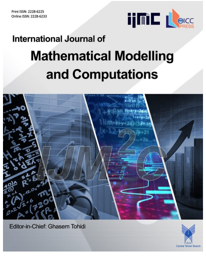

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An Efficient Numerical Method for Solving First Order Pantograph Equations via Shifted Müntz Orthogonal Functions

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Abstract:

Pantograph equations are considered as a special type of delay differential equations with proportional delay and have numerous applications. This paper introduces a collocation method for solving first-order pantograph equations using shifted Müntz (SM) orthogonal functions. Unlike classical polynomial bases, SM functions incorporate logarithmic terms and exhibit real, simple roots in the interval (0,1). Leveraging these roots as collocation points within a domain decomposition framework, we achieve high-precision solutions-particularly advantageous for inherently non-polynomial pantograph solutions. We derive rigorous error estimates, establish method stability, and demonstrate significant accuracy gains over existing techniques. Numerical experiments confirm the method's efficacy, underscoring the superior approximation capability of SM functions for pantograph-type problems. All computations in this study were performed using Maple 2021 software. The codes were executed on a PC equipped with an Intel® Core™ i5-10210U processor (1.60 GHz) and 8 GB DDR4 RAM running Windows 10.

Keywords: Pantograph equation, Shifted Müntz orthogonal functions, Collocation, Stability

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1. Introduction

Delay differential equations (DDEs) appear as mathematical models in a wide range of natural phenomena. Therefore, over the past decades, research and study on DDEs have extended and they have become an important tool for mathematicians and researchers in other disciplines. Pantograph equations are one of the most famous types of DDEs with proportional delays that have a significant impact in describing various phenomena [1–7].

The general form of the first-order pantograph equation is as follows [3]:

$$\begin{cases} \phi'(t) = a\,\phi(t) + \sum_{i=1}^{m} \alpha_i(t)\,\phi(\mu_i t) + \tau(t), t \in [0, L], \\ \phi(0) = y_0, \end{cases} \tag{1}$$

where a is a real constant, $\alpha_i(t)$ and $\tau(t)$ are continuous known functions and $0 < \mu_i \leq 1$. For the mathematical analysis and the numerical scheme, we assume that the functions $\alpha_i(t)$ and $\tau(t)$ are continuous on the interval $[0, L]$. Furthermore, for deriving the error estimates, the exact solution $\phi(t)$ is assumed to be sufficiently smooth, belonging to the Sobolev space $H^r[0, L)$ for some order $r \geq 1$. These equations have many applications in control theory, electrodynamics, astronomy, economics, biology, etc. (see [8–28] and references therein). Quantitative and qualitative studies of pantograph equations have been investigated by many researchers. Li and Liu [29] proved the existence and uniqueness of the pantograph's solution. Authors of [30] proposed a homotopy method to solve delay equations of pantograph type. In [31], the authors used a Legendre spectral method to obtain numerical approximations for pantograph equations. The author of [32] presented an algorithm for solving a pantograph equations by using the homotopy perturbation method. In [9], the authors solved these equations using Chebyshev polynomials. In [33], the collocation method based on the Bernoulli operational matrix has been utilized. In [34], numerical solution of the generalized pantograph equations using the Lagrange coefficients and least squares approximation method was presented. The authors of [35] used the Galerkin spectral method to solve pantograph equations. Two numerical methods using Müntz-Legendre polynomials have been proposed in [3] to solve pantograph equations.

In this paper, our goal is to present a novel numerical method for solving the first-order pantograph equation (1) using a special class of orthogonal Müntz functions. These functions have different structure compared to the Müntz-Legendre polynomials used in [3]. They are defined based on a combination of polynomials and the ln function whilst they have simple and real roots in the interval (0,1). We show that the current method has higher accuracy than the previous methods. We also give some error estimates and assess the numerical stability of the suggested method.

The rest of this paper is organized as follows. In the second section, we introduce the logarithmic Müntz-Legendre functions and review some of their properties. The third section is devoted to a novel collocation method for approximating the exact solution of the pantograph equation (1). In the fourth section, we investigate the stability of the suggested method. In the fifth section, some numerical examples are presented and the obtained results are compared with other methods in the literature. Finally, we give conclusions in the sixth section.

2. Müntz orthogonal functions and their properties

2.1 Müntz orthogonal functions

The general form of Müntz-Legendre polynomials is described in details in [36]. Here, we consider a special class of them that are a combination of polynomials and the ln function.

Definition 2.1 *The set of logarithmic Müntz functions, denoted by $Q_n(t), n = 0, 1, 2, ...$, are defined on the interval $(0, 1)$ as [36]*

$$Q_n(t) = R_n(t) + S_n(t)\ln(t), \quad (n = 0, 1, 2, \dots)$$

where $R_n(t)$ and $S_n(t)$ are algebraic polynomials of degree $\lfloor \frac{n}{2} \rfloor$ and $\lfloor \frac{n-1}{2} \rfloor$ respectively, i.e.,

$$R_n(t) = \sum_{w=0}^{\lfloor \frac{n}{2} \rfloor} a_w^{(n)} t^w, \quad S_n(t) = \sum_{w=0}^{\lfloor \frac{n-1}{2} \rfloor} b_w^{(n)} t^w. \tag{2}$$

Note that $Q_n(1) = R_n(1) = 1$. Further, distinct expression for the polynomial coefficients in (2) are given in the following Theorem [36].

