \end{bmatrix}$$

(14)

4.3. The Derivative Operational Matrix of Integer Order



The derivative operational matrix for the n th-order derivative of the normalized shifted Vieta-Lucas polynomials is defined by

$$\frac{d^n \overline{\mathcal{V}\mathcal{L}_x^L}}{dx^n} \approx \mathcal{D}_x^{(n)} \overline{\mathcal{V}\mathcal{L}_x^L},$$

where $\mathcal{D}_x^{(n)}$ is an $(N+1) \times (N+1)$ matrix, and their entries can be calculated in the following manner

$$d_{i,j}^{(n)} = \left\langle \frac{d^n \overline{\mathcal{V}\mathcal{L}_i^L(x)}}{dx^n}, \overline{\mathcal{V}\mathcal{L}_j^L(x)} \right\rangle_{\omega^L(x)}, \quad i, j = 1, 2, 3, \dots, \quad (15)$$

where the weight function is $\omega^L(x) = \frac{1}{\sqrt{x(L-x)}}$.

4.4. Fractional-order Operational Matrices

The following theorems presented the fractional-order operational matrix for normalized shifted Vieta-Lucas polynomials.

Theorem 4.1. Let

$\overline{\mathcal{V}\mathcal{L}_x^L} = [\overline{\mathcal{V}\mathcal{L}_0^L(x)}, \overline{\mathcal{V}\mathcal{L}_1^L(x)}, \dots, \overline{\mathcal{V}\mathcal{L}_N^L(x)}]^T$ be the vector of normalized shifted Vieta-Lucas polynomials over the interval $[0, L]$, For $n-1 < \alpha(x) \leq n$, the fractional integral of order $\alpha(x)$ can be approximated as

$$\int_0^x \frac{\overline{\mathcal{V}\mathcal{L}_t^L}}{(x-t)^{\alpha(x)}} dt \approx \mathcal{P}_x^{(\alpha(x))} \overline{\mathcal{V}\mathcal{L}_x^L},$$

where $\mathcal{P}_x^{(\alpha(x))}$ denotes an $(N+1) \times (N+1)$ operational matrix with components defined as follows

$$p_{i+1,j+1}^{\alpha(x)} \approx \begin{cases} \frac{\Gamma(1-\alpha(x))}{(\sqrt{\pi})\Gamma(2-\alpha(x))} a_m^{1-\alpha(x)}, & \text{for } n=0, \\ \sum_{i=0}^n \left(\frac{(-1)^i 4^{n-i} (2n)\Gamma(2n-i)}{(\sqrt{2\pi})L^{n-i}\Gamma(i+1)} \times \frac{\Gamma(n-i+1)\Gamma(1-\alpha(x))}{\Gamma(2n-2i+1)\Gamma(n-i-\alpha(x)+2)} \right) a_m^{n-i-\alpha(x)+1}, & \text{for } n=1, 2, \dots, N, \quad m=0, 1, \dots, N. \end{cases}$$

Here, the coefficients a_m^β are determined via the $\mathbb{L}^2_{\omega^L(x)}$ projection

$$x^\beta \approx \sum_{m=0}^N a_m^\beta \overline{\mathcal{V}\mathcal{L}_m^L(x)}, \quad \beta > 0.$$

Proof. According to the definition of normalized shifted Vieta-Lucas polynomials, we get:

$$\begin{aligned} & \int_0^x \frac{\overline{\mathcal{V}\mathcal{L}_n^L(t)}}{(x-t)^{\alpha(x)}} dt \\ &= \int_0^x \frac{\sum_{i=0}^n \frac{(-1)^i 4^{n-i} (2n)\Gamma(2n-i)}{(\sqrt{2\pi})\Gamma(i+1)\Gamma(2n-2i+1)} \left(\frac{t}{L}\right)^{n-i}}{(x-t)^{\alpha(x)}} dt \\ &= \sum_{i=0}^n \left(\frac{(-1)^i 4^{n-i} (2n)\Gamma(2n-i)}{(\sqrt{2\pi})\Gamma(i+1)\Gamma(2n-2i+1)L^{n-i}} \times \int_0^x \frac{t^{n-i}}{(x-t)^{\alpha(x)}} dt \right), \\ & \quad n = 1, 2, 3, \dots, N. \end{aligned} \quad (16)$$

On the other hand, would be have

$$\int_0^x \frac{t^k}{(x-t)^{\alpha(x)}} dt = \frac{\Gamma(k+1)\Gamma(1-\alpha)}{\Gamma(k-\alpha+2)} x^{k-\alpha+1}, \quad \alpha(x) > 0, \quad k > 0,$$

Therefore

$$\begin{aligned} & \int_0^x \frac{t^{n-i}}{(x-t)^{\alpha(x)}} dt \\ &= \frac{\Gamma(n-i+1)\Gamma(1-\alpha(x))}{\Gamma(n-i-\alpha(x)+2)} x^{n-i-\alpha(x)+1}, \\ & \quad i = 0, 1, 2, \dots, \quad n = 1, 2, 3, \dots, N. \end{aligned}$$

Thus, in incorporation of above equation with Eq. (16), we obtain:

$$\begin{aligned} & \int_0^x \frac{\overline{\mathcal{V}\mathcal{L}_n^L(t)}}{(x-t)^{\alpha(x)}} dt \\ &= \sum_{i=0}^n \left(\frac{(-1)^i 4^{n-i} (2n)\Gamma(2n-i)}{(\sqrt{2\pi})L^{n-i}\Gamma(i+1)} \times \frac{\Gamma(n-i+1)\Gamma(1-\alpha(x))}{\Gamma(2n-2i+1)\Gamma(n-i-\alpha(x)+2)} \times a_m^{n-i-\alpha(x)+1} \right), \\ & \quad n = 1, 2, 3, \dots, N. \end{aligned} \quad (17)$$

Now, consider the below approximations in terms of the introduced polynomials:

$$x^{n-i-\alpha(x)+1} \approx \sum_{m=0}^N a_m^{n-i-\alpha(x)+1} \overline{\mathcal{V}\mathcal{L}_m^L(x)}, \quad i = 0, 1, \dots, n, \quad n = 0, 1, \dots, N,$$

By inserting it in Eq.(17), we will have:

$$\int_0^x \frac{\overline{\mathcal{V}\mathcal{L}_n^L(t)}}{(x-t)^{\alpha(x)}} dt$$



$$\begin{aligned}
 &= \sum_{i=0}^n \sum_{m=0}^N \left(\frac{(-1)^i 4^{n-i} (2n) \Gamma(2n-i)}{(\sqrt{2\pi}) L^{n-i} \Gamma(i+1)} \times \right. \\
 &\quad \left. \frac{\Gamma(n-i+1) \Gamma(1-\alpha(x))}{\Gamma(2n-2i+1) \Gamma(n-i-\alpha(x)+2)} \times \right) \overline{\mathcal{V}\mathcal{L}_n^L(x)} \\
 &= \sum_{m=0}^N \left(\sum_{i=0}^n \left(\frac{(-1)^i 4^{n-i} (2n) \Gamma(2n-i)}{(\sqrt{2\pi}) L^{n-i} \Gamma(i+1)} \times \right. \right. \\
 &\quad \left. \left. \frac{\Gamma(n-i+1) \Gamma(1-\alpha(x))}{\Gamma(2n-2i+1) \Gamma(n-i-\alpha(x)+2)} \times \right) \overline{\mathcal{V}\mathcal{L}_n^L(x)} \right).
 \end{aligned}$$

(18)

The expression in parenthesis will be called by $p_{(n+1) \times (m+1)}^{\alpha(x)}$ for $n = 1, 2, \dots, N$, $m = 0, 1, 2, \dots, N$. Also, for $n = 0$ we have:

$$\begin{aligned}
 \int_0^x \frac{\overline{\mathcal{V}\mathcal{L}_0^L(t)}{(x-t)^{\alpha(x)}} dt &= \frac{1}{\sqrt{\pi}} \int_0^x \frac{1}{(x-t)^{\alpha(x)}} dt \\
 &= \frac{\Gamma(1) \Gamma(1-\alpha(x))}{\sqrt{\pi} \Gamma(2-\alpha(x))} x^{1-\alpha(x)},
 \end{aligned}$$

and

$$x^{1-\alpha(x)} \approx \sum_{m=0}^N a_m^{(1-\alpha(x))} \overline{\mathcal{V}\mathcal{L}_x^L(x)}.$$

Therefore

$$\begin{aligned}
 \int_0^x \frac{\overline{\mathcal{V}\mathcal{L}_0^L(t)}{(x-t)^{\alpha(x)}} dt &\approx \frac{\Gamma(1-\alpha(x))}{\sqrt{\pi} \Gamma(2-\alpha(x))} \sum_{m=0}^N a_m^{(1-\alpha(x))} \overline{\mathcal{V}\mathcal{L}_n^L(x)} \\
 &= \sum_{m=0}^N \left(\frac{\Gamma(1-\alpha(x))}{\sqrt{\pi} \Gamma(2-\alpha(x))} a_m^{(1-\alpha(x))} \right) \overline{\mathcal{V}\mathcal{L}_n^L(x)} \\
 &= \sum_{m=0}^N p_{1 \times (m+1)}^{\alpha(x)} \overline{\mathcal{V}\mathcal{L}_n^L(x)}.
 \end{aligned}$$

We show the expression in parenthesis by symbol $p_{(1) \times (m+1)}^{\alpha(x)}$, thus

$$\int_0^x \frac{\overline{\mathcal{V}\mathcal{L}_t^L}}{(x-t)^{\alpha(x)}} dt = \begin{bmatrix} \int_0^x \frac{\overline{\mathcal{V}\mathcal{L}_0^L(t)}{(x-t)^{\alpha(x)}} dt \\ \int_0^x \frac{\overline{\mathcal{V}\mathcal{L}_1^L(t)}{(x-t)^{\alpha(x)}} dt \\ \int_0^x \frac{\overline{\mathcal{V}\mathcal{L}_2^L(x)}{(x-t)^{\alpha(x)}} dt \\ \vdots \\ \int_0^x \frac{\overline{\mathcal{V}\mathcal{L}_N^L(t)}{(x-t)^{\alpha(x)}} dt \end{bmatrix}^T$$

$$\begin{aligned}
 &\approx \begin{bmatrix} \sum_{m=0}^N P_{(1),(m+1)}^{\alpha(x)} \overline{\mathcal{V}\mathcal{L}_m^L(x)} \\ \sum_{m=0}^N P_{(2),(m+1)}^{\alpha(x)} \overline{\mathcal{V}\mathcal{L}_m^L(x)} \\ \vdots \\ \sum_{m=0}^N P_{(N+1),(m+1)}^{\alpha(x)} \overline{\mathcal{V}\mathcal{L}_m^L(x)} \end{bmatrix}^T \\
 &= \begin{bmatrix} p_{1,1}^{\alpha(x)} & \cdots & p_{1,(N+1)}^{\alpha(x)} \\ \vdots & \ddots & \vdots \\ p_{(N+1),1}^{\alpha(x)} & \cdots & p_{(N+1),(N+1)}^{\alpha(x)} \end{bmatrix} \begin{bmatrix} \overline{\mathcal{V}\mathcal{L}_0^L(x)} \\ \overline{\mathcal{V}\mathcal{L}_1^L(x)} \\ \vdots \\ \overline{\mathcal{V}\mathcal{L}_N^L(x)} \end{bmatrix}
 \end{aligned}$$

(19)

4.5. Variable-order Caputo Fractional Derivative Operational Matrices

Theorem 4.2. Let

$\overline{\mathcal{V}\mathcal{L}_x^L} = [\overline{\mathcal{V}\mathcal{L}_0^L(x)}, \overline{\mathcal{V}\mathcal{L}_1^L(x)}, \dots, \overline{\mathcal{V}\mathcal{L}_N^L(x)}]^T$, be the vector of normalized shifted Vieta-Lucas polynomials over the interval $[0, L]$, and suppose that

$$\mathfrak{U}^{(n)}(x) \approx \mathcal{D}_x^{(n)} \overline{\mathcal{V}\mathcal{L}_x^L},$$

where $\mathcal{D}_x^{(n)}$ is the n th derivative operational matrix and the approximation is understood in the sense of the $\mathbb{L}^2_{\omega^L(x)}$ projection. Then, the variable-order Caputo fractional derivative of order $\alpha(x)$, with $n - 1 < \alpha(x) \leq n$, can be approximated by

$${}^c \mathcal{D}_x^{(\alpha(x))} \mathfrak{U}(x) \approx \frac{1}{\Gamma(n-\alpha(x))} \mathcal{D}_x^{(n)} \mathcal{P}_x^{(\alpha(x)-n+1)} \overline{\mathcal{V}\mathcal{L}_x^L} \quad (20)$$

where $\mathcal{P}_x^{(\alpha(x)-n+1)}$ is the fractional integral operational matrix defined in Theorem 4.1.

Proof. By the definition of the variable-order Caputo derivative,

$${}^c \mathcal{D}_x^{(\alpha(x))} \mathfrak{U}(x) \approx \frac{1}{\Gamma(n-\alpha(x))} \int_0^x \frac{\mathfrak{U}^{(n)}(t)}{(x-t)^{\alpha(x)-n+1}} dt,$$

Using the approximation $\mathfrak{U}^{(n)}(x) \approx \mathcal{D}_x^{(n)} \overline{\mathcal{V}\mathcal{L}_x^L}$ and applying Theorem (4.1) for the fractional integral, we obtain

$$\int_0^x \frac{\mathfrak{U}^{(n)}(t)}{(x-t)^{\alpha(x)-n+1}} dt \approx \mathcal{D}_x^{(n)} \mathcal{P}_x^{(\alpha(x)-n+1)} \overline{\mathcal{V}\mathcal{L}_x^L},$$

Therefore,



$${}^c \mathcal{D}_x^{(\alpha(x))} \mathfrak{U}(x) \approx \frac{1}{\Gamma(n-\alpha(x))} \mathcal{D}_x^{(n)} \mathcal{P}_x^{(\alpha(x)-n+1)} \overline{\mathcal{V}\mathcal{L}_x^L}.$$

Remark: This formula provides an analytical approximation of the variable-order Caputo derivative in the finite-dimensional subspace $\{\overline{\mathcal{V}\mathcal{L}_0^L}(x), \overline{\mathcal{V}\mathcal{L}_1^L}(x), \dots, \overline{\mathcal{V}\mathcal{L}_N^L}(x)\}$, and the approximation error decreases as N increases, due to the completeness of the shifted Vieta-Lucas basis in $\mathbb{L}^2_{\omega^L(x)} [0, L]$.

