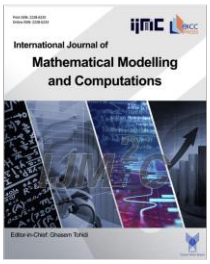

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Defining Fundamental Computational Speed Limits for Quantum Oscillators

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Abstract

This work shows the different effects external magnetic and electric fields have on the computational efficiency of the system by examining the complexity growth rate of a charged quantum oscillator under these conditions. Our work reveals a critical magnetic field threshold above which the complexity behavior qualistically changes and the computational dynamics of the system drastically changes. Moreover, by means of an analysis of the minimum orthogonalization time in an anharmonic oscillator, we derive upper bounds on feasible computation rates, so revealing the practical limitations of quantum computational speed. By identifying specific parameter regimes where complexity development undergoes clear transitions, the study reveals how the interaction of electric and magnetic fields can greatly alter the behavior of quantum systems. These results clarify how external perturbations affect the computational capacity of quantum systems and help us to better grasp the limits of quantum computing. With significant consequences for the design and development of upcoming quantum technologies, this study finally helps us to better understand how outside fields restrict the efficiency of quantum computations.

Keywords: Quantum information, Complexity, Lloyd's bound, Orthogonal states, Quantum computation.

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1. Introduction

Is the maximum speed at which a computer can process information? And are there inherent limits on the amount of data that can be stored? These important and provocative questions still need to be thoroughly explored, especially at the nexus of theoretical physics and computer science. The problems of information processing speed and computer memory capacity—that is, the constraints associated with computing speed and storage—are complex and difficult.

Interestingly, these constraints can be characterized separately according to the available energy and degrees of freedom of the system: in the context of a particular quantum mechanical system, the computational speed limit is related to the number of different states that the system can change through in a given time, and since these states are orthogonal, the problem of establishing speed limits can be solved with the help of quantum mechanical principles.

However, black holes are not completely "black," according to quantum mechanics [1].

When conceptualized as information storage or computational entities containing a finite, well-defined amount of energy, black holes possess the capacity to execute a quantifiable number of operations per unit time [2].

Moreover, there exists a fundamental theoretical upper bound on information processing rates. Specifically, for a quantum system characterized by an average energy E, the maximal number of operations that can be performed within one second is constrained by 2E/πħ. This upper limit is formally recognized as Lloyd's bound [3].

Quantum information theory is indeed a fundamental aspect of quantum mechanics and plays a vital role in the advancement of computer science. Interestingly, a concrete connection between black hole physics and quantum information theory has been proposed. Specifically, in the study of the "physics of black holes", quantum information theory may hold significant importance. For example, Bekenstein [4] argued that black holes impose a theoretical upper limit on information storage, indicating that memory is inherently bounded. Therefore, examining quantum information from a theoretical perspective becomes not only an intriguing endeavor but potentially a critical one. In this context, quantities such as entanglement entropy[5][6][7] and complexity emerge as key factors (for further discussion, see Refs[8][9]).

Notably, black holes can, in principle, define fundamental limits on parameters such as information density, entropy, and computational complexity [10][11][12][13]. Furthermore, in a seminal work, Brown et al. [14] uncovered a remarkable connection between the action within the interior of a black hole and the aforementioned upper bounds on information processing. Specifically, they demonstrated that the action inside a black hole increases at a rate precisely equal to $2E/\pi\hbar$, leading to the conclusion that black holes generate complexity at the maximum possible rate. This naturally raises a critical question: if the system is perturbed, is it possible for it to exceed this established bound?

The primary objective of this paper is to investigate Lloyd's bound as applied to both a quantum harmonic oscillator and a simple anharmonic oscillator under various external conditions. Given the central role of the quantum harmonic oscillator in quantum mechanics, we aim to quantify how the complexity growth rate varies when the system is subjected to perturbations from external electric and magnetic fields. Specifically, this study explores the influence of these fields on the complexity in contrast, within computer science, computational complexity—or simply the complexity of an operation—is commonly defined as the amount of resources needed to solve a problem, or alternatively, as the minimum complexity among all possible algorithms designed to address that problem [15][16]. Furthermore, quantum mechanics provides a standard definition of complexity. Essentially, complexity serves as a measure indicating how difficult a task is.