Theorem 2.2 *Suppose n be an even number ($n = 2r$), then for $0 \le w < r$ we have*

$$a_w^{(2r)} = -\binom{r+w}{r}^2 \binom{r}{w}^2 \left(\frac{2r+1}{2w+1} + 2(r-w) \sum_{\substack{k=0 \\ k \ne w}}^{r-1} \frac{2k+1}{(k-w)(k+w+1)} \right),$$

$$b_w^{(2r)} = -(r-w)\binom{r+w}{r}^2 \binom{r}{w}^2$$

and for $w = r$ we have

$$a_r^{(2r)} = \binom{2r}{r}^2, \quad b_r^{(2r)} = 0.$$

Let n be an odd number ($n = 2r+1$), then for $0 \le w \le r$, we have

$$a_w^{(2r+1)} = \binom{r+w}{r}^2 \binom{r}{w}^2 \left(\frac{2r+1}{2w+1} + 2(r+w+1) \sum_{\substack{k=0 \\ k \ne w}}^{r} \frac{2k+1}{(k-w)(k+w+1)} \right),$$

$$b_w^{(2r+1)} = (r+w+1)\binom{r+w}{r}^2 \binom{r}{w}^2$$

Using the above Theorem, the first few logarithmic Müntz functions are obtained as follows:

$$\begin{aligned}
Q_0(t) &= 1, \\
Q_1(t) &= 1 + \ln(t), \\
Q_2(t) &= -3 + 4t - \ln(t), \\
Q_3(t) &= 9 - 8t + 2(1 + 6t)\ln(t), \\
Q_4(t) &= -11 - 24t + 36t^2 - 2(1 + 18t)\ln(t), \\
Q_5(t) &= 19 + 276t - 294t^2 + 3(1 + 48t + 60t^2)\ln(t).
\end{aligned}$$

Theorem 2.3 *The Müntz functions $Q_n(t)$, $n = 0, 1, 2, \dots$, are orthogonal on the interval $(0, 1)$ and $Q_n(t)$ has exactly n simple and real roots [36].*

2.2 Shifted Müntz functions

Consider the interval $\theta = [0, L]$. We divide it into K subintervals θ_j so that

$$\theta_j = [t_j, t_{j+1}], \quad t_j = j\rho, \quad \rho = \frac{L}{K}, \quad j = 0, 1, \dots, K-1.$$

Then the shifted Müntz (SM) functions on each subinterval θ_j are defined by

$$P_{\theta_j,n}(t) = Q_n\left(\frac{t - t_j}{t_{j+1} - t_j}\right), \quad j = 0, 1, \dots, K-1.$$

2.3 Function approximation

Let $\{P_{\theta_j,n}(t)\}_{n=0}^{N-1}$ be the set of SM functions in the interval $\theta_j = [t_j, t_{j+1}]$ and $Y_j = \text{span}\{P_{\theta_j,n}(t)\}_{n=0}^{N-1}$. Moreover, suppose that $\phi_j \in L^2[t_j, t_{j+1})$ and $I_N\phi_j \in Y_j$ be the best approximation to ϕ_j in the space Y_j, i.e.,

$$\forall y \in Y_j : \quad \|\phi_j - I_N\phi_j\| \le \|\phi_j - y\|.$$

Then, there exist the unique coefficients $C_{j,n}, j = 0, 1, \ldots, K-1, n = 0, 1, \ldots, N-1$, such that

$$\phi_j(t) \approx I_N\phi_j(t) = \sum_{n=0}^{N-1} C_{j,n} P_{\theta_j,n}(t), \tag{3}$$

and these Fourier coefficients are obtained by the formula

$$C_{j,n} = \frac{\int_{\theta_j} \phi_j(t)\, P_{\theta_j,n}(t)\, dt}{\int_{\theta_j} (P_{\theta_j,n}(t))^2\, dt}, \quad j = 0, 1, \ldots, K-1, \quad n = 0, 1, \ldots, N-1$$

Furthermore, if $\Phi(t)$ be an arbitrary element of the Banach space $L^2[0, L]$, the one has

$$\Phi(t) \approx \Phi_N(t) = \sum_{j=0}^{K-1}\sum_{n=0}^{N-1} C_{j,n} P_{\theta_j,n}(t).$$

2.4 Approximation error

Definition 2.4 *The Sobolev space $H^r[0, L)$ of integer order r, is defined by [37],*

$$H^r[0, L) = \left\{ V \in C^{r-1}([0, L]) : \frac{d}{dt} V^{(r-1)} \in L^2[0, L) \right\}.$$

Hence,

$$H^r[t_j, t_{j+1}) = \left\{ V \in C^{r-1}([t_j, t_{j+1}]) \; : \; \frac{d}{dt} V^{(r-1)} \in L^2[t_j, t_{j+1}) \right\}, \quad j = 0, 1, \ldots, K-1.$$

Theorem 2.5 *Suppose that $\Phi \in H^r[0, L)$ and $\phi_j \in H^r[t_j, t_{j+1})$, $j = 0, 1, \ldots, K-1$, with $r \ge 0$ so that $\Phi(t) = \sum_{j=0}^{K-1} \phi_j(t)$. Further, let $Y_j = \text{span}\{P_{\theta_j,n}(t)\}_{n=0}^{N-1}$, $j = 0, 1, \ldots, K-1$. If $I_N\phi_j(t) = \sum_{n=0}^{N-1} C_{j,n} P_{\theta_j,n}(t)$ be the best approximation to ϕ_j in Y_j, then if $r \le N+1$, $\hat{\Phi}(t) = \sum_{j=0}^{K-1} I_N\phi_j(t)$, approximates Φ with the following error bound:*

$$\|\Phi - \hat{\Phi}\|_{L^2[0,L)} \le \xi (NK)^{-r} \|\Phi^{(r)}\|_{L^2[0,L)},$$

and for $1 \le \varepsilon \le r$, we have

$$\|\Phi - \hat{\Phi}\|_{H^\varepsilon[0,L)} \le \xi (NK)^{2\varepsilon - \frac{1}{2} - r} \|\Phi^{(r)}\|_{L^2[0,L)}, \tag{4}$$

where ξ is a constant that depends only on r.