Theorem 4.3. Let $\mathfrak{U}(x) \approx C^T \overline{\mathcal{V}\mathcal{L}_x^L}$, where the approximation is understood in the sense of the $\mathbb{L}^2_{\omega^L(x)}$ projection onto the normalized shifted Vieta-Lucas polynomials. Then, the variable-order Riemann-Liouville fractional integral of order $\alpha(x) > 0$ can be approximated as

$${}^{RL} \mathcal{J}_x^{(\alpha(x))} \mathfrak{U}(x) \approx \frac{1}{\Gamma(\alpha(x))} C^T \mathcal{P}_x^{(1-\alpha(x))} \overline{\mathcal{V}\mathcal{L}_x^L}, \quad (21)$$

Proof. By definition, the variable-order Riemann-Liouville integral is

$${}^{RL} \mathcal{J}_x^{(\alpha(x))} \mathfrak{U}(x) \approx \frac{1}{\Gamma(\alpha(x))} \int_0^x \frac{\mathfrak{U}(x)}{(x-t)^{1-\alpha(x)}} dt,$$

Using the approximation $\mathfrak{U}(x) \approx C^T \overline{\mathcal{V}\mathcal{L}_x^L}$, and Theorem 4.1 for the fractional integral, we obtain

$$\begin{aligned} {}^{RL} \mathcal{J}_x^{(\alpha(x))} \mathfrak{U}(x) &\approx \frac{1}{\Gamma(\alpha(x))} C^T \int_0^x \frac{1}{(x-t)^{1-\alpha(x)}} \overline{\mathcal{V}\mathcal{L}_t^L} dt \\ &= \frac{1}{\Gamma(\alpha(x))} C^T \mathcal{P}_x^{(1-\alpha(x))} \overline{\mathcal{V}\mathcal{L}_x^L}. \end{aligned}$$

Remark: This approximation is exact in the subspace spanned by the first $N+1$ normalized shifted Vieta-Lucas polynomials and converges as $N \rightarrow \infty$.

5. The methodology of proposed method

There is no loss restriction of generality, we can assume the system (1) as follows:

$$\begin{aligned} {}^c \mathcal{D}^{\alpha_k(x)} v_k(x) &= f_k(x) \\ &+ \mathcal{F}_k(x, v_1(x), \dots, \mathcal{D}^{(n_1)} v_1(x), \\ &\quad \dots, v_m(x), \dots, \mathcal{D}^{(n_m)} v_m(x)) \\ &+ \int_0^x K_{1,k}(x, t) \mathcal{G}_{1,k}(x, t, v_1(t), \dots, \mathcal{D}^{(n_1)} v_1(t), \\ &\quad \dots, v_m(t), \dots, \mathcal{D}^{(n_m)} v_m(t)) dt \\ &+ \int_0^L K_{2,k}(x, t) \mathcal{G}_{2,k}(x, t, v_1(t), \dots, \mathcal{D}^{(n_1)} v_1(t), \\ &\quad \dots, v_m(t), \dots, \mathcal{D}^{(n_m)} v_m(t)) dt, \end{aligned} \quad (22)$$

$$\begin{aligned} m_k &= [\alpha_k(x)], & k &= 1, 2, \dots, m, \\ \mathcal{D}^{(i)} v_k(0) &= \gamma_{k,i}, & i &= 1, 2, \dots, n_k - 1, \end{aligned}$$

To solve the above system, we consider the following approximations:

$$\begin{aligned} \mathcal{D}^{(m_k)} v_k(x) &\approx C_k^T \overline{\mathcal{V}\mathcal{L}_x^L}, & k &= 1, 2, \dots, m, \\ {}^c \mathcal{D}^{\alpha_k(x)} v_k(x) &\approx \frac{1}{\Gamma(1-\alpha_k(x))} C_k^T \mathcal{P}_x^{(\alpha_k(x))} \overline{\mathcal{V}\mathcal{L}_x^L}, \\ K_{i,k}(x, t) &\approx (\overline{\mathcal{V}\mathcal{L}_x^L})^T \mathcal{K}_{i,k} \overline{\mathcal{V}\mathcal{L}_x^L}, & i &= 1, 2, \dots, m, \\ \mathcal{G}_{i,k}(v_1(t), \mathcal{D}^{(1)} v_1(t), \dots, \mathcal{D}^{(n_1)} v_1(t), \dots, \mathcal{D}^{(n_m)} v_m(t)) \\ &\approx \mathcal{Y}_{i,k}^T \overline{\mathcal{V}\mathcal{L}_t^L}, \\ \mathcal{F}_k(x, v_1(x), \mathcal{D}^{(1)} v_1(x), \dots, \mathcal{D}^{(n_1)} v_1(x), \dots, \mathcal{D}^{(n_m)} v_m(x)) \\ &\approx F_k^T \overline{\mathcal{V}\mathcal{L}_x^L}, \\ \overline{\mathcal{V}\mathcal{L}_x^L} &= [\overline{\mathcal{V}\mathcal{L}_0^L}(x), \overline{\mathcal{V}\mathcal{L}_1^L}(x), \dots, \overline{\mathcal{V}\mathcal{L}_N^L}(x)]^T, \\ C_k &= [c_{k,0}, c_{k,1}, \dots, c_{k,N}]^T. \end{aligned}$$

(23)

By inserting these approximations in Eq. (22), we have

$$\begin{aligned} \frac{1}{\Gamma(1-\alpha_k(x))} C_k^T \mathcal{P}_x^{(\alpha_k(x))} \overline{\mathcal{V}\mathcal{L}_x^L} &= f_k(x) + F_k^T \overline{\mathcal{V}\mathcal{L}_x^L} \\ &+ \int_0^x (\overline{\mathcal{V}\mathcal{L}_x^L})^T \mathcal{K}_{1,k} \overline{\mathcal{V}\mathcal{L}_t^L} (\overline{\mathcal{V}\mathcal{L}_x^L})^T \mathcal{Y}_{1,k} dt \\ &+ \int_0^L (\overline{\mathcal{V}\mathcal{L}_x^L})^T \mathcal{K}_{2,k} \overline{\mathcal{V}\mathcal{L}_t^L} (\overline{\mathcal{V}\mathcal{L}_t^L})^T \mathcal{Y}_{2,k} dt \\ &= f_k(x) + F_k^T \overline{\mathcal{V}\mathcal{L}_x^L} \\ &+ (\overline{\mathcal{V}\mathcal{L}_x^L})^T \mathcal{K}_{1,k} \int_0^x \overline{\mathcal{V}\mathcal{L}_t^L} (\overline{\mathcal{V}\mathcal{L}_x^L})^T \mathcal{Y}_{1,k} dt \\ &+ (\overline{\mathcal{V}\mathcal{L}_x^L})^T \mathcal{K}_{2,k} \left(\int_0^x \overline{\mathcal{V}\mathcal{L}_t^L} (\overline{\mathcal{V}\mathcal{L}_x^L})^T dt \right) \mathcal{Y}_{2,k} \\ &= f_k(x) + F_k^T \overline{\mathcal{V}\mathcal{L}_x^L} \\ &+ (\overline{\mathcal{V}\mathcal{L}_x^L})^T \mathcal{K}_{1,k} \left(\int_0^x \tilde{\mathcal{Y}}_{1,k} \overline{\mathcal{V}\mathcal{L}_t^L} dt \right) \\ &+ (\overline{\mathcal{V}\mathcal{L}_x^L})^T \mathcal{K}_{2,k} \mathcal{D} \mathcal{Y}_{2,k} \\ &= f_k(x) + F_k^T \overline{\mathcal{V}\mathcal{L}_x^L} \\ &+ (\overline{\mathcal{V}\mathcal{L}_x^L})^T \mathcal{K}_{1,k} \tilde{\mathcal{Y}}_{1,k} \mathcal{P}_x \overline{\mathcal{V}\mathcal{L}_x^L} \\ &+ (\overline{\mathcal{V}\mathcal{L}_x^L})^T \mathcal{K}_{2,k} \mathcal{D} \mathcal{Y}_{2,k}, & k &= 1, 2, \dots, m \end{aligned}$$

By considering the collocation points, $x_i = 1, 2, \dots, N+1$, in the interval $[0, L]$, which are the roots of the Vieta-Lucas polynomials and the following algebraic system can be obtained:

$$\left(\begin{array}{c} \frac{1}{\Gamma(1-\alpha_k(x))} C_k^T \mathcal{P}_x^{(\alpha_k(x))} \overline{\mathcal{V}\mathcal{L}_x^L} - \\ F_k^T \overline{\mathcal{V}\mathcal{L}_x^L} - \\ (\overline{\mathcal{V}\mathcal{L}_x^L})^T \mathcal{K}_{1,k} \tilde{\mathcal{Y}}_{1,k} \mathcal{P}_x \overline{\mathcal{V}\mathcal{L}_x^L} - \\ (\overline{\mathcal{V}\mathcal{L}_x^L})^T \mathcal{K}_{2,k} \mathcal{D} \mathcal{Y}_{2,k} - \\ f_k(x) \end{array} \right)_{x=x_i} \approx 0, \quad (24)$$

$$k = 1, 2, \dots, m, \quad i = 1, 2, \dots, N+1..$$

Finally, the obtained system can be solved by applying an iterative procedure to solve linear or non-linear systems and the unknown vectors C_k , $k = 1, 2, \dots, m$, are estimated. Here, the Newton iterative method is used for solving the non-linear algebraic system. Newton's method is one of the most widely used and efficient numerical methods for solving systems of nonlinear equations. It is based on linearizing the system around an initial estimate and continuously updating it using the derivative (or Jacobian in the multivariate case). A brief description of the computational procedure of the proposed method is provided in Algorithm 1.

Algorithm 1: Procedure for solving the problem

Input: $N, m, f_k(x), \mathcal{F}_k, \mathcal{G}_{1,k}, \mathcal{G}_{2,k}, \alpha_k(x), \gamma_{k,i}$,
 $i = 0, 1, \dots, n_k - 1, \quad k = 1, 2, 3, \dots, m.$

Step 1: Evaluation of the required approximations via truncated normalized and shifted Vieta-Lucas expansions.

Step 2: Constructing the operational matrices corresponding to integral, product, and fractional derivative operators by using initial conditions.

Step 3: Substitution of the approximations into the governing system.

Step 4: Choice of collocation points in the interval $[0, L]$.

Step 5: Computation of the residual functions at the collocation points and construction of the resulting algebraic system.

Step 6: Solve the resulting nonlinear algebraic system using Newton's method, subject to a prescribed tolerance, to obtain the coefficient vectors C_k , $k = 1, 2, \dots, m$.

Output: $v_k(x)$, $k = 1, 2, 3, \dots, m$.

6. Existence and uniqueness

Here, considering system (1) and the fixed-point theorem, the existence and uniqueness of the solution are ascertained. Applying the Riemann-Liouville fractional integral operator on the system (1), the following system can be obtained:

$$v_k(x) = \sum_{i=0}^{n_k-1} \frac{v_k^{(i)}(0)}{i!} x^i + \frac{1}{\Gamma(\alpha_k(x))} \int_0^x (x-t)^{\alpha_k(x)-1} f_k(t) dt + \frac{1}{\Gamma(\alpha_k(x))} \int_0^x (x-t)^{\alpha_k(x)-1} \mathcal{F}_k(t, v_1(t), \mathcal{D}^{(1)}v_1(t), \dots, \mathcal{D}^{(n_1)}v_1(t), \dots, v_m(t), \mathcal{D}^{(1)}v_m(t), \dots, \mathcal{D}^{(n_m)}v_m(t)) dt$$

$$+ \frac{1}{\Gamma(\alpha_k(x))} \int_0^x \int_0^t (x-t)^{\alpha_k(x)-1} \mathcal{G}_{1,k}(t, s, v_1(s), \mathcal{D}^{(1)}v_1(s), \dots, \mathcal{D}^{(n_1)}v_1(s), \dots, v_m(s), \mathcal{D}^{(1)}v_m(s), \dots, \mathcal{D}^{(n_m)}v_m(s)) ds dt + \frac{1}{\Gamma(\alpha_k(x))} \int_0^x \int_0^L (x-t)^{\alpha_k(x)-1} \mathcal{G}_{2,k}(t, s, v_1(s), \mathcal{D}^{(1)}v_1(s), \dots, \mathcal{D}^{(n_1)}v_1(s), \dots, v_m(s), \mathcal{D}^{(1)}v_m(s), \dots, \mathcal{D}^{(n_m)}v_m(s)) ds dt + \mathcal{R}_k(v_k(x)), \quad (25)$$

where

$$n_k = \lceil \alpha_{k,max} \rceil, \quad k = 1, 2, \dots, m, \\ \mathcal{D}^{(i)}v_k(0) = \gamma_{k,i}, \quad i = 0, 1, \dots, n_k - 1, \\ \mathcal{R}_k(x) = \int_0^x \frac{\partial}{\partial x} \left(\frac{(x-t)^{n_k-\alpha_k(x)-1}}{\Gamma(n_k-\alpha_k(x))} \right) \left(v_k^{(n_k)}(t) - v_k^{(n_k)}(0) \right) dt, \quad k = 1, 2, \dots, m.$$