More specifically, in the context of quantum computation, a quantum state's complexity can be defined as the smallest number of basic quantum operations (gates) needed to create that state from a fixed reference state.

Additionally, some aspects of the late-time dynamics of eternal black hole geometries have been suggested to be significantly elucidated by quantum complexity [15].
In Reference [18], the authors examine the foundational concept of Lloyd's bound within the framework of holographic complexity, emphasizing a critical distinction between orthogonalization and simple gates.

They argue that these concepts are instrumental in effectively characterizing holographic complexity. Fundamentally, the capacity of a physical system to store or process information is intrinsically linked to the number of accessible states within that system.
dynamics of the quantum system, with a focus on testing Lloyd's bound in the context of an anharmonic oscillator.

The layout of this paper is as follows: In Section 2, we briefly consider the orthogonality time for two simple uncoupled harmonic oscillators in the presence of magnetic and electric fields. In particular, we explore the minimum time needed for a given state to evolve into an orthogonal state. In Section 3, we look for any upper bound on the complexification. Section 4 extends our calculations to an anharmonic oscillator using perturbation theory and investigates the behavior of the complexity growth rate; finally, the concluding section summarizes the results of this study.

2. Charged Harmonic Oscillator

Consider a system comprising two uncoupled harmonic oscillators, each characterized by mass m and chargeq, subjected to external electric and magnetic fields. The corresponding Hamiltonian for this system is expressed as (See also Ref. [19] for related work):

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right)$$

$$+\frac{1}{2}m\left(\omega^2 + \frac{q^2B^2}{4m^2c^2}\right)\left(x^2 + y^2\right)$$

$$+\frac{qB}{2mc}\left(yp_x - xp_y\right) - \frac{q^2}{2m\omega^2}\mathcal{E}^2$$

The energy eigenvalue for a two-dimensional oscillator, with the definitions $\frac{qB}{2m} \equiv \omega_c$, $\frac{q\mathcal{E}}{2m} \equiv \omega_e$ is given by:

$$E = \sqrt{\omega^2 + \omega_c^2}\left(n_1 + n_2 + 1\right)$$

$$-\omega_c\left(n_1 - n_2\right) - 2m\frac{\omega_e^2}{\omega^2}$$

Furthermore, the Margolus – Levitin bound provides a way to determine how quickly a quantum state can change into an orthogonal state. The following relation [19] expresses this temporal limitation [19]:

$$S(\tau_\perp) \equiv \langle\psi_0|\psi_{\tau_\perp}\rangle$$

$$= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left|c_{n_1,n_2}\right|^2 e^{-iE_{n_1,n_2}\tau_\perp} ,$$

This is an example of a fundamental quantum dynamics limitation. In this case, $|\psi_0\rangle$ represents the initial quantum state, while $|\psi_{\tau_\perp}\rangle$ represents the state that evolved at time $\tau_\perp$. Solving the equation $S(\tau_\perp) = 0.$ yields the minimal evolution time $\tau_\perp$.

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$$1-\frac{2}{\pi}[x+\sin(x)]\le\cos(x),$$
$$1-\frac{2\boldsymbol{E}\,\tau_\perp}{\pi}+\frac{2}{\pi}Im(S)\le Re(S) \qquad (1)$$

In the above relations we have used an algebraic inequality. Also, $\boldsymbol{E}$ represents the average energy in the state $|\psi_0\rangle$ .By imposing the orthogonality condition, which requires that $ReS(\tau_\perp)=ImS(\tau_\perp)=0$, one can infer that the minimum time$(\tau_\perp)$ necessary for the quantum state $|\psi_0\rangle$ evolve into an orthogonal state is given by the following relation:

$$\tau_\perp=\frac{\pi}{2\boldsymbol{E}}$$

Average energy determined by:

$N_t=\sum_{n=0}^{N}(n+1)$, $n=n_1+n_2$ and $c_{n_1,n_2}=\sqrt{\frac{1}{N_t}}$, for

$$\boldsymbol{E}=\langle\psi_0|H|\psi_0\rangle$$
$$=\left(1+\frac{2N}{3}\right)\sqrt{\omega^2+\omega_c^2}-2m\frac{\omega_e^2}{\omega^2}$$

