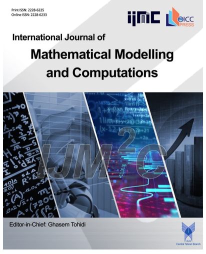

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Research Article

Müntz Function Collocation Method for 2D Fredholm Integral Equations of the Second Kind

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Abstract:

This paper presents a novel numerical approach for solving two-dimensional Fredholm integral equations of the second kind (2DFIEs) using Müntz functions and a collocation method. In the proposed approach, both the unknown function and the integral kernel are approximated by Müntz-type orthogonal functions, and the integral equation is transformed into a system of linear equations using two-dimensional collocation points; the approximate solution is then obtained by solving this linear system. The approximation of bivariate functions is analyzed in the Sobolev space. Several numerical examples are provided to demonstrate the accuracy and efficiency of the method. The numerical results confirm that the two-dimensional Müntz collocation method offers a flexible and effective tool for solving complex 2DFIEs, with potential applications in scientific and engineering problems involving integral equations.

Keywords: 2D Fredholm integral equation; Collocation Method; Müntz Function; Approximation

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1. Introduction

The present work addresses the numerical treatment of 2D Fredholm integral equations (2DFIEs) of the second kind, expressed as

$$f(x,y) - \mu \int_0^1 \int_0^1 k(x,y,s,t) f(s,t)\,ds\,dt = g(x,y), \quad (x,y) \in [0,1]^2. \tag{1}$$

where $\mu \in \mathbb{R}\backslash\{0\}$, and the functions k and g are continuous on $[0,1]^2$ and $[0,1]$. The function f is the unknown solution [1].

Equations of this type appear in many applied fields. They provide mathematical descriptions of problems in electromagnetic wave scattering, image rendering, and aerodynamic modeling [2, 3]. They may also emerge naturally when boundary value problems are reformulated in integral form [4, 5].

Numerical strategies for such problems vary widely. Among them are local schemes based on piecewise polynomials [4, 6], global tensor-operator approaches [7, 8], and cubature rules that rely on hyper-interpolation [9]. Hybrid approximation frameworks have also been investigated, such as Galerkin methods with Legendre-Block-Pulse [10], Chebyshev-Block-Pulse [11], and Fourier-Block-Pulse bases [12], as well as approaches using sine–cosine wavelets [13]. All of these techniques typically assume that data values of k and g can be computed on structured point sets within S.

If only scattered samples of the known functions are available, alternative strategies become more suitable. Examples include mesh-free and spectral collocation methods [14], Galerkin schemes using wavelets [15], or polynomial interpolation techniques [16]. Comparisons of different radial basis function (RBF) methods for such equations can be found in [17].

In this study, our objective is to obtain an approximate solution for the Fredholm integral equation of the second kind (1) by employing Müntz functions together with Müntz collocation points [18]. This methodology provides a flexible framework for approximating the solutions of integral equations while simultaneously offering a deeper understanding of their intrinsic behavior and properties. The use of this approach not only enhances the accuracy of numerical solutions for such equations but also contributes to the advancement of modern computational techniques. Consequently, this research is expected to take a significant step toward the development of more powerful analytical and numerical tools for solving complex mathematical problems and their applications in science and engineering.

2. Preliminary Concepts

2.1 Müntz Orthogonal Functions and Their Features

Definition 2.1 *Consider the set* $\{\rho_\eta(x)\}_{\eta=0}^{\infty}$ *representing a sequence of Müntz orthogonal functions, orthogonal over the interval* $[0,1]$ *with weight* $w(x)=1$. *These functions are defined as follows (Milovanović [19]):*

$$\rho_\eta(x) = \Psi_\eta(x) + \ln(x)\Phi_\eta(x), \quad x \in [0,1], \quad \eta = 0,1,2,\ldots \tag{2}$$