The above system can be considered as the below vector form:

$$\mathcal{V}(x) = \mathcal{T}(\mathcal{V}(x)), \quad (26)$$

where in

$$\mathcal{V}(x) = [v_1(x), v_2(x), \dots, v_m(x)]^T,$$

$$\mathcal{T}(\mathcal{V}(x)) = \begin{bmatrix} T_1(\mathcal{V}(x)) \\ T_2(\mathcal{V}(x)) \\ \vdots \\ T_m(\mathcal{V}(x)) \end{bmatrix}$$

and

$$T_k \mathcal{V}(x) = \sum_{i=0}^{n_k-1} \frac{v_k^{(i)}(0)}{i!} x^i + \frac{1}{\Gamma(\alpha_k(x))} \int_0^x (x-t)^{\alpha_k(x)-1} f_k(t) dt + \frac{1}{\Gamma(\alpha_k(x))} \int_0^x (x-t)^{\alpha_k(x)-1} \mathcal{F}_k(t, v_1(t), \mathcal{D}^{(1)}v_1(t), \dots, \mathcal{D}^{(n_1)}v_1(t), \dots, v_m(t), \mathcal{D}^{(1)}v_m(t), \dots, \mathcal{D}^{(n_m)}v_m(t)) dt + \frac{1}{\Gamma(\alpha_k(x))} \int_0^x \int_0^t (x-t)^{\alpha_k(x)-1} \mathcal{G}_{1,k}(t, s, v_1(s), \dots, \mathcal{D}^{(1)}v_1(s), \dots, \mathcal{D}^{(n_1)}v_1(s), \dots, v_m(s), \mathcal{D}^{(1)}v_m(s), \dots, \mathcal{D}^{(n_m)}v_m(s)) ds dt + \frac{1}{\Gamma(\alpha_k(x))} \int_0^x \int_0^L (x-t)^{\alpha_k(x)-1} \mathcal{G}_{2,k}(t, s, v_1(s), \dots, \mathcal{D}^{(1)}v_1(s), \dots, \mathcal{D}^{(n_1)}v_1(s), \dots, v_m(s), \mathcal{D}^{(1)}v_m(s), \dots, \mathcal{D}^{(n_m)}v_m(s)) ds dt + \mathcal{R}(v_k(x)), \quad k = 1, 2, \dots, m.$$

Theorem 5.1. To get asymptotic results, it is necessary to consider some restrictions. The basic assumptions would be as follows:

I. Suppose that derivative operators $\mathfrak{D}^{(s)}$, are contraction operators on $[0, L]$.

$$\begin{aligned} & \|\mathfrak{D}^{(s)}v_k(x) - \mathfrak{D}^{(s)}\omega_k(x)\|_{\omega^L(x)} \\ & \leq \eta_k^{(s)} \|v_k(x) - \omega_k(x)\|_{\omega^L(x)}, \end{aligned} \quad (27)$$

$s = 1, 2, \dots, n_k, \quad k = 1, 2, \dots, m,$

where $\eta_k^{(s)}$ are positive constants.

II. Assuming that

$$M = \max_{1 \leq k \leq m} \|K_k(x, t)\|_{\omega^L(x)},$$

$$K_k(x, t) = \frac{\partial}{\partial x} \left(\frac{(x-t)^{n_k - \alpha_k(x) - 1}}{\Gamma(n_k - \alpha_k(x))} \right), \quad k = 1, 2, \dots, m, \quad (28)$$

III. Let $0 < \kappa < 1$ and,

$$\kappa = \max_{x \in [0, L]} \left\{ \bigcup_{k=1}^m \left\{ \left(\frac{\zeta_1 L^{\alpha_k(x)}}{\Gamma(\alpha_k(x)+1)} + \frac{\zeta_2 L^{\alpha_k(x)}}{\Gamma(\alpha_k(x)+2)} + \frac{\zeta_3 L^{\alpha_k(x)}}{\Gamma(\alpha_k(x)+1)} \right) \bar{N} \right\} + 2M\zeta_4 L \right\}, \quad (29)$$

where $\zeta_1, \zeta_2, \zeta_3$ and ζ_4 are positive constant values that are determined during the proof and $\bar{N} = \sum_{k=1}^m \eta_k + m$. According to the existing assumptions, system (25) has a unique solution.

Proof. We assume $\chi = \mathcal{C}^L([0, L])$, where $\mathcal{L} = \text{Max}\{n_k, k = 1, 2, \dots, m\}$ and $(\chi, \|\cdot\|_\infty)$ be a Banach space. Consider the mapping $\mathcal{T}(\mathcal{V}(x)): \chi \rightarrow \chi$. Our proof starts with applying Lipschitz's condition for $\mathcal{F}_k, k = 1, 2, \dots, m$ thereby

$$\begin{aligned} & \left\| \mathcal{F}_k(x, v_1(x), \mathfrak{D}^{(1)}v_1(x), \dots, \mathfrak{D}^{(n_m)}v_m(x)) - \mathcal{F}_k(x, \omega_1(x), \mathfrak{D}^{(1)}\omega_1(x), \dots, \mathfrak{D}^{(n_m)}\omega_m(x)) \right\|_{\omega^L(x)} \\ & \leq \alpha_k^{1,0}(x) \|v_1(x) - \omega_1(x)\|_{\omega^L(x)} + \\ & \quad \dots + \alpha_k^{1,n_1}(x) \|\mathfrak{D}^{(n_1)}v_1(x) - \mathfrak{D}^{(n_1)}\omega_1(x)\|_{\omega^L(x)} + \\ & \quad \dots + \alpha_k^{m,0}(x) \|v_n(x) - \omega_n(x)\|_{\omega^L(x)} + \\ & \quad \dots + \alpha_k^{m,n_m}(x) \|\mathfrak{D}^{(n_m)}v_m(x) - \mathfrak{D}^{(n_m)}\omega_m(x)\|_{\omega^L(x)} \\ & = \sum_{i=1}^m \sum_{j=0}^{n_i} \alpha_k^{i,j}(x) \|\mathfrak{D}^{(j)}v_i(x) - \mathfrak{D}^{(j)}\omega_i(x)\|_{\omega^L(x)}, \end{aligned}$$

where $\alpha_k^{i,j}(x)$ are positive functions on $[0, L]$. We conclude from the first assumption that

$$\begin{aligned} & \left\| \mathcal{F}_k(x, v_1(x), \mathfrak{D}^{(1)}v_1(x), \dots, \mathfrak{D}^{(n_m)}v_m(x)) - \mathcal{F}_k(x, \omega_1(x), \mathfrak{D}^{(1)}\omega_1(x), \dots, \mathfrak{D}^{(n_m)}\omega_m(x)) \right\|_{\omega^L(x)} \\ & \leq \alpha_k^{1,0}(x) \|v_1(x) - \omega_1(x)\|_{\omega^L(x)} \\ & \quad + \dots + \eta_1^{(1)} \alpha_k^{1,1}(x) \|v_1(x) - \omega_1(x)\|_{\omega^L(x)} \\ & \quad + \dots + \eta_1^{(n_1)} \alpha_k^{1,n_1}(x) \|v_1(x) - \omega_1(x)\|_{\omega^L(x)} \\ & \quad + \dots + \alpha_k^{m,0}(x) \|v_m(x) - \omega_m(x)\|_{\omega^L(x)} \\ & \quad + \dots + \eta_m^{(1)} \alpha_k^{m,1}(x) \|v_m(x) - \omega_m(x)\|_{\omega^L(x)} \\ & \quad + \dots + \eta_m^{(n_m)} \alpha_k^{m,n_m}(x) \|v_m(x) - \omega_m(x)\|_{\omega^L(x)}. \end{aligned}$$

By considering

$$\begin{aligned} \zeta_1 &= \max_{x \in [0, L]} \left\{ \bigcup_{i,k=1}^m \left\{ \alpha_k^{i,0}(x), \eta_k^{(1)} \alpha_k^{i,1}(x), \dots, \eta_k^{(n_k)} \alpha_k^{i,n_k}(x) \right\} \right\}, \\ \bar{N} &= \sum_{k=1}^m \eta_k + m, \end{aligned}$$

would be have

$$\begin{aligned} & \left\| \mathcal{F}_k(x, v_1(x), \mathfrak{D}^{(1)}v_1(x), \dots, \mathfrak{D}^{(n_m)}v_m(x)) - \mathcal{F}_k(x, \omega_1(x), \mathfrak{D}^{(1)}\omega_1(x), \dots, \mathfrak{D}^{(n_m)}\omega_m(x)) \right\|_{\omega^L(x)} \\ & \leq \zeta_1 \left(\sum_{j=1}^m (n_j + 1) \|v_j(x) - \omega_j(x)\|_{\omega^L(x)} \right) \\ & \leq \zeta_1 \bar{N} \|v(x) - \omega(x)\|_{\omega^L(x)}, \end{aligned} \quad (30)$$

where

$$\|\mathcal{V}(x) - \mathcal{W}(x)\|_{\omega^L(x)} = \max_{x \in [0, L]} \left\{ \|v_j(x) - \omega_j(x)\|_{\omega^L(x)}, k = 1, 2, \dots, m \right\},$$

In the same manner by using the assumptions I and II, the following relations can be concluded.

$$\begin{aligned} & \left\| \mathcal{G}_{1,k}(x, t, v_1(x), \mathfrak{D}^{(1)}v_1(x), \dots, \mathfrak{D}^{(n_m)}v_m(x)) - \mathcal{G}_{1,k}(x, t, \omega_1(x), \mathfrak{D}^{(1)}\omega_1(x), \dots, \mathfrak{D}^{(n_m)}\omega_m(x)) \right\|_{\omega^L(x)} \\ & \leq \zeta_2 \bar{N} \|\mathcal{V}(x) - \mathcal{W}(x)\|_{\omega^L(x)}, \end{aligned} \quad (31)$$

$$\begin{aligned} & \left\| \mathcal{G}_{2,k}(x, t, v_1(x), \mathfrak{D}^{(1)}v_1(x), \dots, \mathfrak{D}^{(n_m)}v_m(x)) - \mathcal{G}_{2,k}(x, t, \omega_1(x), \mathfrak{D}^{(1)}\omega_1(x), \dots, \mathfrak{D}^{(n_m)}\omega_m(x)) \right\|_{\omega^L(x)} \\ & \leq \zeta_3 \bar{N} \|\mathcal{V}(x) - \mathcal{W}(x)\|_{\omega^L(x)}, \end{aligned} \quad (32)$$

$$\|R(v_k(x)) - R(\omega_k(x))\|_{\omega^L(x)} =$$

$$\begin{aligned}
 & \left\| \int_0^x \mathbf{K}_k(x, t) \left[\begin{array}{l} (v_k^{(n_k)}(t) - \omega_k^{(n_k)}(t)) \\ + (v_k^{(n_k)}(0) - \omega_k^{(n_k)}(0)) \end{array} \right] dt \right\|_{\omega^L(x)} \\
 & \leq \mathbf{M} \int_0^x \left(\begin{array}{l} \|v_k^{(n_k)}(t) - \omega_k^{(n_k)}(t)\|_{\omega^L(x)} \\ + \|v_k^{(n_k)}(0) - \omega_k^{(n_k)}(0)\|_{\omega^L(x)} \end{array} \right) dt \\
 & \leq 2\mathbf{M}\zeta_4 L \|\mathcal{V}(x) - \mathcal{W}(x)\|_{\omega^L(x)},
 \end{aligned} \tag{33}$$

where

$$\zeta_2 = \max_{(x,t) \in [0,L]^2} \left\{ \bigcup_{i,k=1}^m \left(\beta_k^{i,0}(x), \eta_k^{(1)} \beta_k^{i,1}(x), \dots, \eta_k^{(n_k)} \beta_k^{i,n_k}(x) \right) \right\},$$

$$\zeta_3 = \max_{(x,t) \in [0,L]^2} \left\{ \bigcup_{i,k=1}^m \left(\gamma_k^{i,0}(x), \eta_k^{(1)} \gamma_k^{i,1}(x), \dots, \eta_k^{(n_k)} \gamma_k^{i,n_k}(x) \right) \right\},$$

$$\zeta_4 = \max_{(x,t) \in [0,L]^2} \left\{ \eta_1^{(n_1)}, \eta_2^{(n_2)}, \dots, \eta_m^{(n_m)} \right\}.$$

For $\mathcal{V}(x), \mathcal{W}(x) \in \mathcal{X}$, we have

$$\begin{aligned}
 & \|T_k(\mathcal{V}(x)) - T_k(\mathcal{W}(x))\|_{\omega^L(x)} \leq \\
 & \frac{1}{\Gamma(\alpha_k(x))} \int_0^x (x-t)^{\alpha_k(x)-1} \|\mathcal{F}_k(t, v_1(t), \dots, \mathfrak{D}^{(n_m)} v_m(t)) \\
 & \quad - \mathcal{F}_k(t, \omega_1(t), \dots, \mathfrak{D}^{(n_m)} \omega_m(t))\|_{\omega^L(x)} dt + \\
 & \frac{1}{\Gamma(\alpha_k(x))} \int_0^x \int_0^t |(x-t)^{\alpha_k(x)-1} \\
 & \quad \times \|\mathcal{G}_{1,k}(t, v_1(t), \dots, \mathfrak{D}^{(n_m)} v_m(t)) \\
 & \quad - \mathcal{G}_{1,k}(t, \omega_1(t), \dots, \mathfrak{D}^{(n_m)} \omega_m(t))\|_{\omega^L(x)} ds dt + \\
 & \frac{1}{\Gamma(\alpha_k(x))} \int_0^x \int_0^L |(x-t)^{\alpha_k(x)-1} \\
 & \quad \times \|\mathcal{G}_{2,k}(t, v_1(t), \dots, \mathfrak{D}^{(n_m)} v_m(t)) \\
 & \quad - \mathcal{G}_{2,k}(t, \omega_1(t), \dots, \mathfrak{D}^{(n_m)} \omega_m(t))\|_{\omega^L(x)} ds dt + \\
 & \|\mathcal{R}(v_k(x)) - \mathcal{R}(\omega_k(x))\|_{\omega^L(x)}.
 \end{aligned}$$