Proof. Let $L_N\phi_j$ be the truncated Legendre series for the function ϕ_j and $\theta_j = [t_j, t_{j+1}] = [\frac{jL}{K}, \frac{(j+1)L}{K}]$. According to equation (5.4.11) in [37], for $r \le N+1$ we have

$$\|\phi_j - L_N\phi_j\|^2_{L^2(\theta_j)} \le \xi N^{-2r} K^{-2r} \|\phi_j^{(r)}\|^2_{L^2(\theta_j)}.$$

As $L_N\phi_j$ is the best approximation to ϕ_j in the L^2-norm, [37], we can deduce that

$$\begin{aligned}\|\phi_j - I_N\phi_j\|^2_{L^2(\theta_j)} &= \|\phi_j - L_N\phi_j\|^2_{L^2(\theta_j)} \\ &\le \xi (NK)^{-2r} \|\phi_j^{(r)}\|^2_{L^2(\theta_j)}.\end{aligned}$$

Consequently,

$$\begin{aligned}\|\Phi - \hat{\Phi}\|^2_{L^2[0,L)} &= \sum_{j=0}^{K-1} \|\phi_j - I_N\phi_j\|^2_{L^2(\theta_j)} \\ &\le \xi (NK)^{-2r} \|\Phi^{(r)}\|^2_{L^2[0,L)}.\end{aligned}$$

as desired. Equation (4) is proved using equation (5.5.11) in [37] in a similar manner.

Theorem 2.6 *Let $\Phi(t)$ and $\Phi_N(t)$ are the exact and approximate solutions of equation (1), respectively. Then, the residual error E_N for equation (1) has the following bound:*

$$\|E_N\|_{L^2[0,L)} \le \eta \|\Phi^{(r)}\|_{L^2[0,L)}.$$

Where,

$$\eta = \xi \left((NK)^{2\varepsilon - \frac{1}{2}} + |a| + \sum_{i=1}^{m} \|\alpha_i(t)\| \right) (NK)^{-r}.$$

Proof. According to equation (1), we have

$$\begin{aligned}\|E_N\|_{L^2[0,L)} &= \Big\|\Phi'(t) - a\Phi(t) - \sum_{i=1}^{m} \alpha_i(t)\Phi(\mu_i t) - \tau(t) - \Phi'_N(t) \\ &\quad - a\Phi_N(t) + \sum_{i=1}^{m} \alpha_i(t)\Phi_N(\mu_i t) + \tau(t)\Big\|_{L^2[0,L)} \\ &= \Big\|\Phi'(t) - \Phi'_N(t) - a(\Phi(t) - \Phi_N(t)) \\ &\quad - \sum_{i=1}^{m} \alpha_i(t)(\Phi(\mu_i t) - \Phi_N(\mu_i t))\Big\|_{L^2[0,L)}.\end{aligned}$$

Hence,

$$\begin{aligned}\|E_N\|_{L^2[0,L)} &\le \|\Phi'(t) - \Phi'_N(t)\|_{L^2[0,L)} \\ &\quad + |a| \|\Phi(t) - \Phi_N(t)\|_{L^2[0,L)} \\ &\quad + \sum_{i=1}^{m} \|\alpha_i(t)\| \; \|\Phi(\mu_i t) - \Phi_N(\mu_i t)\|_{L^2[0,L)} \\ &\le \|\Phi(t) - \Phi_N(t)\|_{H^\varepsilon[0,L)} \\ &\quad + |a| \|\Phi(t) - \Phi_N(t)\|_{L^2[0,L)} \\ &\quad + \sum_{i=1}^{m} \|\alpha_i(t)\| \; \|\Phi(\mu_i t) - \Phi_N(\mu_i t)\|_{L^2[0,L)}.\end{aligned}$$

According to Theorem 2.5, we have

$$\begin{aligned}\|E_N\|_{L^2[0,L)} &\le \xi (NK)^{2\varepsilon - \frac{1}{2} - r} \|\Phi^{(r)}\|_{L^2[0,L)} \\ &\quad + |a|\, \xi (NK)^{-r} \|\Phi^{(r)}\|_{L^2[0,L)} \\ &\quad + \sum_{i=1}^{m} \|\alpha_i(t)\|\, \xi (NK)^{-r} \|\Phi^{(r)}\|_{L^2[0,L)} \\ &\le \xi \Big((NK)^{2\varepsilon - \frac{1}{2}} + |a| + \sum_{i=1}^{m} \|\alpha_i(t)\| \Big) \\ &\quad \times (NK)^{-r} \|\Phi^{(r)}\|_{L^2[0,L)}.\end{aligned}$$

The proof ends.

3. Solving the first-order pantograph equations

In this section, we derive a collocation method based on the SM functions for the numerical solution of the pantograph equation (1). To this end, we first divide the original interval $\theta = [0, L]$ into K subintervals θ_j so that

$$\theta_j = [t_j, t_{j+1}],\ t_j = j\rho,\ \rho = \frac{L}{K},\ j = 0, 1, \dots, K-1,$$

and consider the solution of equation (1) as a piecewise function,

$$\Phi(t) = \begin{cases} \phi_0(t), & t \in \theta_0, \\ \phi_1(t), & t \in \theta_1, \\ \vdots & \\ \phi_j(t), & t \in \theta_j, \\ \vdots & \\ \phi_{K-1}(t), & t \in \theta_{K-1}. \end{cases}$$

If $\phi_0(t)$ be the solution of equation (1) in the interval $\theta_0 = [0, \rho]$, then

$$t \in \theta_0 \ \Rightarrow\ 0 < \mu_i t \le \mu_i \rho \le \rho \ \Rightarrow\ \mu_i t \in \theta_0.$$

And in this subinterval, equation (1) is replaced with

$$\begin{cases} \phi_0'(t) - a\,\phi_0(t) - \sum_{i=1}^{m} \alpha_i(t)\,\phi_0(\mu_i t) - \tau(t) = 0, \\ \phi_0(0) = \phi_{-1}(0) = y_0. \end{cases} \tag{5}$$