Here, $\Psi_\eta(x)$ *and* $\Phi_\eta(x)$ *are polynomials of degree* $\lfloor \frac{\eta}{2} \rfloor$ *and* $\lfloor \frac{\eta-1}{2} \rfloor$, *respectively:*

$$\Psi_\eta(x) = \sum_{\nu=0}^{\lfloor \frac{\eta}{2} \rfloor} \alpha_\nu^{(\eta)} x^\nu, \qquad \Phi_\eta(x) = \sum_{\nu=0}^{\lfloor \frac{\eta-1}{2} \rfloor} \beta_\nu^{(\eta)} x^\nu. \tag{3}$$

For even indices $\eta = 2\theta$, *the coefficients satisfy*

$$\begin{aligned} \alpha_\nu^{(2\theta)} &= -\binom{\theta+\nu}{\theta}^2 \binom{\theta}{\nu}^2 \\ &\times \left(\frac{2\theta+1}{2\nu+1} + 2(\theta-\nu) \sum_{\substack{j=0 \\ j\neq\nu}}^{\theta-1} \frac{2j+1}{(j-\nu)(j+\nu+1)} \right), \\ &\qquad 0 \le \nu \le \theta-1. \end{aligned} \tag{4}$$

$$\beta_\nu^{(2\theta)} = -(\theta-\nu)\binom{\theta+\nu}{\theta}^2 \binom{\theta}{\nu}^2, \qquad 0 \le \nu \le \theta-1. \tag{5}$$

and for $\nu = \theta$:

$$\alpha_\theta^{(2\theta)} = \binom{2\theta}{\theta}^2, \qquad \beta_\theta^{(2\theta)} = 0. \tag{6}$$

For odd indices $\eta = 2\theta+1$, *the coefficients are given by*

$$\begin{aligned} \alpha_\nu^{(2\theta+1)} &= \binom{\theta+\nu}{\theta}^2 \binom{\theta}{\nu}^2 \\ &\times \left(\frac{2\theta+1}{2\nu+1} + 2(\theta+\nu+1) \sum_{\substack{j=0 \\ j\neq\nu}}^{\theta-1} \frac{2j+1}{(j-\nu)(j+\nu+1)} \right), \\ &\qquad 0 \le \nu \le \theta. \end{aligned} \tag{7}$$

$$\beta_\nu^{(2\theta+1)} = (\theta+\nu+1)\binom{\theta+\nu}{\theta}^2 \binom{\theta}{\nu}^2, \qquad 0 \le \nu \le \theta. \tag{8}$$

Following Definition 2.1, the first few Müntz functions can be expressed explicitly as:

$$\begin{aligned} &\rho_0(x) = 1, \\ &\rho_1(x) = 1 + \ln(x), \\ &\rho_2(x) = -3 + 4x - \ln(x), \\ &\rho_3(x) = 9 - 8x + 2\ln(x)(1+6x), \\ &\rho_4(x) = -11 - 24x + 36x^2 - 2\ln(x)(1+18x), \\ &\vdots \end{aligned} \tag{9}$$

Theorem 2.2 *Each Müntz function* $\rho_\eta(x)$ *over* $[0,1]$ *possesses precisely* η *distinct simple zeros (Milovanović [19]).*

Theorem 2.3 *The set of Müntz functions forms an orthogonal system and fulfills*

$$\int_0^1 \rho_i(x)\rho_j(x)dx = \begin{cases} \delta_j, & i = j, \\ 0, & i \neq j, \end{cases} \tag{10}$$

where

$$\delta_j = \begin{cases} \frac{1}{j+1}, & j \text{ even}, \\ \frac{1}{j}, & j \text{ odd}. \end{cases} \tag{11}$$

2.2 2D Approximation by Müntz Functions

Assume that $f \in L^2([0,1]\times[0,1])$ and that n is a positive integer. In this case, the bivariate function $f(x,y)$ can be approximated by Müntz-type basis functions as [20]

$$f(x,y) \simeq \widetilde{f}(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{n} c_{ij}\,\rho_i(x)\,\rho_j(y), \tag{12}$$

where the unknown coefficients $c_{i,j}$ are obtained through

$$c_{ij} = \frac{1}{\delta_i\,\delta_j}\int_0^1\int_0^1 f(x,y)\,\rho_i(x)\,\rho_j(y)\,dx\,dy, \qquad i,j = 0,1,\ldots,n. \tag{13}$$