According to the relations (30) to (33), can be obtained

$$\begin{aligned}
 & \|T_k(\mathcal{V}(x)) - T_k(\mathcal{W}(x))\|_{\omega^L(x)} \\
 & \leq \zeta_1 \bar{\mathcal{N}} \|\mathcal{V}(x) - \mathcal{W}(x)\|_{\omega^L(x)} \times \\
 & \quad \left(\frac{1}{\Gamma(\alpha_k(x))} \int_0^x |(x-t)^{\alpha_k(x)-1} dt \right) + \\
 & \bar{\mathcal{N}} \|\mathcal{V}(x) - \mathcal{W}(x)\|_{\omega^L(x)} \times \\
 & \quad \left(\frac{1}{\Gamma(\alpha_k(x))} \int_0^x \int_0^t |(x-t)^{\alpha_k(x)-1} ds dt \right) + \\
 & \zeta_3 \bar{\mathcal{N}} \|\mathcal{V}(x) - \mathcal{W}(x)\|_{\omega^L(x)} \times \\
 & \quad \left(\frac{1}{\Gamma(\alpha_k(x))} \int_0^x \int_0^L |(x-t)^{\alpha_k(x)-1} ds dt \right) + \\
 & 2\mathbf{M}\zeta_4 L \|\mathcal{V}(x) - \mathcal{W}(x)\|_{\omega^L(x)} \leq
 \end{aligned}$$

$$\left(\begin{array}{l} \frac{\zeta_1 L^{\alpha_k(x)}}{\Gamma(\alpha_k(x)+1)} + \\ \frac{\zeta_2 L^{\alpha_k(x)}}{\Gamma(\alpha_k(x)+1)} + \\ \frac{\zeta_3 L^{\alpha_k(x)}}{\Gamma(\alpha_k(x)+1)} \bar{\mathcal{N}} + \\ 2\mathbf{M}\zeta_4 L \end{array} \right) \|\mathcal{V}(x) - \mathcal{W}(x)\|_{\omega^L(x)}, \tag{34}$$

$k = 1, 2, \dots, m.$

Therefore, by using the assumption III, we have

$$\|T_k(\mathcal{V}(x)) - T_k(\mathcal{W}(x))\|_{\omega^L(x)} \leq \kappa \|\mathcal{V}(x) - \mathcal{W}(x)\|_{\omega^L(x)}$$

where

$$\kappa = \max_{x \in [0,L]} \left\{ \bigcup_{k=1}^m \left(\begin{array}{l} \left(\frac{\zeta_1 L^{\alpha_k(x)}}{\Gamma(\alpha_k(x)+1)} + \right. \\ \left. \frac{\zeta_2 L^{\alpha_k(x)}}{\Gamma(\alpha_k(x)+2)} + \right. \\ \left. \frac{\zeta_3 L^{\alpha_k(x)}}{\Gamma(\alpha_k(x)+1)} \right) \bar{\mathcal{N}} \right. \\ \left. + 2\mathbf{M}\zeta_4 L \right) \right\}$$

As a result, it is proved that the operator T , is a κ -contraction mapping and, according to the Banach fixed point theorem, it has a unique fixed point, which is the solution of system (24).

7. Error analysis and stability

In this part, the stability and convergence analysis of the normalized shifted Vieta-Lucas method are investigated in $\mathbb{L}^2_{\omega^L(x)}$ space.

Theorem 7.1. Let $\mathfrak{U}(x) \in \mathfrak{C}^{N+1}[a, b]$, the error bound for the approximate solution in terms of normalized shifted Vieta-Lucas polynomials, $\mathfrak{U}_{\bar{\mathcal{V}}L}(x)$, would be as follows:

$$\|e(x)\| = \|\mathfrak{U}(x) - \mathfrak{U}_{\bar{\mathcal{V}}L}(x)\|_{\omega^L(x)} \leq \frac{\sqrt{\pi} \mathcal{M} L^{N+1}}{(N+1)!}, \tag{35}$$

where $\mathcal{M} = \max_{x \in [0,L]} |\mathfrak{U}^{(N+1)}(x)|$.

Proof. Assume the approximate solution as below

$$\mathfrak{U}_{\bar{\mathcal{V}}L}(x) \approx \sum_{n=0}^N c_n \bar{\mathcal{V}}L_n^L(x), \quad x \in [0, L],$$

Consider the following interpolation function of Newton's divided difference method for $\mathfrak{U}(x)$ by using points x_i , $i = 0, 1, 2, \dots, N$, when $x_i \in [0, L]$

$$\begin{aligned}
 P_N(x) & \approx \mathfrak{U}(x_0) + \mathfrak{U}[x_0, x_1](x - x_0) \\
 & \quad + \mathfrak{U}[x_0, x_1, x_2](x - x_0)(x - x_1)
 \end{aligned}$$



$$+ \dots + \mathfrak{U}[x_0, x_1, \dots, x_N](x - x_0) \dots (x - x_{N-1}),$$

Where $\mathfrak{U}[x_0, x_1, \dots, x_k]$, $k = 0, 1, 2, \dots, N$, indicating Newton's divided difference of order k and the error is calculated as follows

$$\begin{aligned} & \mathfrak{U}(x) - P_N(x) \\ &= \frac{(x - x_0)(x - x_1) \dots (x - x_N)}{(N + 1)!} \mathfrak{U}^{(N+1)}(\eta_x), \\ & \eta_x \in [0, L]. \end{aligned}$$

We have

$$|\mathfrak{U}(x) - P_N(x)| \leq \frac{\mathcal{M} \prod_{i=0}^N |x - x_i|}{(N + 1)!},$$

$$\text{with } \mathcal{M} = \max_{x_i \in [0, L]} |\mathfrak{U}^{(N+1)}(x)|.$$

Because of $\forall x_i \in [0, L]$, $|x - x_i| \leq L$, then the following bound for the error of interpolation function can be obtained

$$|\mathfrak{U}(x) - P_N(x)| \leq \frac{\mathcal{M} L^{N+1}}{(N+1)!}, \quad (36)$$

According to the least square property of orthogonal polynomials, for normalized shifted Vieta-Lucas polynomials would be have

$$\begin{aligned} & \|\mathfrak{U}(x) - \mathfrak{U}_{\overline{VL}_k}(x)\|_{\omega^L(x)}^2 \\ &= \int_0^L \omega^L(x) (\mathfrak{U}(x) - \mathfrak{U}_{\overline{VL}_k}(x))^2 dx \\ &\leq \frac{\mathcal{M}^2 L^{2N+2}}{((N + 1)!)^2} \int_0^L \omega^L(x) dx. \end{aligned}$$

Finally, the error bound would be obtained as the below

$$\|\mathfrak{U}(x) - \mathfrak{U}_{\overline{VL}_k}(x)\|_{\omega^L(x)} \leq \frac{\sqrt{\pi} \mathcal{M} L^{N+1}}{(N + 1)!},$$

$$\text{where } \int_0^L \omega^L(x) dx = \sqrt{\pi}.$$

Theorem 7.2. Suppose that $v_{\overline{VL}_k}(x)$, $k = 1, 2, \dots, m$, are approximate solutions of system (1) based on normalized shifted Vieta-Lucas polynomials, $e_k(x) = v_k(x) - v_{\overline{VL}_k}(x)$, are the absolute errors and $r_k(x)$, $k = 1, 2, \dots, m$, are the residual functions of approximate solutions. Assuming

$$0 < \left(\frac{\zeta_{1L} \alpha_k(x)}{\Gamma(\alpha_k(x)+1)} + \frac{\zeta_{2L} \alpha_k(x)}{\Gamma(\alpha_k(x)+1)} + \frac{\zeta_{3L} \alpha_k(x)}{\Gamma(\alpha_k(x)+1)} \right) \overline{\mathcal{N}} < 1, \quad k = 1, 2, \dots, m,$$

then

$$\|e(x)\|_{\omega^L(x)} \leq \mathfrak{K} \|r(x)\|_{\omega^L(x)},$$

\mathfrak{K} is a positive constant and

$$\begin{aligned} \|e(x)\|_{\omega^L(x)} &= \max_{x \in [0, L]} \left\{ \|e_k(x)\|_{\omega^L(x)} \right\}_{k=1, 2, \dots, m}, \\ \|r(x)\|_{\omega^L(x)} &= \max_{x \in [0, L]} \left\{ \|r_k(x)\|_{\omega^L(x)} \right\}_{k=1, 2, \dots, m}. \end{aligned} \quad (37)$$

Proof. For two solutions $v_{\overline{VL}_k}(x)$ and $v_k(x)$ would be have

$$\begin{aligned} v_{\overline{VL}_k}(x) + e_k(x) &= \sum_{i=0}^{n_k-1} \frac{v_k^{(i)}(0)}{i!} x^i \\ &+ \frac{1}{\Gamma(\alpha_k(x))} \int_0^1 (x-t)^{\alpha_k(x)-1} f_k(t) dt \\ &+ \frac{1}{\Gamma(\alpha_k(x))} \int_0^x (x-t)^{\alpha_k(x)-1} \mathcal{F}_k(t, v_1(t), \dots, \mathfrak{D}^{(n_m)} v_m(t)) dt \\ &+ \frac{1}{\Gamma(\alpha_k(x))} \int_0^x \int_0^t (x-t)^{\alpha_k(x)-1} \mathcal{G}_{1,k}(t, s, v_1(s), \dots, \mathfrak{D}^{(n_m)} v_m(s)) ds dt \\ &+ \frac{1}{\Gamma(\alpha_k(x))} \int_0^x \int_0^L (x-t)^{\alpha_k(x)-1} \mathcal{G}_{2,k}(t, s, v_1(s), \dots, \mathfrak{D}^{(n_m)} v_m(s)) ds dt \\ &+ \mathcal{R}_k(v_k(x)), \quad k = 1, 2, \dots, m, \end{aligned} \quad (38)$$

and

$$\begin{aligned} v_{\overline{VL}_k}(x) &= r_k(x) + \sum_{i=0}^{n_k-1} \frac{v_k^{(i)}(0)}{i!} x^i \\ &+ \frac{1}{\Gamma(\alpha_k(x))} \int_0^1 (x-t)^{\alpha_k(x)-1} f_k(t) dt \\ &+ \frac{1}{\Gamma(\alpha_k(x))} \int_0^x (x-t)^{\alpha_k(x)-1} \mathcal{F}_k(t, v_{\overline{VL}_1}(t), \dots, \mathfrak{D}^{(n_m)} v_{\overline{VL}_m}(t)) dt \\ &+ \frac{1}{\Gamma(\alpha_k(x))} \int_0^x \int_0^t (x-t)^{\alpha_k(x)-1} \mathcal{G}_{1,k}(t, s, v_{\overline{VL}_1}(s), \dots, \mathfrak{D}^{(n_m)} v_{\overline{VL}_m}(s)) ds dt \\ &+ \frac{1}{\Gamma(\alpha_k(x))} \int_0^x \int_0^L (x-t)^{\alpha_k(x)-1} \mathcal{G}_{2,k}(t, s, v_{\overline{VL}_1}(s), \dots, \mathfrak{D}^{(n_m)} v_{\overline{VL}_m}(s)) ds dt \\ &+ \mathcal{R}_k(v_{\overline{VL}_k}(x)), \quad k = 1, 2, \dots, m. \end{aligned} \quad (39)$$

By subtracting (35) from (34), we have:

$$e_k(x) = v_k(x) - v_{\overline{VL}_k}(x) = -r_k(x)$$



$$\begin{aligned}
 & + \frac{1}{\Gamma(\alpha_k(x))} \int_0^x (x-t)^{\alpha_k(x)-1} (\mathcal{F}_k(x, v_1(t), \\
 & \quad \dots, \mathfrak{D}^{(n_m)} v_m(t)) - \mathcal{F}_k(t, v_{\overline{\mathcal{V}}_1}(t), \dots, \\
 & \quad \mathfrak{D}^{(n_m)} v_{\overline{\mathcal{V}}_m}(t))) dt \\
 & + \frac{1}{\Gamma(\alpha_k(x))} \int_0^x \int_0^t (x-t)^{\alpha_k(x)-1} (\mathcal{G}_{1,k}(t, s, v_1(s), \\
 & \quad \dots, \mathfrak{D}^{(n_1)} v_m(s)) - \mathcal{G}_{1,k}(t, s, v_{\overline{\mathcal{V}}_1}(s), \dots, \\
 & \quad \mathfrak{D}^{(n_m)} v_{\overline{\mathcal{V}}_m}(s))) ds dt \\
 & + \frac{1}{\Gamma(\alpha_k(x))} \int_0^x \int_0^L (x-t)^{\alpha_k(x)-1} (\mathcal{G}_{2,k}(t, s, v_1(s), \\
 & \quad \dots, \mathfrak{D}^{(n_1)} v_m(s)) - \mathcal{G}_{2,k}(t, s, v_{\overline{\mathcal{V}}_1}(s), \dots, \\
 & \quad \mathfrak{D}^{(n_m)} v_{\overline{\mathcal{V}}_m}(s))) ds dt \\
 & + (\mathcal{R}_k(v_k(x)) - \mathcal{R}_k(v_{\overline{\mathcal{V}}_k}(x))), \quad k = 1, 2, \dots, m, \\
 (40)
 \end{aligned}$$

According to results of section 5, can be obtained the following results

$$\begin{aligned}
 & \left\| \mathcal{F}_k(x, v_1(x), \dots, \mathfrak{D}^{(n_m)} v_m(x)) - \right. \\
 & \left. \mathcal{F}_k(x, v_{\overline{\mathcal{V}}_1}(x), \dots, \mathfrak{D}^{(n_m)} v_{\overline{\mathcal{V}}_m}(x)) \right\|_{\omega^L(x)} \\
 & \leq \zeta_1 \left(\sum_{j=1}^m (n_j + 1) \|e_j(x)\|_{\omega^L(x)} \right) \\
 & \leq \zeta_1 \overline{\mathcal{N}} \|e(x)\|_{\omega^L(x)},
 \end{aligned}$$

$$\begin{aligned}
 & \left\| \mathcal{G}_{1,k}(x, t, v_1(t), \dots, \mathfrak{D}^{(n_m)} v_m(t)) - \right. \\
 & \left. \mathcal{G}_{1,k}(x, t, v_{\overline{\mathcal{V}}_1}(t), \dots, \mathfrak{D}^{(n_m)} v_{\overline{\mathcal{V}}_m}(t)) \right\|_{\omega^L(x)} \\
 & \leq \zeta_2 \overline{\mathcal{N}} \|e(x)\|_{\omega^L(x)},
 \end{aligned}$$