To approximate the solution of equation (5), we first approximate $\phi_0'(t)$ using equation (3) as

$$\phi_0'(t) = \sum_{n=0}^{N-1} C_{0,n}\, P_{\theta_0,n}(t). \tag{6}$$

Integrating equation (6) from 0 to t, yields

$$\phi_0(t) = \int_0^t \phi_0'(x)\,dx + \phi_{-1}(0), \qquad t \in \theta_0, \tag{7}$$

and using equations (6) and (7), we get

$$\phi_0(t) = \int_0^t \sum_{n=0}^{N-1} C_{0,n} P_{\theta_0,n}(x)dx + y_0. \tag{8}$$

Now, by substituting equations (6)-(8) into equation (5), we obtain

$$\begin{aligned} &\sum_{n=0}^{N-1} C_{0,n} P_{\theta_0,n}(t) \\ &- a\left(\int_0^t \sum_{n=0}^{N-1} C_{0,n} P_{\theta_0,n}(x)\,dx + y_0\right) \\ &- \sum_{i=1}^{m} \alpha_i(t)\left(\int_0^t \sum_{n=0}^{N-1} C_{0,n} P_{\theta_0,n}(x)\,dx + y_0\right) \\ &- \tau(t) = 0. \end{aligned} \tag{9}$$

Accordingly, if $\phi_j(t)$ be the solution of equation (1) in the interval $\theta_j = [j\rho, (j+1)\rho]$, then

$$\begin{aligned} t \in \theta_j &\Rightarrow j\rho\mu_i < \mu_i t \le (j+1)\rho\mu_i \le (j+1)\rho \\ &\Rightarrow \mu_i t \in \bigcup_{k=0}^{j} \theta_k. \end{aligned}$$

And the problem is replaced with

$$\begin{cases} \phi_j'(t) - a\phi_j(t) - \sum_{i=1}^{m} \alpha_i(t)\phi_{s_j}(\mu_i t) - \tau(t) = 0, \\ \phi_j(j\rho) = \phi_{j-1}(j\rho). \end{cases} \tag{10}$$

And the index $s_j \in \{0, 1, \dots, j\}$ is such that $\mu_i t \in \theta_{s_j}$. Similarly to equations (6)-(8), we utilize equation (3) to write

$$\phi_j'(t) = \sum_{n=0}^{N-1} C_{j,n} P_{\theta_j,n}(t), \tag{11}$$

$$\phi_j(t) = \int_{j\rho}^t \phi_j'(x)\,dx + \phi_{j-1}(j\rho), \quad t \in \theta_j, \tag{12}$$

$$\phi_j(t) = \int_{j\rho}^t \sum_{n=0}^{N-1} C_{j,n} P_{\theta_j,n}(x)\,dx + \phi_{j-1}(j\rho). \tag{13}$$

Further, by substituting equations (11)-(13) into (10), we get

$$\begin{aligned} &\sum_{n=0}^{N-1} C_{j,n} P_{\theta_j,n}(t) \\ &- a\left(\int_{j\rho}^t \sum_{n=0}^{N-1} C_{j,n} P_{\theta_j,n}(x)\,dx + \phi_{j-1}(j\rho)\right) \\ &- \sum_{i=1}^{m} \alpha_i(t)\left(\int_{s_j\rho}^t \sum_{n=0}^{N-1} C_{s_j,n} P_{\theta_{s_j},n}(x)\,dx + \phi_{s_j-1}(s_j\rho)\right) \\ &- \tau(t) = 0. \end{aligned} \tag{14}$$

In addition, if $\phi_{K-1}(t)$ be the solution of the equation (1) in the interval
$\theta_{K-1} = [(K-1)\rho, L]$, then

$$\begin{aligned} t \in \theta_{K-1} &\Rightarrow \mu_i(K-1)\rho < \mu_i t \le \mu_i L \le L \\ &\Rightarrow \mu_i t \in \bigcup_{k=0}^{K-1} \theta_k. \end{aligned}$$

And the reduced problem is

$$\begin{cases} \phi_{K-1}'(t) - a\phi_{K-1}(t) - \sum_{i=1}^{m} \alpha_i(t)\phi_{s_K}(\mu_i t) - \tau(t) = 0, \\ \phi_{K-1}((K-1)\rho) = \phi_{K-2}((K-1)\rho). \end{cases} \tag{15}$$

Where the index $s_K \in \{0, 1, \dots, K-1\}$ is such that $\mu_i t \in \theta_{s_K}$. Again, we use equation (3) to write

$$\phi_{K-1}'(t) = \sum_{n=0}^{N-1} C_{K-1,n} P_{\theta_{K-1},n}(t). \tag{16}$$

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By integrating equation (16) over $((K-1)\rho, t)$ we arrive at

$$\phi_{K-1}(t) = \int_{(K-1)\rho}^{t} \phi'_{K-1}(x)\, dx + \phi_{K-2}((K-1)\rho), \quad t \in \theta_{K-1}. \tag{17}$$

Equations (16)-(17), result

$$\phi_{K-1}(t) = \int_{(K-1)\rho}^{t} \sum_{n=0}^{N-1} C_{K-1,n}\, P_{\theta_{K-1},n}(x)\, dx + \phi_{K-2}(K-1)\rho, \quad t \in \theta_{K-1}. \tag{18}$$

substituting equations (16)-(18) into (15), yields

$$\sum_{n=0}^{N-1} C_{K-1,n} P_{\theta_{K-1},n}(t) - a \times \left(\int_{(K-1)\rho}^{t} \sum_{n=0}^{N-1} C_{K-1,n} P_{\theta_{K-1},n}(x)\, dx + \phi_{K-2}((K-1)\rho) \right) - \sum_{i=1}^{m} \alpha_i(t) \left(\int_{s_K\rho}^{t} \sum_{n=0}^{N-1} C_{s_K,n} P_{\theta_{s_K},n}(x)\, dx + \phi_{s_K-1}(s_K\rho) \right) - \tau(t) = 0. \tag{19}$$