Theorem 2.4 *Suppose that* $f \in H^{r_1,r_2}([0,1]^2)$ *with integer* $r_1, r_2 \ge 0$, *where*

$$\begin{aligned} H^{r_1,r_2}(\Omega) &= \Big\{ v \in L^2(\Omega) : D_x^{\alpha_1} D_y^{\alpha_2} v \in L^2(\Omega), \\ &\qquad |\alpha_1| \le r_1,\ |\alpha_2| \le r_2 \Big\}, \\ \Omega &= [0,1]\times[0,1], \end{aligned}$$

denotes the Sobolev space of order r_1 and r_2 in two dimensions. Here, $D_x^\alpha v$ denotes the derivative of order α of the function v with respect to x.

Let

$$\widetilde{f}(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{n} c_{ij}\,\rho_i(x)\,\rho_j(y)$$

be the best approximation of f in span$\{\rho_i(x)\rho_j(y) : 0 \le i, j \le n\}$. Then, if $r_1, r_2 \le n+1$, we have

$$\|f - \widetilde{f}\|_{L^2([0,1]^2)} \le C(n+1)^{-r_1}(n+1)^{-r_2} \times \|D_x^{r_1} D_y^{r_2} f\|_{L^2([0,1]^2)},$$

and for $1 \le \mu \le \min\{r_1, r_2\}$,

$$\|f - \widetilde{f}\|_{H^\mu([0,1]^2)} \le C(n+1)^{2\mu-\frac{1}{2}-r_1}(n+1)^{2\mu-\frac{1}{2}-r_2} \times \|D_x^{r_1} D_y^{r_2} f\|_{L^2([0,1]^2)},$$

where C is a constant depending only on r_1 and r_2.

Remark 2.5 *If x_α, $\alpha = 1, 2, \ldots, n+1$ are the roots of $\rho_{n+1}(x)$, then the roots of $\rho_{n+1}(x)\rho_{n+1}(y)$ are given by*

$$(x_\alpha, x_\beta), \quad \alpha, \beta = 1, 2, \ldots, n+1. \qquad (14)$$

These are the roots in the 2-dimensional domain.

3. Description of the method

Here, we propose a 2-dimensional collocation approach for the numerical solution of the 2DFIEs.

Let us consider 2DFIE given in (1). Substituting expression (12) into (1) yields the following form:

$$\sum_{i=0}^{n}\sum_{j=0}^{n} c_{ij}\,\rho_i(x)\,\rho_j(y) - \mu\int_0^1\int_0^1 k(x,y,s,t)\sum_{i=0}^{n}\sum_{j=0}^{n} c_{i,j}\rho_i(s)\,\rho_j(t)dsdt = g(x,y), \quad (x,y) \in [0,1]^2, \qquad (15)$$

Approximating the kernel k by means of Müntz functions, we have

$$k(x,y,s,t) \simeq \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{l=0}^{n}\sum_{m=0}^{n} k_{ijlm}\,\rho_i(x)\,\rho_j(y)\,\rho_l(s)\,\rho_m(t), \qquad (16)$$

By substituting this approximation into equation (15), we obtain

$$\sum_{i=0}^{n}\sum_{j=0}^{n} c_{ij}\,\rho_i(x)\,\rho_j(y) - \mu\int_0^1\int_0^1 \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{l=0}^{n}\sum_{m=0}^{n} k_{ijlm}\rho_i(x)\rho_j(y)\rho_l(s)\rho_m(t) \sum_{p=0}^{n}\sum_{q=0}^{n} c_{pq}\,\rho_p(s)\,\rho_q(t)\,dsdt = g(x,y), \qquad (17)$$

Hence, it follows that

$$\sum_{i=0}^{n}\sum_{j=0}^{n} \rho_i(x)\,\rho_j(y) \times \left(c_{ij} - \mu\int_0^1\int_0^1 \sum_{l=0}^{n}\sum_{m=0}^{n} k_{ijlm}\,\rho_l(s)\,\rho_m(t) \times \sum_{p=0}^{n}\sum_{q=0}^{n} c_{pq}\,\rho_p(s)\,\rho_q(t)\,dsdt\right) = g(x,y), \qquad (18)$$