(41)

$$\begin{aligned}
 & \left\| \mathcal{G}_{2,k}(x, t, v_1(t), \dots, \mathfrak{D}^{(n_m)} v_m(t)) - \right. \\
 & \left. \mathcal{G}_{2,k}(x, t, v_{\overline{\mathcal{V}}_1}(t), \dots, \mathfrak{D}^{(n_m)} v_{\overline{\mathcal{V}}_m}(t)) \right\|_{\omega^L(x)} \\
 & \leq \zeta_3 \overline{\mathcal{N}} \|e(x)\|_{\omega^L(x)},
 \end{aligned}$$

$$\begin{aligned}
 & \left\| \mathcal{R}_k(v_k(x)) - \mathcal{R}_k(v_{\overline{\mathcal{V}}_k}(x)) \right\|_{\omega^L(x)} \\
 & \leq 2M\zeta_4 L \|e(x)\|_{\omega^L(x)},
 \end{aligned}$$

that

$$\|e(x)\|_{\omega^L(x)} = \max_{x \in [0, L]} \left\{ \|e_k(x)\|_{\omega^L(x)} \right\}, \quad k = 1, 2, \dots, m,$$

and $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ were yields in Theorem 5.1. Therefore

$$\|e(x)\|_{\omega^L(x)} \leq \|r_k(x)\|_{\omega^L(x)} +$$

$$\left(\begin{array}{c} \left(\frac{\zeta_1 L^{\alpha_k(x)}}{\Gamma(\alpha_k(x)+1)} + \right. \\ \frac{\zeta_2 L^{\alpha_k(x)}}{\Gamma(\alpha_k(x)+1)} + \\ \left. \frac{\zeta_3 L^{\alpha_k(x)}}{\Gamma(\alpha_k(x)+1)} \right) \overline{\mathcal{N}} + \\ 2M\zeta_4 L \end{array} \right) \|e(x)\|_{\omega^L(x)}, \quad k = 1, 2, \dots, m.$$

By considering

$$\mathcal{C} = \max_{x \in [0, L]} \left\{ \begin{array}{c} \left(\frac{\zeta_1 L^{\alpha_k(x)}}{\Gamma(\alpha_k(x)+1)} + \right. \\ \frac{\zeta_2 L^{\alpha_k(x)}}{\Gamma(\alpha_k(x)+1)} + \\ \left. \frac{\zeta_3 L^{\alpha_k(x)}}{\Gamma(\alpha_k(x)+1)} \right) \overline{\mathcal{N}} + \\ 2M\zeta_4 L \end{array} \right\}, \quad k = 1, 2, \dots, m.$$

We have $(1 - \mathcal{C})\|e(x)\|_{\omega^L(x)} \leq \|r(x)\|_{\omega^L(x)}$ and by using the assumption of the theorem, the proof of stability is completed.

Theorem 7.3. Suppose $v_k(x) \in \mathfrak{C}^{N+1}[0, L]$, $k = 1, 2, \dots, m$. By considering the assumption of theorem 7.2, if $n \rightarrow \infty$ then

$$\|v_k(x) - v_{\overline{\mathcal{V}}_k}(x)\|_{\omega^L(x)} \rightarrow 0, \quad k = 1, 2, \dots, m.$$

Proof. For the functions $v_k(x)$, $k = 1, 2, \dots, m$ we consider the following approximations in terms of normalized Vieta-Lucas polynomials on the interval $[0, L]$ as follows

$$v_{\overline{\mathcal{V}}_k}(x) \approx \sum_{n=0}^N c_{k,n} \overline{\mathcal{V}}_n^L(x) = C_k^T \overline{\mathcal{V}}_x^L, \quad (42)$$

$$k = 1, 2, \dots, m.$$

That

$$\overline{\mathcal{V}}_x^L = [\overline{\mathcal{V}}_0^L(x), \overline{\mathcal{V}}_1^L(x), \dots, \overline{\mathcal{V}}_N^L(x)]^T,$$

$$C_k = [c_{k,0}, c_{k,1}, \dots, c_{k,N}]^T.$$

According to theorem 7.2, we have

$$\begin{aligned}
 & \|v_k(x) - v_{\overline{\mathcal{V}}_k}(x)\|_{\omega^L(x)} \leq \\
 & \|r_k(x)\|_{\omega^L(x)} + \left(\begin{array}{c} \frac{\zeta_1 L^{\alpha_k(x)}}{\Gamma(\alpha_k(x)+1)} + \\ \frac{\zeta_2 L^{\alpha_k(x)}}{\Gamma(\alpha_k(x)+1)} + \\ \frac{\zeta_3 L^{\alpha_k(x)}}{\Gamma(\alpha_k(x)+1)} \end{array} \right) \times \\
 & \left(\sum_{j=1}^m (n_j + 1) \|v_j(x) - v_{\overline{\mathcal{V}}_j}(x)\|_{\omega^L(x)} \right) + \\
 & 2M\zeta_4 L \|v_j(x) - v_{\overline{\mathcal{V}}_j}(x)\|_{\omega^L(x)}, \quad k = 1, 2, \dots, m.
 \end{aligned}$$

Under the result of theorem 7.1, for the considered approximations would be have

$$\|e_k(x)\|_{\omega^L(x)} = \|v_k(x) - v_{\overline{\mathcal{V}L}_k}(x)\|_{\omega^L(x)} \leq \frac{\sqrt{\pi}M_k L^{N+1}}{(N+1)!}$$

$$k = 1, 2, \dots, m. \tag{43}$$

Thus

$$\|v_k(x) - v_{\overline{\mathcal{V}L}_k}(x)\|_{\omega^L(x)} \leq \left(\frac{\zeta_1 L^{\alpha_k(x)}}{\Gamma(\alpha_k(x)+1)} + \frac{\zeta_2 L^{\alpha_k(x)}}{\Gamma(\alpha_k(x)+1)} + \frac{\zeta_3 L^{\alpha_k(x)}}{\Gamma(\alpha_k(x)+1)} \right) \times$$

$$\left(\sum_{j=1}^m (n_j + 1) \frac{\sqrt{\pi}M_k L^{N+1}}{(N+1)!} \right) + 2M\zeta_4 L \frac{\sqrt{\pi}M_k L^{N+1}}{(N+1)!},$$

$$k = 1, 2, \dots, m.$$

Let $\mathcal{M} = \text{Max}\{M_k, k = 1, 2, \dots, m\}$, and we will have

$$\|v_k(x) - v_{\overline{\mathcal{V}L}_k}(x)\|_{\omega^L(x)} \leq \left(\frac{\zeta_1 L^{\alpha_k(x)}}{\Gamma(\alpha_k(x)+1)} + \frac{\zeta_2 L^{\alpha_k(x)}}{\Gamma(\alpha_k(x)+1)} + \frac{\zeta_3 L^{\alpha_k(x)}}{\Gamma(\alpha_k(x)+1)} \right) \frac{\sqrt{\pi}M\mathcal{N}L^{N+1}}{(N+1)!},$$

$$k = 1, 2, \dots, m.$$

When $N \rightarrow \infty$, then $\lim_{N \rightarrow \infty} \frac{L^{N+1}}{(N+1)!} = 0$, and

$$\|r_k(x)\|_{\omega^L(x)} \rightarrow 0, \quad k = 1, 2, \dots, m.$$

Therefore, the following desired results are obtained

$$\|v_k(x) - v_{\overline{\mathcal{V}L}_k}(x)\|_{\omega^L(x)} \rightarrow 0, \quad k = 1, 2, \dots, m.$$

8. Numerical Results

In order to demonstrate the efficiency of the proposed approach, this section considers two nonlinear models. The method employs approximations constructed from the normalized shifted Vieta-Lucas functions, as outlined in earlier sections, and utilizes the collocation technique to transform the problem into a system of nonlinear algebraic equations. To obtain a numerical solution, the Newton iterative algorithm is applied. All simulations were carried out using MAPLE 22 on an Asus laptop powered by an Intel Core i7

processor, with calculations maintained to ten-digit precision. The root-mean-square error norm is evaluated as follows:

$$\mathcal{L}_2^N v_k(x) = \left(\int_0^L (v_k(x) - v_{\overline{\mathcal{V}L}_k}^N(x))^2 dx \right)^{\frac{1}{2}},$$

$$k = 1, 2, \dots, m, \quad N = 2, 3, \dots$$

where $v_k(x)$ and $v_{\overline{\mathcal{V}L}_k}^N(x)$ are the exact and approximate solutions. The convergence rate of the technique is obtained with the following rule:

$$C\mathcal{R}_{v_k}^N = \left| \frac{\text{Ln} \left(\frac{\mathcal{L}_2^N v_k(x)}{\mathcal{L}_2^{N-1} v_k(x)} \right)}{\text{Ln} \left(\frac{N}{N-1} \right)} \right|,$$

$$k = 1, 2, \dots, m, \quad N = 2, 3, \dots$$

Example.8.1. Consider the following system of fractional Volterra-Fredholm integro-differential equations

$$\begin{cases} {}^C\mathcal{D}_x^{\alpha_1(x)} u(x) = f_1(x) - 3v'(x) + \int_0^x \frac{(x+t)u(t)w(t)}{\sqrt{x-t}} dt, \\ {}^C\mathcal{D}_x^{\alpha_2(x)} v(x) = f_2(x) - \int_0^L e^x t u'(t) dt + 4 \int_0^x x^2 t w(t) dt, \\ {}^C\mathcal{D}_x^{\alpha_3(x)} w(x) = f_3(x) - u^2(x) + \int_0^x \frac{v(t) - 2w(t)}{\sqrt[3]{x-t}} dt, \end{cases}$$

$$\alpha_1(x) = \frac{\sin x}{3}, \quad \alpha_2(x) = \frac{\cos x}{5}, \quad \alpha_3(x) = \frac{x}{2},$$

$$0 < \alpha_i(x) < 1, \quad i = 1, 2, 3, \quad [0, L] = [0, 1],$$

and with initial values as follows

$$u(0) = 2, \quad v(0) = -1, \quad w(0) = 1.$$

The exact solutions of system are $u(x) = 2 \cos x$, $v(x) = x^4 - 3x + 1$, $w(x) = \sin x + 1$. and by using them the functions $f_i(x)$, $i = 1, 2, 3$ can be calculated. According to the proposed method, we have:

$$u'(x) \approx C_1^T \overline{\mathcal{V}L}_x^L,$$

$$u(x) = C_1^T \mathcal{P}_x \overline{\mathcal{V}L}_x^L + u(0) = (C_1^T \mathcal{P}_x + U_1^T) \overline{\mathcal{V}L}_x^L = Y_1^T \overline{\mathcal{V}L}_x^L,$$

$$v'(x) \approx C_2^T \overline{\mathcal{V}L}_x^L,$$

$$v(x) = C_2^T \mathcal{P}_x \overline{\mathcal{V}L}_x^L + v(0) = (C_2^T \mathcal{P}_x + V_1^T) \overline{\mathcal{V}L}_x^L = Y_2^T \overline{\mathcal{V}L}_x^L,$$

$$w'(x) \approx C_3^T \overline{\mathcal{V}L}_x^L,$$

$$w(x) = C_3^T \mathcal{P}_x \overline{\mathcal{V}L}_x^L + w(0) = (C_3^T \mathcal{P}_x + W_1^T) \overline{\mathcal{V}L}_x^L = Y_3^T \overline{\mathcal{V}L}_x^L,$$

$$x + t \approx (\overline{\mathcal{V}L}_x^L)^T \mathcal{K}_1 \overline{\mathcal{V}L}_t^L,$$

$$te^x \approx (\overline{\mathcal{V}L}_x^L)^T \mathcal{K}_2 \overline{\mathcal{V}L}_t^L,$$