Finally, let $S_r, r = 1, \ldots, N$ be the roots of the Müntz function $Q_N(t)$ on the interval $(0,1)$. By considering $\gamma_{r_0} = S_r\rho + t_0, \quad \ldots, \quad \gamma_{r_j} = S_r\rho + t_j$ and $\gamma_{r_{K-1}} = S_r\rho + t_{K-1}$ as the collocation points in the subintervals $\theta_0, \theta_j (1 \le j \le K-2)$ and θ_{K-1} and collocating equations (9), (14) and (19) at these points, we obtain the following collocation conditions:

$$\begin{cases} \sum_{n=0}^{N-1} C_{0,n} P_{\theta_0,n}(\gamma_{r_0}) - a\left(\int_0^{\gamma_{r_0}} \sum_{n=0}^{N-1} C_{0,n} P_{\theta_0,n}(x)\, dx + y_0 \right) \\ \qquad - \sum_{i=1}^{m} \alpha_i(\gamma_{r_0}) \left(\int_0^{\gamma_{r_0}} \sum_{n=0}^{N-1} C_{0,n} P_{\theta_0,n}(x)\, dx + y_0 \right) \\ \qquad - \tau(\gamma_{r_0}) = 0, \\ \sum_{n=0}^{N-1} C_{j,n} P_{\theta_j,n}(\gamma_{r_j}) - a\left(\int_{j\rho}^{\gamma_{r_j}} \sum_{n=0}^{N-1} C_{j,n} P_{\theta_j,n}(x)\, dx + \phi_{j-1}(j\rho) \right) \\ \qquad - \sum_{i=1}^{m} \alpha_i(\gamma_{r_j}) \left(\int_{s_j\rho}^{\gamma_{r_j}} \sum_{n=0}^{N-1} C_{s_j,n} P_{\theta_{s_j},n}(x)\, dx + \phi_{s_j-1}(s_j\rho) \right) \\ \qquad - \tau(\gamma_{r_j}) = 0, \\ \qquad \vdots \\ \sum_{n=0}^{N-1} C_{K-1,n} P_{\theta_{K-1},n}(\gamma_{r_{K-1}}) - a\left(\int_{(K-1)\rho}^{\gamma_{r_{K-1}}} \sum_{n=0}^{N-1} C_{K-1,n} P_{\theta_{K-1},n}(x)\, dx + \phi_{K-2}((K-1)\rho) \right) \\ \qquad - \sum_{i=1}^{m} \alpha_i(\gamma_{r_{K-1}}) \left(\int_{s_K\rho}^{\gamma_{r_{K-1}}} \sum_{n=0}^{N-1} C_{s_K,n} P_{\theta_{s_K},n}(x)\, dx + \phi_{s_K-1}(s_K\rho) \right) \\ \qquad - \tau(\gamma_{r_{K-1}}) = 0, \quad r = 1, \ldots, N. \end{cases} \tag{20}$$

By solving the system of algebraic equations (20), the unknown coefficients $C_{j,n}$, $j = 0, 1, ..., K-1, \quad n = 0, 1, ..., N-1$, in equations (8), (13) and (18) are obtained.

4. Stability analysis

In this section, we investigate the RN-stability of the proposed method for first-order pantograph equations. The concept of RN-stability means that the propagation of the initial error can be controlled in long time calculations. Things start with the following Definition [3, 38].

Definition 4.1 *A numerical method for the pantograph equation is called RN-stable, if the approximate solutions $\phi_N(t)$ and $\psi_N(t)$, with the initial values $\phi_N(0) = \phi_0$ and $\psi_N(0) = \psi_0$, satisfy the following condition:*

$$|\phi_N((j+1)\rho) - \psi_N((j+1)\rho)| \le |\phi_0 - \psi_0|,$$

for each subinterval $\theta_j = [t_j, t_{j+1}], \quad j = 0, 1, \ldots, K-1$, with the length ρ.

Theorem 4.2 *According to the assumptions of Definition 4.1, the proposed SM collocation method for solving the pantograph equation is RN-stable.*

Proof. For simplicity of statement, we consider the pantograph equation as

$$\begin{cases} \phi'(t) = G(t, \phi(t), \phi(\mu t)), & t \in [0, L] \\ \phi(0) = \phi_0, \end{cases} \tag{21}$$

with the following condition

$$\big(G(t, \phi(t), \phi(\mu t)) - G(t, \psi(t), \psi(\mu t))\big) \cdot (\phi - \psi) \le 0. \tag{22}$$

Suppose $\phi_{N,j}(t)$ and $\psi_{N,j}(t)$ be the numerical solutions of equation (21) on the interval θ_j corresponding to the initial values $\phi_{N,0}(t) = \phi_0$ and $\psi_{N,0}(t) = \psi_0$. Also, let $\Delta_{N,j}(t) = \phi_{N,j}(t) - \psi_{N,j}(t)$. By substituting $\phi_{N,j}$ and $\psi_{N,j}$ into equation (21) and subtracting the results, we arrive at

$$\begin{cases} \Delta'_{N,j}(\gamma_{r_j}) = G(\gamma_{r_j}, \phi_{N,j}(\gamma_{r_j}), \phi_{N,s}(\mu\gamma_{r_j})) - \\ G(\gamma_{r_j}, \psi_{N,j}(\gamma_{r_j}), \psi_{N,s}(\mu\gamma_{r_j})), \\ \Delta_{N,j}(j\rho) = \Delta_{N,j-1}(j\rho), \\ \quad r = 1, \ldots, N, \quad j = 0, 1, \ldots, K-1, \end{cases} \tag{23}$$

where, γ_{r_j} are the collocation points in θ_j, and $\Delta_{N,-1}(0) = \phi_0 - \psi_0$, and the index s is such that $\mu\gamma_{r_j} \in \theta_s$.