By substituting the 2-dimensional collocation points (x_α, y_β), $\alpha, \beta = 1, 2, \ldots, n+1$ introduced in relation (14) into the above expression, we obtain:

$$\sum_{i=0}^{n}\sum_{j=0}^{n} \rho_i(x_\alpha)\,\rho_j(y_\beta) \times \left(c_{ij} - \mu\int_0^1\int_0^1 \sum_{l=0}^{n}\sum_{m=0}^{n} k_{ijlm}\,\rho_l(s)\,\rho_m(t) \times \sum_{p=0}^{n}\sum_{q=0}^{n} c_{pq}\,\rho_p(s)\,\rho_q(t)\,dsdt\right) = g(x_\alpha, y_\beta), \qquad (19)$$

Equations (19) form a linear system consisting of $(n+1)^2$ equations and $(n+1)^2$ unknowns c_{ij}. By solving this system, the coefficients c_{ij} can be determined. Once the c_{ij} are known, the approximation of $f(x,y)$ is obtained from relation (12).

4. Numerical solution of 2DFIEs using Müntz functions

In this section, three numerical examples are presented to illustrate the effectiveness and performance of the proposed method.

In examples where the integration domain is different, the integral equation can be mapped back to the interval $[-1,1]^2$ using simple linear transformations.

Example 4.1 *Consider the following integral equation:*

$$f(x,y) - \int_0^1\int_0^1 (x+y+s+t)f(s,t)\,ds\,dt = (4x-1)(5-4y+2\ln y) - y - x - \frac{3}{2}. \qquad (20)$$

The exact solution is

$$f(x,y) = (4x-1)(5-4y+2\ln y).$$

This example was solved using the proposed method with $n = 5$. The absolute error of the approximate solution is displayed in Figure 1. As illustrated in the figure, the numerical approximation coincides with the exact solution.

This example has also been solved using the collocation method based on Legendre functions, and the corresponding numerical results are reported in Table 1. Figure 2 illustrates the absolute error of the Legendre collocation method for $n = 15$. From the results, it is evident that for integral equations involving logarithmic

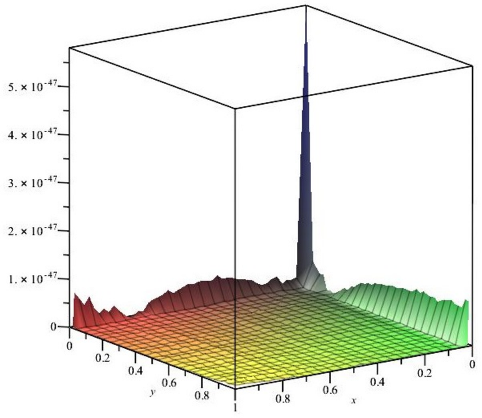

Figure 1. The absolute error of the approximate solution for $n = 5$

Table 1. Absolute error of the Legendre collocation method for Example 1

n	$e_n(f)$
5	2.5×10^{0}
10	7.0×10^{-1}
15	4.5×10^{-1}

functions, the Muntz collocation method yields a more accurate approximation.

Example 4.2 *Let us consider the integral equation*

$$f(x,y)-\int_0^1\int_0^1 (s+t)e^{x+y} f(s,t)\,ds\,dt = xy-\frac{1}{3}e^{x+y}, \tag{21}$$

where the exact solution is

$$f(x,y) = xy.$$

Example 2, similarly to Example 1, was solved using the proposed method with n = 5. Figure 3 illustrates the absolute error of the approximate solution. As observed, the approximate solution exhibits excellent agreement with the exact solution.