$$\begin{aligned}
 x^2 t &\approx (\overline{\mathcal{V}\mathcal{L}_x^L})^T \mathcal{K}_3 \overline{\mathcal{V}\mathcal{L}_t^L}, \\
 u(x) \cdot w(x) &= (\mathcal{Y}_1^T \overline{\mathcal{V}\mathcal{L}_x^L}) ((\overline{\mathcal{V}\mathcal{L}_x^L})^T) \mathcal{Y}_3 = \mathcal{Y}_4^T \overline{\mathcal{V}\mathcal{L}_x^L}, \\
 u(x)^2 &= \mathcal{Y}_1^T \overline{\mathcal{V}\mathcal{L}_x^L} ((\overline{\mathcal{V}\mathcal{L}_x^L})^T) \mathcal{Y}_1, \\
 \int_0^x \frac{(x+t)u(t)w(t)}{\sqrt{x-t}} dt \\
 &= (\overline{\mathcal{V}\mathcal{L}_x^L})^T \mathcal{K}_1 \int_0^x \frac{\overline{\mathcal{V}\mathcal{L}_t^L} (\overline{\mathcal{V}\mathcal{L}_t^L})^T \mathcal{Y}_4}{\sqrt{x-t}} dt \\
 &= (\overline{\mathcal{V}\mathcal{L}_x^L})^T \mathcal{K}_1 \tilde{\mathcal{C}}_{1x} \int_0^x \frac{\overline{\mathcal{V}\mathcal{L}_t^L}}{\sqrt{x-t}} dt \\
 &= (\overline{\mathcal{V}\mathcal{L}_x^L})^T \mathcal{K}_1 \tilde{\mathcal{C}}_{1x} \mathcal{P}_x^{(\frac{1}{2})} \overline{\mathcal{V}\mathcal{L}_x^L}, \\
 \int_0^x x^2 t w(t) dt &= (\overline{\mathcal{V}\mathcal{L}_x^L})^T \mathcal{K}_3 \int_0^x \overline{\mathcal{V}\mathcal{L}_t^L} (\overline{\mathcal{V}\mathcal{L}_x^L})^T \mathcal{Y}_3 dt \\
 &= (\overline{\mathcal{V}\mathcal{L}_x^L})^T \mathcal{K}_3 \tilde{\mathcal{C}}_{2x} \int_0^x \overline{\mathcal{V}\mathcal{L}_t^L} dt \\
 &= (\overline{\mathcal{V}\mathcal{L}_x^L})^T \mathcal{K}_3 \tilde{\mathcal{C}}_{2x} \mathcal{P}_x \overline{\mathcal{V}\mathcal{L}_x^L}, \\
 \int_0^x \frac{v(t) - 2w(t)}{\sqrt[3]{x-t}} dt &= (\mathcal{Y}_2 - 2\mathcal{Y}_3)^T \int_0^x \frac{\overline{\mathcal{V}\mathcal{L}_t^L}}{\sqrt[3]{x-t}} dt \\
 &= (C_2 - 2C_3)^T \mathcal{P}_x^{(\frac{1}{3})} \overline{\mathcal{V}\mathcal{L}_x^L}, \\
 \int_0^1 e^x t u'(t) dt &= (\overline{\mathcal{V}\mathcal{L}_x^L})^T \mathcal{K}_2 \left(\int_0^1 \overline{\mathcal{V}\mathcal{L}_t^L} (\overline{\mathcal{V}\mathcal{L}_t^L})^T dt \right) C_1 \\
 &= (\overline{\mathcal{V}\mathcal{L}_x^L})^T \mathcal{K}_2 \mathcal{D} C_1, \\
 {}^c \mathcal{D}_x^{\alpha_1(x)} u(x) &\approx \frac{1}{\Gamma(1 - \alpha_1(x))} C_1^T \mathcal{P}_x^{(\alpha_1(x))} \overline{\mathcal{V}\mathcal{L}_x^L}, \\
 {}^c \mathcal{D}_x^{\alpha_2(x)} v(x) &\approx \frac{1}{\Gamma(1 - \alpha_2(x))} C_2^T \mathcal{P}_x^{(\alpha_2(x))} \overline{\mathcal{V}\mathcal{L}_x^L}, \\
 {}^c \mathcal{D}_x^{\alpha_3(x)} w(x) &\approx \frac{1}{\Gamma(1 - \alpha_3(x))} C_3^T \mathcal{P}_x^{(\alpha_3(x))} \overline{\mathcal{V}\mathcal{L}_x^L}.
 \end{aligned}$$

By placing the considered approximations in the equation system, the following non-linear algebraic system is obtained:

$$\left\{ \begin{aligned}
 &\frac{1}{\Gamma(1 - \alpha_1(x))} C_1^T \mathcal{P}_x^{(\alpha_1(x))} \overline{\mathcal{V}\mathcal{L}_x^L} + 3C_2^T \overline{\mathcal{V}\mathcal{L}_x^L} \\
 &\quad - (\overline{\mathcal{V}\mathcal{L}_x^L})^T \mathcal{K}_1 \tilde{\mathcal{C}}_{1x} \mathcal{P}_x^{(\frac{1}{2})} \overline{\mathcal{V}\mathcal{L}_x^L} - f_1(x) \approx 0, \\
 &\frac{1}{\Gamma(1 - \alpha_2(x))} C_2^T \mathcal{P}_x^{(\alpha_2(x))} \overline{\mathcal{V}\mathcal{L}_x^L} + (\overline{\mathcal{V}\mathcal{L}_x^L})^T \mathcal{K}_2 \mathcal{D} C_1 \\
 &\quad - 4(\overline{\mathcal{V}\mathcal{L}_x^L})^T \mathcal{K}_3 \tilde{\mathcal{C}}_{2x} \mathcal{P}_x \overline{\mathcal{V}\mathcal{L}_x^L} - f_2(x) \approx 0, \\
 &\frac{1}{\Gamma(1 - \alpha_3(x))} C_3^T \mathcal{P}_x^{(\alpha_3(x))} \overline{\mathcal{V}\mathcal{L}_x^L} - C_1^T \overline{\mathcal{V}\mathcal{L}_x^L} (\overline{\mathcal{V}\mathcal{L}_x^L})^T C_1 \\
 &\quad - (C_2 - 2C_3)^T \mathcal{P}_x^{(\frac{1}{3})} \overline{\mathcal{V}\mathcal{L}_x^L} - f_3(x) \approx 0.
 \end{aligned} \right.$$

By selecting $N = 8$ and applying the collocation approach, the resulting nonlinear equation system is tackled using the Newton iterative method. The following outcomes are then derived for the algebraic system:

$$\begin{aligned}
 C_1 &= [-1.59494778350666, -1.06587152774304, \\
 &\quad 0.0735558687966520, 0.0112792114904673, \\
 &\quad -0.000386976087644809, \\
 &\quad -0.0000358924600568123, 1.043213231 \times 10^{-7}, \\
 &\quad 1.636406664 \times 10^{-7}, -4.782110330 \times 10^{-7}], \\
 C_2 &= [3.10179418346809, 2.34996404663482, \\
 &\quad 0.939985582988100, 0.156664242993795, \\
 &\quad 9.869774288 \times 10^{-9}, 1.779904461 \times 10^{-8}, \\
 &\quad 9.720089266 \times 10^{-9}, 5.036350359 \times 10^{-9}, \\
 &\quad 7.799882154 \times 10^{-9}], \\
 C_3 &= [1.45976553042304, -0.291143789569143, \\
 &\quad -0.0673224704718614, 0.00308163908608228, \\
 &\quad 0.000352646640976222, -9.455844315 \times 10^{-6}, \\
 &\quad 1.797593776 \times 10^{-6}, 3.997897464 \times 10^{-7}, \\
 &\quad 1.385991603 \times 10^{-6}],
 \end{aligned}$$

Hence, the approximate solutions of system will be as below:

$$\begin{aligned}
 u_{\overline{\mathcal{V}\mathcal{L}}}(x) &\approx -1.00000974185226x^2 \\
 &\quad + 5.18171925423675 \times 10^{-7}x \\
 &\quad + 0.0000829438309642378x^3 \\
 &\quad + 0.0829902354375030x^4 \\
 &\quad + 0.000781247937435942x^5 \\
 &\quad - 0.00377763604536389x^6 \\
 &\quad + 0.000670785135173023x^7 \\
 &\quad - 0.000133699953888045x^8 \\
 &\quad + 1.99999998925753, \\
 v_{\overline{\mathcal{V}\mathcal{L}}}(x) &\approx 4.16765149364069 \times 10^{-7}x^2 \\
 &\quad - 3.00000002030494x \\
 &\quad - 3.22735126738956 \times 10^{-6}x^3 \\
 &\quad + 1.00001179465568x^4 \\
 &\quad - 0.0000230784051090481x^5 \\
 &\quad + 0.0000260467333493173x^6 \\
 &\quad - 0.0000160112239442302x^7 \\
 &\quad + 4.11486842136445 \times 10^{-6}x^8 \\
 &\quad - 0.99999999800930, \\
 w_{\overline{\mathcal{V}\mathcal{L}}}(x) &\approx 7.01282546547356 \times 10^{-6}x^2 \\
 &\quad + 1.00000032242698x \\
 &\quad - 0.166766492740069x^3 \\
 &\quad + 0.000523802298355497x^4 \\
 &\quad + 0.00699398718471345x^5 \\
 &\quad + 0.00178843196843631x^6 \\
 &\quad - 0.00140265060766058x^7 \\
 &\quad + 0.000326641731679030x^8 \\
 &\quad + 0.999999969199938.
 \end{aligned}$$

The numerical results of exact and approximate solutions based on normalized Vieta-Lucas polynomials, evaluated for selected various values of x and $N = 8$, are presented in Table 1. The \mathcal{L}_2^N error and convergence rate for selected values of N are shown in Table 2. In the Table 3 to 5, the \mathcal{L}_2^N error is presented for the cases $N = 5, 7, 9$.



Additionally, plots of the absolute errors are displayed in Figure 1.

Table 1 presents the \mathcal{L}_2^N errors and the corresponding convergence rates \mathcal{CR}^N for different values of N . As seen, the \mathcal{L}_2^N errors for all solution components $u(x)$, $v(x)$, and $w(x)$ decrease rapidly as N increases, which clearly demonstrates the high accuracy and strong convergence properties of the shifted Vieta-Lucas polynomial approximation. Moreover, the convergence rates grow significantly with increasing N , indicating that the proposed numerical method achieves spectral-like accuracy. This exponential decay in the error confirms that the method is both stable and efficient for solving system of fractional integro-differential equations.

As observed from Table 2, the numerical values obtained using the shifted Vieta-Lucas polynomial approximation ($u_{\overline{VL}}(x)$, $v_{\overline{VL}}(x)$, and $w_{\overline{VL}}(x)$) are in excellent agreement with the corresponding exact solutions. The discrepancies between the approximate and exact results are negligible, typically on the order of 10^{-10} to 10^{-12} , confirming the high precision and numerical stability of the proposed method. This demonstrates that the shifted Vieta-Lucas polynomial approach provides an efficient and accurate scheme for solving systems of fractional integro-differential equations. The smooth convergence behavior and minimal error values further verify the reliability of the method even for relatively small values of N .

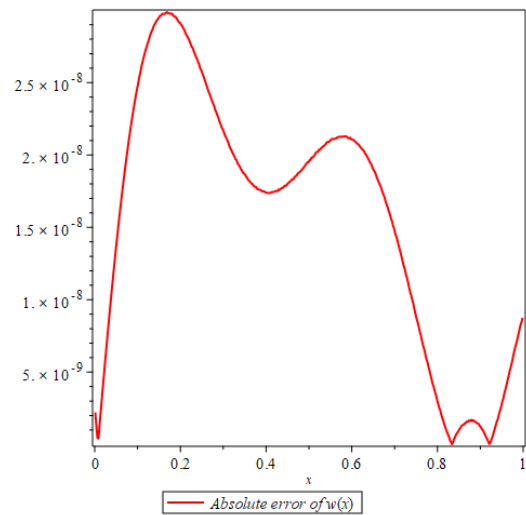
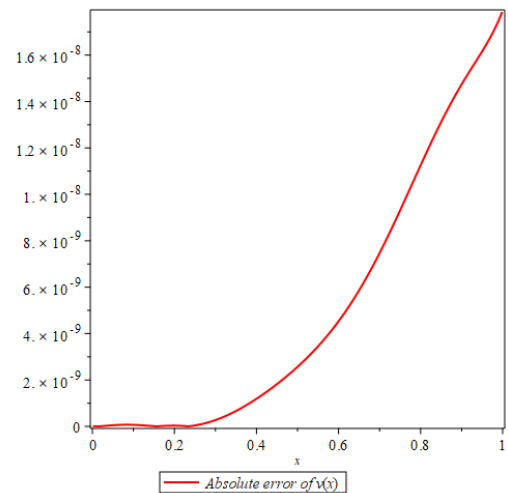
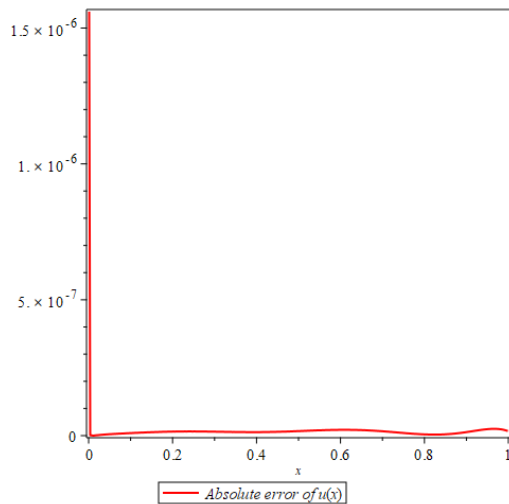


Figure1: Graphs of absolute error in Example 1.

Table1: \mathcal{L}_2^N error and convergence rate for some value of N in Example 1.