Inequality Eq.(1) is satisfied by $\tau_\perp$, which in this case represents the shortest time required for the system to evolve into an orthogonal state.

$$\tau_\perp\ge\frac{\pi}{2\left(1+\frac{2N}{3}\right)\sqrt{\omega^2+\omega_c^2}-4m\frac{\omega_e^2}{\omega^2}}$$

In the large-N limit and near the boundary regime, the inequality $\omega_e\le\frac{\sqrt{3+2N}}{\sqrt{6m}}\omega^{\frac{3}{2}}$ ensures that the system maintains non-negative energy.

Building upon the previous discussion, computational complexity is formally defined as the minimal number of elementary quantum gates required to perform a specific computational task. In the framework of quantum computation, this is interpreted as the minimum number of unitary operations necessary to evolve a system from a given reference state to a target state in Hilbert space.

This notion of complexity captures key structural and dynamical characteristics of quantum systems and has become a useful diagnostic in studies of quantum gravity and holography. By invoking the uncertainty principle, one can derive an upper bound on the rate of growth of complexity (often referred to as the rate of complexification), which will be explored in the following section.

3. Upper Bound on Complexification

Consider a single computational step represented by the gate G, which requires a finite time interval Δt to complete its operation [19]. In the framework of quantum mechanics, the gate G can be expressed as a unitary evolution generated by a Hamiltonian $\boldsymbol{E}$, such that $G=e^{iH\Delta}$, where the average energy associated with the gate is given by $\boldsymbol{E}=\langle H\rangle$. The duration required to perform the operation is inversely related to the energy $\boldsymbol{E}$; that is, a higher energy leads to a shorter execution time for the gate. Defining the total operation time for n such steps as $T\equiv n\Delta t$, and denoting the total computation time of the system as t, one obtains the following relation:

$$t\ge T \qquad (2)$$

The variable T in classical computation denotes the time needed for a specific computational task to be finished by the fastest classical computer. Nonetheless, the Margolus–Levitin theorem in quantum mechanics provides a fundamental bound on the smallest time needed for a quantum system to change between two orthogonal states. This suggests that, depending on the system's energy availability, each basic logical operation (or gate) must be completed in a specific amount of time. This quantum speed limit thus restricts the number of logical operations that can take place in each discrete time step.

$$\Delta t\ge\frac{\pi}{2\boldsymbol{E}} \qquad (3)$$

This expression can be reformulated into the following form:

$$\frac{1}{\Delta t}\le\frac{2\boldsymbol{E}}{\pi}$$

the relation above imposes an upper bound on the instantaneous rate at which computational operations can be performed. Specifically, the complexity of a given quantum state must be less than or equal to n, where n denotes the number of logical gates required by the most efficient classical algorithm to generate that state. By invoking assumption (3), one can derive a constraint on the rate of change of complexity, expressed as follows:

$$\dot{C}\equiv\frac{\Delta C}{\Delta t}\ge\frac{\pi}{2\boldsymbol{E}}\ ,$$

it is important to note that Lloyd's framework serves as the foundation for our assumption regarding orthogonal states. According to this formulation, quantum gates act on essentially classical states and are viewed as the quantum-mechanical equivalents of classical logic gates. The gates are designed to evolve the associated quantum states into mutually orthogonal ones as well, since distinct classical states are by definition orthogonal. More specifically, the time evolution is thought of as performing a sequential calculation, as explained by [17]:

$$U(t)=T\prod_i U(t_{i+1},t_i),$$

Each discrete evolution step corresponds to the application of an orthogonalizing quantum gate. Consequently, the quantum state transitions to an orthogonal state whenever the circuit complexity increments by one unit. By definition, the growth rate of circuit complexity exceeds that of the intrinsic state complexity, which leads to the following inequality:

$$\dot{C} \leq \frac{1}{\tau_\perp},$$

it can be straightforwardly expressed that:

$$\dot{C} \leq \left(E_m \sqrt{1 + \frac{\omega_c^2}{\omega^2} - \frac{4m\omega_e^2}{\pi\omega^2}} \right),$$

as we have defined $E_m = \frac{2}{\pi}\left(1 + \frac{2N}{3}\right)\omega$ in the above relation.