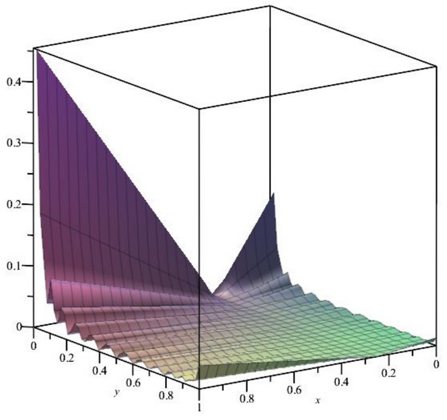

Figure 2. The absolute error of the Legendre collocation method for $n = 15$

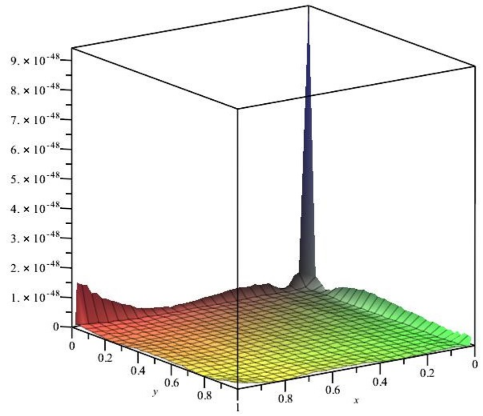

Figure 3. The absolute error of the approximate solution for $n = 5$

Table 2. Absolute and approximate absolute errors for Example 3

n	$e_n(f)$	$\widetilde{e}_n(f)$
5	6.3×10^{-3}	5.7×10^{-3}
10	2.6×10^{-5}	9.5×10^{-4}
20	8.7×10^{-7}	1.7×10^{-6}

Example 4.3 *Consider the equation*

$$\begin{aligned} f(x,y) - 6\int_0^1\int_0^1 \left(sx^2+ty^2\right) f(s,t)\,ds\,dt \\ = xe^y + (2-2e)x^2 - 3y^2. \end{aligned} \tag{22}$$

where the exact solution is

$$f(x,y) = xe^y.$$

.

In this example, as in the previous ones, the exact solution is not available. If the exact solution f is known, the relative error is computed by

$$e_n(f) = \|f-\widetilde{f}\|_\infty.$$

However, if the exact solution of the considered Fredholm integral equation is unknown, the relative error is estimated as

$$\widetilde{e}_n(f) = \|\widetilde{f}_{2n}-\widetilde{f}_n\|_\infty.$$

Figure 4 depicts the absolute error associated with the approximate solution, highlighting the accuracy of the proposed method.

5. Conclusion

n this study, we developed a Müntz function-based collocation method for the numerical solution of two-dimensional Fredholm integral equations of the second kind. The proposed approach effectively approximates both the unknown function and the kernel using Müntz orthogonal functions and determines the coefficients through a system of linear equations derived from

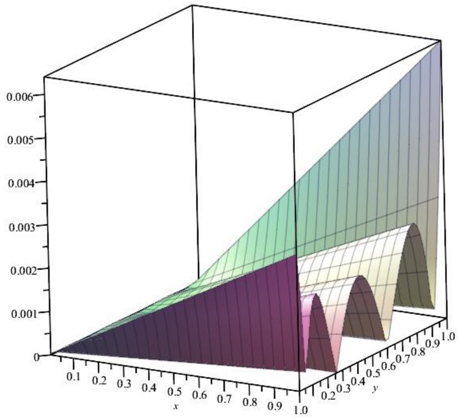

Figure 4. The absolute error of the approximate solution for $n = 5$

two-dimensional collocation points. Theoretical analysis confirms the convergence properties of the method in Sobolev norms, and numerical experiments demonstrate its high accuracy and computational efficiency. The method performs well for problems with both known and unknown exact solutions, highlighting its robustness and flexibility. Overall, the Müntz collocation approach provides a powerful tool for tackling complex 2DFIEs and has potential applications in various scientific and engineering fields.

Authors contributions

All the authors have participated sufficiently in the intellectual content, conception and design of this work or the analysis and interpretation of the data (when applicable), as well as the writing of the manuscript.

Availability of data and materials

The data that support the findings of this study are openly available through the text.

Conflict of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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