Suppose that $\langle \cdot, \cdot \rangle_{L^2(\theta_j)}$ and $\langle \cdot, \cdot \rangle_{\theta_j,N}$ denote the continuous and discrete inner products on θ_j, respectively.

Multiplying both sides of equation (23) by $\Delta_{N,j}(\gamma_{r_j})$ and summing the resulting expressions for $r = 1, \ldots, N$, and taking into account the inequality (22), we obtain

$$\begin{aligned} \langle \Delta'_{N,j}, \Delta_{N,j} \rangle_{\theta_j,N} &= \langle G(\cdot, \phi_{N,j}, \phi_{N,s}) \\ &\quad - G(\cdot, \psi_{N,j}, \psi_{N,s}), \Delta_{N,j} \rangle_{\theta_j,N} \\ &\le 0. \end{aligned} \tag{24}$$

Now, we can write

$$\begin{aligned} (\Delta'_{N,j}, \Delta_{N,j})(t) := &\sum_{i_1=0}^{2\lfloor N/2 \rfloor - 1} a_{i_1 j} t^{i_1} \\ &+ \left(\sum_{i_2=0}^{\lfloor N/2 \rfloor + \lfloor (N-1)/2 \rfloor - 1} a_{i_2 j} t^{i_2} \right) \ln t \\ &+ \left(\sum_{i_3=0}^{2\lfloor (N-1)/2 \rfloor - 1} a_{i_3 j} t^{i_3} \right) (\ln t)^2. \end{aligned}$$

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We also have

$$\int_{\theta_j} t^{i_1}\, dt \le d_{i_1 j} \sum_{r=0}^{i_1} C_{rj}^{(1)}\, \gamma_{r_j}^{i_1},$$
$$\int_{\theta_j} t^{i_2} \ln t\, dt \le d_{i_2 j} \sum_{r=0}^{i_2} C_{rj}^{(2)}\, \gamma_{r_j}^{i_2} \ln \gamma_{r_j},$$
$$\int_{\theta_j} t^{i_3} (\ln t)^2\, dt \le d_{i_3 j} \sum_{r=0}^{i_3} C_{rj}^{(3)}\, \gamma_{r_j}^{i_3} (\ln \gamma_{r_j})^2,$$

where $di_1 j$, $di_2 j$ and $di_3 j$ are positive constants. Therefore, there exists the positive constant $M_{N,j}$ such that

$$\left\langle \Delta_{N,j}^{'}, \Delta_{N,j} \right\rangle_{L^2(\theta_j)} \le M_{N,j} \left\langle \Delta_{N,j}^{'}, \Delta_{N,j} \right\rangle_{\theta_{j,N}}. \quad (25)$$

On the other hand, the following relation holds true:

$$(\Delta_N((j+1)\rho))^2 - (\Delta_N(j\rho))^2 = 2\left\langle \Delta_{N,j}^{'}, \Delta_{N,j} \right\rangle_{L^2(\theta_j)}. \quad (26)$$

Hence, combination of (24), (25), and (26) yields

$$(\Delta_N((j+1)\rho))^2 - (\Delta_N(j\rho))^2 \le 0.$$

As a result,

$$|\Delta_N((j+1)\rho)| \le |\Delta_N(j\rho)|$$
$$\Rightarrow \quad |\phi_N((j+1)\rho) - \psi_N((j+1)\rho)| \le |\phi_N(j\rho) - \psi_N(j\rho)|,$$
$$j = 0, 1, \dots, K-1.$$

which leads to

$$|\phi_N((j+1)\rho) - \psi_N((j+1)\rho)| \le |\phi_0 - \psi_0|.$$

This proves the RN-stability of the proposed method introduced in the previous section.

5. Examples

In this section, five examples are given to demonstrate the stability and accuracy of the suggested method.

Example 5.1 *First let us consider the following problem adopted from [3]*

$$\begin{cases} \phi'(t) = 1 - 2(\phi(0.5t))^2, & t \in [0,1] \\ \phi(0) = 0. \end{cases}$$

The exact solution to this problem is $\phi(t) = \sin(t)$. Figure 1 plots the exact and approximate solutions for this example. In Table 1, the absolute errors of the suggested method are compared with those in[3] for different values of t and N. It is observed that, using the present method more accurate numerical results are obtained.

Example 5.2 *Consider the following equation [34]*

$$\begin{cases} \phi'(t) = \frac{1}{2} e^{\frac{t}{2}} \phi\left(\frac{t}{2}\right) + \frac{1}{2}\phi(t), & t \in [0,1] \\ \phi(0) = 1, \end{cases}$$

with the exact solution $\phi(t) = e^t$.

As we can see in Table 2, a comparison between the present method and the methods in [9, 34, 39–42] has been made, which demonstrate the superiority of the present method.

Example 5.3 *Next, we consider the pantograph equation [34]*

$$\begin{cases} \phi'(t) = -\phi(t) + \frac{1}{10}\phi\left(\frac{t}{5}\right) - \frac{1}{10} e^{-\frac{t}{5}}, & t \in [0,1] \\ \phi(0) = 1, \end{cases}$$

with the exact solution $\phi(t) = e^{-t}$.

This problem has also been solved in [24, 34, 39]. Figure 2 illustrates the absolute errors for Example 5.3. Comparison of the numerical results is made in Table 3.