N	$\mathcal{L}_2^N u(x)$	CR_u^N	$\mathcal{L}_2^N v(x)$	CR_v^N	$\mathcal{L}_2^N w(x)$	CR_w^N
1	1.984543451	-	0.1732105675	-	0.6551809175	-
2	0.09388312624	4.4017974 81	0.08931267369	0.955590133 9	0.1323598585	2.30742768 5
3	0.004425725251	7.5336112 24	0.005754679767	6.762924722	0.007061809966	7.228298986
4	0.0003147998080	17.192025 88	0.00001715099542	20.21576255	0.00004299488685	17.73268466
5	$1.199642676 \times 10^{-6}$	14.642269 75	$4.910315192 \times 10^{-7}$	15.92384432	$1.481877641 \times 10^{-6}$	15.09239778
6	$7.303115034 \times 10^{-8}$	15.351409 91	$3.462736061 \times 10^{-8}$	14.54498430	$1.348638496 \times 10^{-7}$	13.14600168
7	$5.939001970 \times 10^{-8}$	1.3412842 11	$3.555895948 \times 10^{-8}$	0.172221168 7	$9.821628660 \times 10^{-8}$	2.057037276
8	$3.164213150 \times 10^{-8}$	4.7152708 67	$1.734110146 \times 10^{-8}$	5.377856428	$4.965133127 \times 10^{-8}$	5.108512956

Table2: Comparison of numerical results of approximate and exact for $N = 8$ for Example 1.

x_i	$u_{exact}(x)$	$u_{\overline{VL}}(x)$	$v_{exact}(x)$	$v_{\overline{VL}}(x)$	$w_{exact}(x)$	$w_{\overline{VL}}(x)$
0.1	1.990008331	1.99000832972415	-1.2999	-1.29989999991796	1.099833417	1.0998334290496 9
0.2	1.960133156	1.96013316761958	-1.5984	-1.59840000005118	1.198669331	1.19866935575793
0.3	1.910672978	1.91067301079735	-1.8919	-1.89190000030946	1.295520207	1.29552025840400
0.4	1.842121988	1.84212203443672	-2.1744	-2.17439999901930	1.389418342	1.38941843383756
0.5	1.755165124	1.75516517725240	-2.4375	-2.437499999524898	1.479425539	1.47942563375635
0.6	1.650671230	1.65067127597128	-2.6704	-2.67039998890714	1.564642473	1.56464252423145
0.7	1.529684375	1.52968438747455	-2.8599	-2.85989998067870	1.644217687	1.64421769170096
0.8	1.393413419	1.39341339409566	-2.9904	-2.99039997240166	1.717356091	1.71735608115508
0.9	1.243219937	1.24321993365683	-3.0439	-3.04389996682935	1.783326910	1.78332688393720

Table3: Comparison of \mathcal{L}_2^N error of $u(x)$ with some methods in Example1.

metho d	Roots of $P_{N+1}(x)$			Uniform points		
	$N = 5$	$N = 7$	$N = 9$	$N = 5$	$N = 7$	$N = 9$
NVL	1.16771×10^{-4}	1.42572×10^{-7}	8.65586×10^{-8}	4.10376×10^{-5}	6.80473×10^{-7}	1.31268×10^{-6}
Ch2K	1.00753	3.28409×10^{-2}	134.88665	4.20190×10^{-5}	0.14911	1.00870×10^{-6}
Ch5K	8.91902	2.54195×10^{-7}	1.31146	6.3052×10^{-5}	7.29148×10^{-7}	7.68616×10^{-7}
Ch7K	2.5728×10^{-4}	6.52536×10^{-7}	200.26859	9.5323×10^{-5}	6.37075×10^{-7}	6.18514×10^{-5}

Table4: Comparison of \mathcal{L}_2^N error of $v(x)$ with some methods in Example1.

metho d	Roots of $P_{N+1}(x)$			Uniform points		
	$N = 5$	$N = 7$	$N = 9$	$N = 5$	$N = 7$	$N = 9$
NVL	7.12600×10^{-5}	8.22390×10^{-8}	5.12293×10^{-8}	2.47888×10^{-5}	4.12176	7.45076×10^{-7}
Ch2K	0.19004	1.92827×10^{-2}	148.57050	2.17826×10^{-5}	0.10200	5.85335×10^{-7}

Ch5K	3.02808 × 10 ⁻⁵	8.38704 × 10 ⁻⁸	0.14528	3.80945 × 10 ⁻⁵	4.70872 × 10 ⁻⁷	4.47213 × 10 ⁻⁷
Ch7K	6.74294 × 10 ⁻⁵	1.66404 × 10 ⁻⁷	674.10191	5.66871 × 10 ⁻⁵	4.45118 × 10 ⁻⁷	8.49265 × 10 ⁻⁷

Table5: Comparison of L_2^N error of $w(x)$ with some methods in Example1.

metho d	Roots of $P_{N+1}(x)$			Uniform points		
	$N = 5$	$N = 7$	$N = 9$	$N = 5$	$N = 7$	$N = 9$
NVL	1.8892 × 10 ⁻⁴	2.36396 × 10 ⁻⁷	1.38919 × 10 ⁻⁷	6.47318 × 10 ⁻⁵	1.10177 × 10 ⁻⁶	2.0434 × 10 ⁻⁶
Ch2K	0.98502	5.26473 × 10 ⁻²	904.37480	6.67503 × 10 ⁻⁵	0.25303	1.6087 × 10 ⁻⁶
Ch5K	5.47885 × 10 ⁻⁵	1.63533 × 10 ⁻⁷	1.43377	9.55803 × 10 ⁻⁵	1.19016 × 10 ⁻⁶	1.2462 × 10 ⁻⁶
Ch7K	1.13884 × 10 ⁻⁴	3.16399 × 10 ⁻⁷	2383.41137	5.668710 × 10 ⁻⁵	1.1195 × 10 ⁻⁶⁷	2.3278 × 10 ⁻⁶

A comprehensive comparison of the L_2^N errors obtained using the proposed shifted Vieta-Lucas method and several existing techniques namely Chebyshev polynomials of the second kind (Ch2K), fifth kind (Ch5K), and seventh kind (Ch7K) is presented in Tables 3–5 for various collocation levels ($N = 5, 7, 9$) and two types of node distributions: the roots of $P_{N+1}(x)$ and uniform nodes. The results consistently demonstrate that the shifted Vieta-Lucas approximation yields significantly smaller L_2^N errors across all test cases and collocation settings. In particular, when the collocation points are chosen as the roots of $P_{N+1}(x)$, the proposed method exhibits rapid error reduction as N increases, confirming its spectral convergence and numerical robustness. For all test functions $u(x)$, $v(x)$ and $w(x)$ the error decreases smoothly and monotonically with increasing collocation order, in contrast to the irregular and occasionally unstable behavior observed in the Chebyshev-based approaches. These findings indicate that the shifted Vieta-Lucas scheme maintains high stability, precision, and efficiency, even for large N . Consequently, the proposed shifted Vieta-Lucas polynomial method can be regarded as a superior and reliable computational framework for solving fractional integro-differential systems, achieving excellent accuracy and numerical stability compared with conventional spectral techniques

Example.8.2. As the second model, the following equations will be considered:

$$\begin{cases} {}^C \mathcal{D}_x^{\alpha_1(x)} u(x) = f_1(x) - v^2(x) \\ \quad + \int_0^L (x+t)u'(t)v'(t)dt \\ \quad - \int_0^x \sin(x+t)u(t)dt, \\ {}^C \mathcal{D}_x^{\alpha_2(x)} v(x) = f_2(x) - \int_0^L u(t)v''(t)dt \\ \quad + \int_0^x x^2 \cos t(u(t) + v(t))dt. \end{cases}$$

$$\alpha_1(x) = \frac{\sin x + \cos x}{3}, \quad \alpha_2(x) = \frac{3}{2},$$

$$[0, L] = [0, 2], \quad 0 < \alpha_1(x) < 1,$$

$$u(x) = e^x, \quad v(x) = \cos x,$$

$$u(0) = 1, \quad v(0) = 1, \quad v'(0) = 0.$$

The functions $f_i(x)$, $i = 1, 2$ can be calculated using the analytical solutions. By assuming

$$u'(x) \approx C_1^T \overline{\mathcal{V}} \mathcal{L}_x^L, \quad v''(x) \approx C_2^T \overline{\mathcal{V}} \mathcal{L}_x^L,$$

and operational matrices, the following non-linear algebraic equations system is obtained:

$$\begin{cases} \frac{1}{\Gamma(1 - \alpha_1(x))} C_1^T \mathcal{P}_x^{(\alpha_1(x))} \overline{\mathcal{V}} \mathcal{L}_x^L + \chi_3^T \overline{\mathcal{V}} \mathcal{L}_x^L (\overline{\mathcal{V}} \mathcal{L}_x^L)^T \chi_3 \\ \quad - (\overline{\mathcal{V}} \mathcal{L}_x^L)^T \mathcal{K}_1 \mathcal{D} y_1 + (\overline{\mathcal{V}} \mathcal{L}_x^L)^T \mathcal{K}_1 \mathcal{P}_x \overline{\mathcal{V}} \mathcal{L}_x^L - f_1(x) \approx 0, \\ \frac{1}{\Gamma(1 - \alpha_2(x))} C_1^T \mathcal{P}_x^{(\alpha_2(x))} \overline{\mathcal{V}} \mathcal{L}_x^L + \chi_1^T \mathcal{D} C_2 \\ \quad - (\overline{\mathcal{V}} \mathcal{L}_x^L)^T \mathcal{K}_3 - \tilde{C}_{2_x} - f_2(x) \approx 0. \end{cases}$$

By choosing $N = 10$, collocation points and solving the above system, the following results for unknown vectors C_1 and C_2 would be obtained:



$$C_1 = [6.099941754, 3.850834782, 0.9249460081, 0.1510474010, 0.1864873815, 0.001848419826, 0.0001524092900, 0.00001033189883, 2.337719789 \times 10^{-7}, -1.175016730 \times 10^{-7}, -1.093482024 \times 10^{-7}],$$

$$C_2 = [-0.7328007181, 0.9281786980, 0.1556166364, -0.04126470199, -0.003355282403, 0.0005264783846, 0.00002770486951, -3.282906649 \times 10^{-6}, -4.380108612 \times 10^{-7}, -1.880488970 \times 10^{-8}, -9.660365554 \times 10^{-8}].$$

Thus, the approximate results as below

$$u_{\overline{VL}}(x) \approx 0.999999996 + 1.000002477 x + 0.4999903586 x^2 + 0.166697135 x^3 + 0.04158045452 x^4 + 0.008510944305 x^5 - 0.0000703462897099999 x^8,$$

$$v_{\overline{VL}}(x) \approx 0.999999999427769 + 9.16500191126100 \times 10^{-10} x - 0.500001014536689 x^2 + 5.03087356389317 \times 10^{-6} x^3 + 0.0416517009885213 x^4 + 0.0000264255804168566 x^5 - 0.00141750426845290 x^6 + 0.0000191199647642690 x^7 + 0.0000172260694083868 x^8 + 1.56985201713515 \times 10^{-6} x^9 - 3.874183657 \times 10^{-7} x^{10}.$$

Considering the numerical data presented in Tables 4 and 5, as well as the absolute error distribution illustrated in Figure 2 and the comparative plots between the exact and approximate solutions shown in Figure 3, it can be concluded that the proposed shifted Vieta-Lucas collocation method demonstrates excellent accuracy and convergence characteristics. From Table 4, the approximate values of $u_{\overline{VL}}(x)$ and $v_{\overline{VL}}(x)$ are almost identical to the exact solutions for all tested points $x_i \in [0,1]$, confirming the high precision of the method even for a relatively small number of collocation points ($N = 10$). Furthermore, Table 5 reveals a rapid reduction in the \mathcal{L}_2^N error as N increases, which indicates spectral convergence of the proposed technique. The convergence rates (\mathcal{CR}_u^N and \mathcal{CR}_v^N) show a significant increase for higher N , demonstrating the stability and consistency of the algorithm.

These results validate that the shifted Vieta-Lucas approach efficiently handles systems of variable-order fractional integro-differential equations. The method maintains numerical stability while achieving superior accuracy compared to existing polynomial-based collocation techniques, confirming its effectiveness and

potential for solving more nonlinear fractional systems.

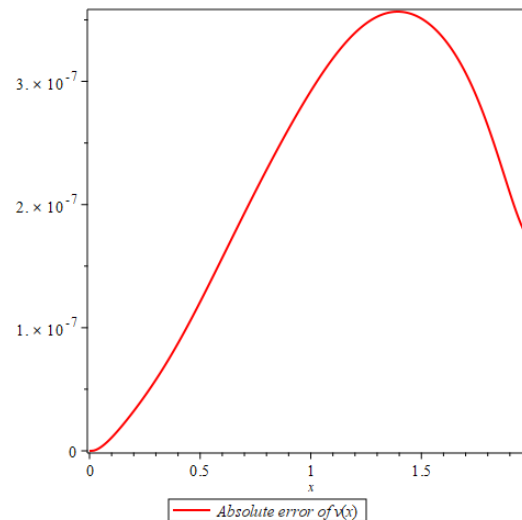
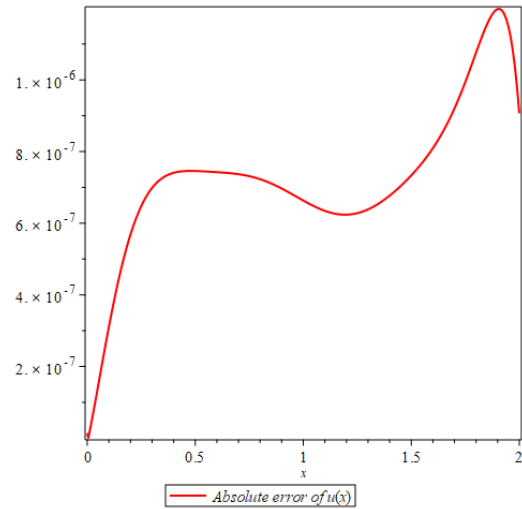


Figure2: Graphs of absolute error in Example 2.

8. Conclusion

This study introduces a class of orthogonal polynomials and investigates their application in solving functional equations. The proposed approach employs normalized and shifted Vieta-Lucas polynomials to approximate the solutions of systems of variable-order fractional integro-differential equations. To reduce the original problem to a solvable algebraic form, operational matrices are combined with the collocation method. The resulting nonlinear algebraic system is subsequently solved using Newton's iterative method. Key analytical aspects, including the existence and uniqueness of solutions, as well as the stability and convergence of the approach, are thoroughly examined. A computational algorithm based on these orthogonal polynomials is developed and validated through two benchmark problems. The numerical results demonstrate the



high accuracy and reliability of the method when compared with analytical solutions, confirming its suitability for approximating variable-order fractional systems. One of the primary advantages of using orthogonal bases is their ability to transform complex differential systems, whether linear or nonlinear, into algebraic formulations. The initial terms in the orthogonal expansion capture most of the essential information of the target function, enabling accurate approximations with a limited number of basis functions. This reduction minimizes the need for large expansions while maintaining high precision. Moreover, the sparsity of the associated operational matrices significantly simplifies algebraic computations. Nevertheless, solving the resulting nonlinear algebraic systems may present computational challenges. To reduce memory and processing costs, Newton's method is employed. Although it may exhibit slower convergence in some cases, it enhances overall computational efficiency. Another crucial aspect is the selection of appropriate collocation points, as this choice directly affects both the accuracy and stability of the solution.