Example 5.4 *To demonstrate the applicability and efficiency of the present method for problems defined in larger intervals, we consider the following problem adopted from [9]:*

$$\begin{cases} \phi'(t) = -\phi(t) + 0.1\phi(0.8t) + 0.5\phi'(0.8t) \\ \qquad +(0.32t - 0.5)e^{-0.8t} + e^{-t}, & t \in [0,10] \\ \phi(0) = 0. \end{cases}$$

The exact solution to this problem is $\phi(t) = te^{-t}$. In Table 4, the numerical results of the present method are compared with the method in [9].

Example 5.5 *Finally consider the following equation [9]*

$$\begin{cases} \phi'(t) = -\frac{5}{4} e^{-\frac{t}{4}} \phi\left(\frac{4t}{5}\right), & t \in [0,1] \\ \phi(0) = 1, \end{cases}$$

with the exact solution $\phi(t) = e^{-1.25t}$. In Table 5, the exact and approximate solutions for different values of t are compared with the methods in [9] and [43].

6. Conclusion

In this article, a type of the first-order pantograph equations has been solved numerically via the SM orthogonal functions and a multi-domain collocation method. It was shown that, in the domain decomposition strategy, higher accuracy can be achieved with a smaller number of collocation points in subintervals. Moreover, Since the exact solution to many pantograph equations is not a polynomial, the behavior of their solutions cannot be analyzed using classical polynomials such as Legendre, Chebyshev, etc. The advantage of solving the problem using SM orthogonal functions lies in the fact that these functions are not polynomials and, in many cases, can approximate the solutions of pantograph equations more effectively than classical polynomials. Furthermore, considering the SM functions as the trial functions in the collocation method and their roots of as the collocation points, lead to more accurate numerical results compared to some other methods in the literature. In addition,

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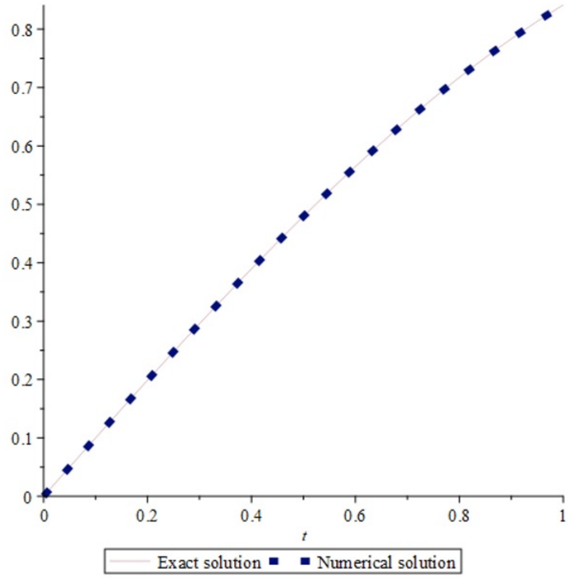

Figure 1. Exact and approximate solutions for N=20 and K=10 for Example 5.1

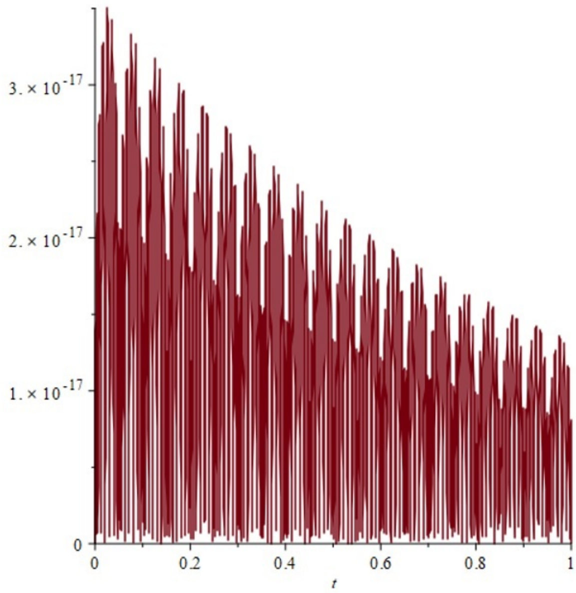

Figure 2. Absolute errors for N=10 of Example 5.3

the numerical results of Section 5 are in agreement with the theoretical results given in Sections 2 and 4. Given the high efficacy of the proposed method in solving first-order pantograph equations, extending this approach to higher-order pantograph equations is suggested as a future research direction.

Authors contributions
All the authors have participated sufficiently in the intellectual content, conception and design of this work or the analysis and interpretation of the data (when applicable), as well as the writing of the manuscript.
Availability of data and materials
The data that support the findings of this study are available from the corresponding author, upon reasonable request.

Conflict of interests
The author states that there is no conflict of interest.

Table 1. Numerical results of Example 5.1 for $K = 10$. (Note: $e\text{-}c \equiv 10^{-c}$)

t	N=10			N=20			N=30		
	ML [3]	SML [3] $h = 0.1$	Present method	ML [3]	SML [3] $h = 0.1$	Present method	ML [3]	SML [3] $h = 0.1$	Present method
0.2	6.1×10^{-8}	8.0×10^{-14}	1.5×10^{-19}	8.9×10^{-12}	7.5×10^{-16}	2.2×10^{-36}	3.2×10^{-14}	4.4×10^{-18}	2.0×10^{-50}
0.4	3.4×10^{-8}	2.5×10^{-14}	3.0×10^{-19}	6.0×10^{-12}	1.8×10^{-16}	2.0×10^{-36}	4.6×10^{-14}	9.2×10^{-18}	7.0×10^{-50}
0.6	6.9×10^{-7}	2.2×10^{-14}	2.4×10^{-19}	1.5×10^{-12}	4.0×10^{-16}	2.0×10^{-35}	7.4×10^{-14}	3.9×10^{-17}	7.0×10^{-50}
0.8	1.6×10^{-8}	3.1×10^{-14}	4.3×10^{-19}	9.4×10^{-12}	6.0×10^{-16}	6.2×10^{-36}	3.0×10^{-14}	5.5×10^{-17}	3.0×10^{-50}
1.0	4.7×10^{-8}	7.2×10^{-15}	1.5×10^{-19}	5.2×10^{-13}	1.0×10^{-16}	5.1×10^{-35}	2.1×10^{-14}	7.3×10^{-18}	3.7×10^{-49}

Table 2. Comparison of the absolute errors for Example 5.2

t	N=9				N=10				
	TPM[39]	CPM[9]	[34]	Present method	BCM[42]	ECM[41]	[34]	ADM[40]	Present method
0.2	7.05×10^{-15}	1.02×10^{-12}	1.09×10^{-12}	9.6×10^{-19}	1.03×10^{-13}	4.02×10^{-5}	1.07×10^{-14}	0	3.5×10^{-21}
0.4	1.06×10^{-11}	7.56×10^{-13}	4.22×10^{-13}	2.3×10^{-18}	1.29×10^{-13}	4.98×10^{-5}	2.54×10^{-14}	2.22×10^{-16}	8.5×10^{-21}
0.6	2.94×10^{-10}	7.00×10^{-13}	3.43×10^{-13}	4.1×10^{-18}	1.58×10^{-13}	6.08×10^{-5}	2.59×10^{-14}	2.22×10^{-16}	1.5×10^{-20}
0.8	3.86×10^{-9}	6.37×10^{-13}	1.03×10^{-12}	6.5×10^{-18}	1.81×10^{-13}	7.42×10^{-5}	1.20×10^{-14}	1.33×10^{-15}	2.4×10^{-20}
1.0	2.90×10^{-8}	3.87×10^{-13}	1.22×10^{-15}	9.8×10^{-18}	4.23×10^{-12}	7.54×10^{-5}	2.50×10^{-15}	4.88×10^{-15}	3.6×10^{-20}

Table 3. Comparison of the absolute errors for Example 5.3

t	**TPM[39]**		**VIM[24]**		**[34]**			**Present method**		
	$N = 10$	$N = 12$	$m = 3$	$m = 4$	$N = 10$	$N = 12$	$N = 14$	$N = 10$	$N = 12$	$N = 14$
0.5	2.0×10^{-10}	1.24×10^{-10}	1.47×10^{-10}	2.40×10^{-15}	3.38×10^{-16}	6.54×10^{-20}	1.12×10^{-23}	4.5×10^{-21}	1.1×10^{-24}	3.1×10^{-28}
0.25	1.0×10^{-10}	9.74×10^{-11}	9.79×10^{-12}	7.91×10^{-17}	7.35×10^{-15}	9.21×10^{-19}	1.12×10^{-21}	3.0×10^{-21}	7.1×10^{-25}	1.9×10^{-28}
0.125	1.0×10^{-10}	7.00×10^{-11}	6.31×10^{-13}	2.53×10^{-18}	4.85×10^{-15}	3.19×10^{-18}	3.13×10^{-22}	3.0×10^{-17}	1.2×10^{-20}	5.9×10^{-25}
0.0625	1.0×10^{-10}	9.14×10^{-11}	4.00×10^{-14}	8.03×10^{-20}	7.54×10^{-15}	1.91×10^{-18}	2.76×10^{-22}	4.6×10^{-19}	1.7×10^{-22}	5.9×10^{-26}
0.03125	1.0×10^{-10}	5.28×10^{-11}	2.52×10^{-15}	2.52×10^{-21}	6.24×10^{-16}	1.45×10^{-18}	7.30×10^{-22}	7.3×10^{-18}	1.6×10^{-20}	2.9×10^{-24}
0.015625	0	1.95×10^{-11}	1.58×10^{-12}	7.00×10^{-23}	5.10×10^{-15}	1.37×10^{-18}	1.22×10^{-22}	3.0×10^{-17}	1.5×10^{-20}	6.1×10^{-24}

Table 4. Comparison of the absolute errors for Example 5.4

t	[9]			Present method		
	$m = 4$	$m = 8$	$m = 16$	$N = 4$	$N = 8$	$N = 16$
2	1.63e-1	3.30e-3	5.24e-9	6.85e-5	3.22e-8	3.85e-15
4	0.14e-1	3.30e-3	3.38e-9	4.37e-6	1.85e-8	1.54e-15
6	1.79e-1	2.40e-3	3.12e-9	1.74e-5	5.91e-10	1.36e-16
8	0.01e-1	1.01e-3	8.41e-9	6.19e-6	1.23e-9	2.32e-16
10	0.76e-1	0.89e-3	5.43e-9	3.98e-6	5.35e-10	7.24e-17

Table 5. Comparison between the exact and approximate solution for Example 5.5

t	Exact solution	[43]	[9] $m = 8$	Present method $N = 8$	[9] $m = 10$	Present method $N = 10$
0.000	1.000000000000000	1.0000000	1.0000000000	1.000000000000	1.000000000000	1.000000000000000
0.125	0.8553453273074225	0.8553451	0.8553532720	0.855345327307	0.8553453273074	0.855345327307422
0.250	0.7316156289466417	0.7316117	0.7316156290	0.731615628944	0.7316156289464	0.731615628946636
0.375	0.6257840096045911	0.6257781	0.6257840094	0.625784009606	0.6257840096045	0.625784009604594
0.500	0.5352614285189902	0.5352547	0.5352614285	0.535261428518	0.5352614285189	0.535261428518990
0.625	0.4578333617716142	0.4578240	0.4578333619	0.457833361771	0.4578333617715	0.457833361771614
0.750	0.3916056266767989	0.3915973	0.3916056265	0.391605626675	0.3916056266768	0.391605626676795
0.875	0.3349580429252949	0.3349489	0.3349580429	0.334958042926	0.3349580429252	0.334958042925296
1.000	0.2865047968601901	0.2864956	0.2865047968	0.286504796860	0.2865047968601	0.286504796860190

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