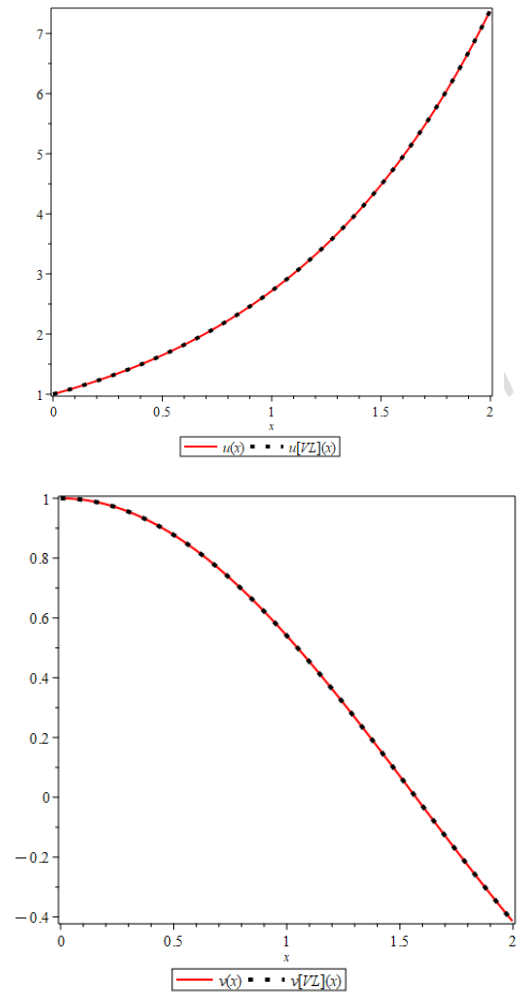


Figure3: Comparing the graph of the exact and approximate solutions in Example 2.

Table6: Comparison of numerical results of approximate and exact for $N = 10$ for Example2.

x_i	$u_{exact}(x)$	$u_{\overline{VL}}(x)$	$v_{exact}(x)$	$v_{\overline{VL}}(x)$
0.1	1.105170918	1.10517053649319	0.9950041653	0.99500417105813 0
0.2	1.221402758	1.22140227334706	0.9800665778	0.98006659706331 8
0.3	1.349858808	1.34985826727498	0.95533644891	0.95533652467223 9
0.4	1.491824698	1.49182410840082	0.9210609940	0.92106104902611 0
0.5	1.648721271	1.64872064884808	0.8775825619	0.87758263907751 8
0.6	1.822118800	1.82211816992806	0.8253356149	0.82533571541772 5
0.7	2.013752707	2.013722087366834	0.7648421873	0.76484231067675 1
0.8	2.225540928	2.22554032625938	0.6967067093	0.69670685411780 5
0.9	2.459603111	2.45960252479531	0.6216099683	0.62161013221989 1

Table7: The L_2^N error and convergence rate for some value of N in Example 2.

N	$L_2^N u(x)$	CR_u^N	$L_2^N v(x)$	CR_v^N
-----	--------------	----------	--------------	----------



0.1	0.64798434422	-	0.1907978733	-
0.2	0.1215678242	2.414197517	0.03600913284	2.405610329
0.3	0.01721221369	4.821260971	0.003440108283	5.791545563
0.4	0.001553515368	8.360266838	0.0004259953051	7.260897025
0.5	0.0001239751514	11.32990155	0.00002565211212	12.59190506
0.6	$8.846746295 \times 10^{-6}$	14.48008375	$2.546980119 \times 10^{-6}$	12.668373310
0.7	$5.553457669 \times 10^{-7}$	17.95784550	$1.32997215 \times 10^{-7}$	20.79153324
0.8	$6.730430531 \times 10^{-7}$	1.439499524	$1.912592662 \times 10^{-6}$	4.613111791
0.9	$1.128329978 \times 10^{-6}$	4.386749082	$1.087429447 \times 10^{-6}$	14.75545228
10	$3.944838808 \times 10^{-7}$	9.974473236	$1.814928218 \times 10^{-7}$	16.99266344

Acknowledgement

The authors would like to thank the editorial board of the journal and the anonymous reviewers for their constructive comments and suggestions, which led to a substantial improvement of the manuscript.

Authors Contribution

All the authors have participated sufficiently in the intellectual content, conception and design of this work or the analysis and interpretation of the data (when applicable), as well as the writing of the manuscript.

Availability of data and materials

The data that support the findings of this study are available from the corresponding author, upon reasonable request.

Conflict of interests

The author states that there is no conflict of interest.

References

- [1] Podlubny, I. (1999). *Fractional Differential Equations*. Academic Press. SanDiago.
- [2] El-Shahed, M., Nieto, J., & Ahmed, A. (2017). Fractional-Order Model for Biocontrol of the Lesser Date Moth in Palm Trees and Its Discretization. *Advances in Difference Equations*, 295, 193-209. <https://doi.org/10.1186/s13662-017-1349-1>
- [3] Li, M. (2018). Three Classes of Fractional Oscillators. *Symmetry-Basel*, 10(2), 40-56. <https://doi.org/10.3390/sym10020040>
- [4] Li, M., (2020). Multi-Fractional Generalized Cauchy Process and Its Application to Teletraffic. *Physica A: Statistical Mechanics and its Applications*, 550, 105-127. <https://doi.org/10.1016/j.physa.2019.123982>
- [5] Li, M., & Zhao, Y. L. (2018). A Fast Energy Conserving Finite Element Method for the Nonlinear Fractional Schrodinger Equation with Wave Operator. *Applied Mathematics and Computation*, 338, 758-773. <https://doi.org/10.1016/j.amc.2018.06.010>
- [6] Li, M., Gu, X. M., Huang, C., Fei, M., & Zhang, G. (2018). A fast linearized Conservative Finite Element Method for the Strongly Coupled Nonlinear Fractional Schrodinger Equations, *Computational Physics*, 358, 256-282. <https://doi.org/10.1016/j.jcp.2017.12.044>.
- [7] Li, M., Huang, C., & Zhao, Y. L. (2020). Fast conservative numerical Algorithm for the Coupled Fractional Klein-Gordon-Schrodinger Equation. *Numerical Algorithms*, 84, 1081-1119. <https://doi.org/10.1007/s11075-019-00793-9>.
- [8] Mirzaee, F., Samadyar, N. (2019). Numerical solution of Time Fractional stochastic Korteweg-de Vries Equation via Implicit Meshless Approach. *Iranian Journal of Science and Technology*, 43: 2905-2912. <https://doi.org/10.1007/s40995-019-00763-9>
- [9] Mirzaee, F., & Samadyar, N. (2019). Numerical Solution based on Two-dimensional Orthonormal Bernstein Polynomials for Solving Some Classes of Two-Dimensional Nonlinear Integral Equations of Fractional Order. *Applied Mathematics and Computation*, 344-345, 191-203. <https://doi.org/10.1016/j.amc.2018.10.020>
- [10] Mirzaee, F., & Samadyar, N. (2021). Implicit Meshless Method to Solve 2D Fractional Stochastic Tricomi-Type Equation Defined on Irregular Domain Occurring in Fractal Transonic Flow. *Numerical Methods for Partial Differential Equations*, 37 1781-1799. <https://doi.org/10.1002/num.22608>
- [11] Mirzaee, F., & Samadyar, N. (2021). Combination of Finite Difference Method and Meshless Method based on Radial Basis Functions to Solve Fractional Stochastic Advection-Diffusion Equations. *Engineering with Computers*, 36 1673-1686. <https://doi.org/10.1007/s00366-019-00789-y>.
- [12] Mirzaee, F., Sayevand, K., Rezaeei, S., & Samadyar, N. (2020). Finite Difference and Spline Approximation for Solving Fractional Stochastic Advection-Diffusion Equation. *Iranian Journal of Science and Technology*, 45, 607-617. <https://doi.org/10.1007/s40995-020-01036-6>



- [13] Saemi, F., Ebrahimi, H., & Shafiee, M. (2020). An Effective Scheme for Solving System of Fractional Volterra-Fredholm Integro-Differential Equations based on the Müntz-Legendre Wavelets. *Computational and Applied Mathematics*, 374, 273-289. <https://doi.org/10.1016/j.cam.2020.112773>
- [14] Wang, L., & Chen, Y. M. (2020). Shifted-Chebyshev-polynomial-based numerical algorithm for fractional order polymer visco-elastic rotating beam. *Chaos Solutions and Fractals*, 132, 109585-109601. <https://doi.org/10.1016/j.chaos.2019.109585>
- [15] Xu, T., Chen, W., Lu, S., & Chen, H. (2016). An Implicit Midpoint Difference Scheme for the Fractional Ginzburg-Landau Equation. *Computational Physics*, 312, 31-49. <https://doi.org/10.1016/j.jcp.2016.02.018>
- [16] Babaei, A., Jafari, H., & Banihashemi, S. (2020). Numerical Solution of Variable Order Fractional Nonlinear Quadratic integro-Differential Equations based on the Sixth-kind Chebyshev Collocation Method. *Computational and Applied Mathematics*, 377, 112908-112931. <https://doi.org/10.1016/j.cam.2020.112908>
- [17] Babaei, A., Parsa Moghadam, B., Banihashemi, S., & Tenreiro, J. (2020). Numerical Solution of Variable-Order Fractional Integro-Partial Differential Equations via Sinc Collocation Method based on Dingle and double Exponential Transformations. *Communications in Nonlinear Science and Numerical Simulation*, 82, 104985-105010. <https://doi.org/10.1016/j.cnsns.2019.104985>
- [18] Hosseininia, M., Heydari, M. H., Maalek Ghaini, F. M., & Avazzadeh, Z. (2019). A Wavelet Method to Solve Nonlinear Variable-Order Time Fractional 2D Klein-Gordon Equation. *Computers and Mathematics with Applications*, 78, 3713-3730. <https://doi.org/10.1016/j.camwa.2019.06.008>
- [19] Hosseininia, M., Heydari, M. H., & Avazzadeh, Z. (2021). Numerical Study of the Variable-Order Fractional Version of the Nonlinear Fourth-Order 2D Diffusion-Wave Equation via 2D Chebyshev Wavelets. *Engineering with Computers*, 37, 3319-3328. <https://doi.org/10.1007/s00366-020-00995-z>
- [20] Xie, J., Huang, Q., & Zhao, F. (2017). Numerical Solution of Nonlinear Volterra-Fredholm-Hammerstein Integral Equations in Two-Dimensional Space Based on Block Pulse Functions. *Computational and Applied Mathematics*, 317, 565-572. <https://doi.org/10.1016/j.cam.2016.12.028>
- [21] Xie, J., & Yi, M. (2019). Numerical Research of Nonlinear System of Fractional Volterra-Fredholm Integral Equations via Block-Pulse Functions and Error Analysis. *Computational and Applied Mathematics*, 345, 159-167. <https://doi.org/10.1016/j.cam.2018.06.008>
- [22] Xie, J., Ren, Z., & Li, Y. (2019). Numerical Scheme for Solving System of Fractional Partial Differential Equations with Volterra-Type Integral Term Through Two-Dimensional Block-Pulse Functions. *Numerical Methods for Partial Differential Equations*, 35, 1890-1903. <https://doi.org/10.1002/num.22383>
- [23] Nagy, A. M., & El-Sayed, A. A. (2019). An Accurate Numerical Technique for Solving Two-Dimensional Time Fractional Order Diffusion Equation. *International Journal Modelling and Simulation*, 39, 214-221. <https://doi.org/10.1080/02286203.2019.1592285>
- [24] Nagy, A. M., Sweilam, N. H., & El-Sayed, A. A. (2018). New Operational Matrix for Solving Multi-Term Variable Order Fractional Differential Equations. *Journal of The Computational and Nonlinear Dynamics*, 13, 11001-11021. <https://doi.org/10.1115/1.4037922>
- [25] Sweilam, N. H., Khader, M., & Al-Bar, R. F. (2008). Homotopy Perturbation Method for linear and Nonlinear System of Fractional Integro-Differential Equations. *International Journal of Modelling and Simulation*, 1, 73-87. <https://doi.org/10.1016/j.enganabound.2015.02.002>
- [26] Ganji, D., & Rajabi, A. (2006). Assessment of Homotopy-Perturbation and Perturbation Methods in Heat Radiation Equation. *International Communications in Heat and Mass Transfer*, 33, 391-400. <https://doi.org/10.1016/j.icheatmasstransfer.2005.11.001>
- [27] Ganji, D., & Sadighi, A. (2006). Applications of He's Homotopy-Perturbation Method to Nonlinear Coupled Systems of Reaction-Diffusion Equations. *International Journal of Nonlinear Sciences and Numerical Simulation*, 7, 411-418. <https://doi.org/10.1515/IJNSNS.2006.7.4.411>
- [28] Riahi Beni, N., (2024). Vieta-Lucas operational matrix Technique for Fractional variable-order Integro-Differential Equations. *Journal of Mahani Mathematical Research*, 14(2), 147-173. <https://doi.org/10.22103/JMMR.2025.24074.1699>
- [29] Noor, Z. A., Talib, I., Abdjljewad, T., & Alqudah, M. A. (2022). Numerical study of Caputo Fractional-order Differential Equations by Developing New Operational Matrices of Vieta-Lucas Polynomials. *Fractal and Fractional*, 6(2), 79. <https://doi.org/10.3390/fractalfract6020079>
- [30] Agarwal, P., & El-Sayed, A. A. (2023). Operational Matrix Approach for Solving Variable-Order Fractional Integro-Differential Equations. *Fractional Order Systems and Applications in Engineering*, 301-317. <https://doi.org/10.1016/b978-0-32-390953-2.00025-6>
- [31] Gokcen Kocer, E. (2016). Bivariate Vieta-Fibonacci and Bivariate Vieta-Lucas Polynomials. *Mathematics*, 12(04), 44-50. <https://doi.org/10.9790/5728-1204024450>
- [32] Heydari, M. H., Avazzadeh, Z., & Razzaghi, M. (2021). Vieta-Lucas Polynomials for the Coupled Nonlinear Variable-Order Fractional Ginzburg-Landau Equation. *Applied Numerical Mathematics*, 165, 442-458. <https://doi.org/10.1016/j.apnum.2021.03.007>
- [33] Agarwal, P., & El-Sayed, A. A. (2020). Vieta-Lucas Polynomials for Solving a Fractional-Order Mathematical Physics Model. *Advances in Difference Equations*, 626, 754-772. <https://doi.org/10.1186/s13662-020-03085-y>