The rate of normalized complexification of in terms of ω is depicted in Fig 1 for different values of mass, and this quantity is also shown for m = 1 and different values of ω_c and $\omega_e$in Fig 2.

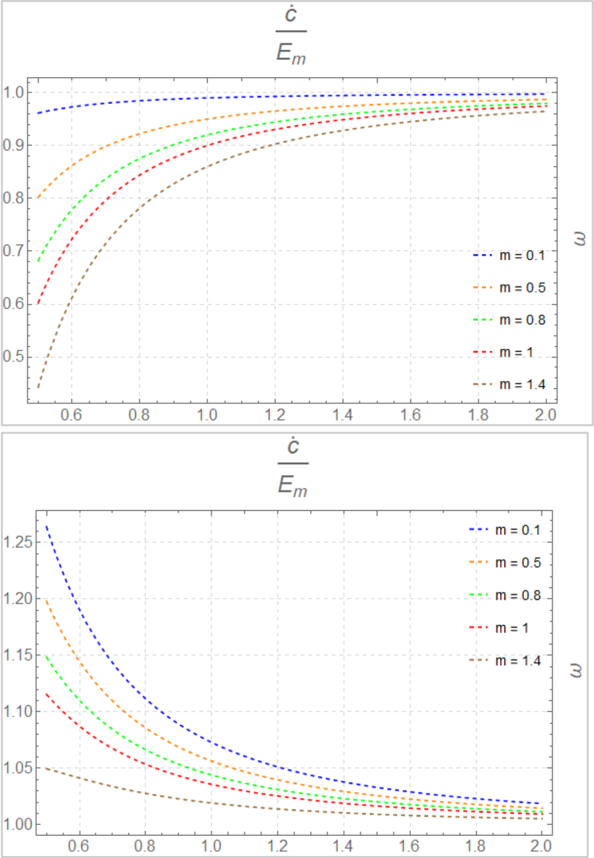

Figure 1. Rate of normalized complexification of the quantum harmonic oscillator for different values of mass for, up plot: $\omega_c = 0.03$ and $\omega_e = 0.28$ and down plot:$\omega_c = 0.4$ and $\omega_e = 0.18$.

There exists a critical value of the magnetic field beyond which the rate of complexity undergoes a qualitative transition in its behavior, characterized by the following expression:

$$\omega_c^{cri} = \frac{2\sqrt{3}}{(3+2N)\omega^2} \times \sqrt{m\omega_e^2[(3+2N)\omega^3 + 3m\omega_e^2]} \tag{4}$$

Now, in the next section, we will try to generalize our calculations using perturbation theory for an anharmonic Oscillator.

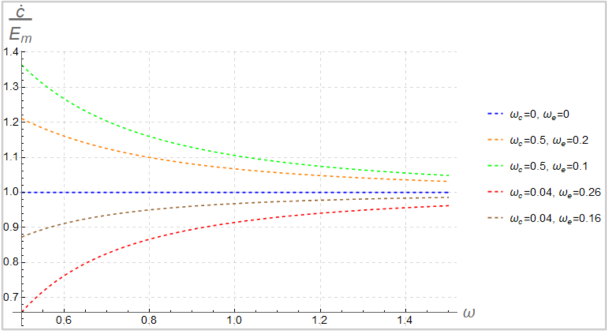

Figure 2. Rate of normalized complexification of the quantum harmonic oscillator. When the magnetic feld induction reached a definite critical value given by Eq.(4), the behavior of this rate changed. Note that we set and m = 1.

4. Anharm onic Oscillator

In the case of an anharmonic oscillator, where the influence of a weak external electric field is described by the perturbation term $H_P = q\mathcal{E}x + \lambda x^4$, the one-dimensional Hamiltonian takes the following form
(Note that from now on we have used the natural units $\hbar = c = 1$.):

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 + q\mathcal{E}x - \lambda x^4,$$

The lowest non-vanishing order of the energy shift is then expressed as (see Appendix A):

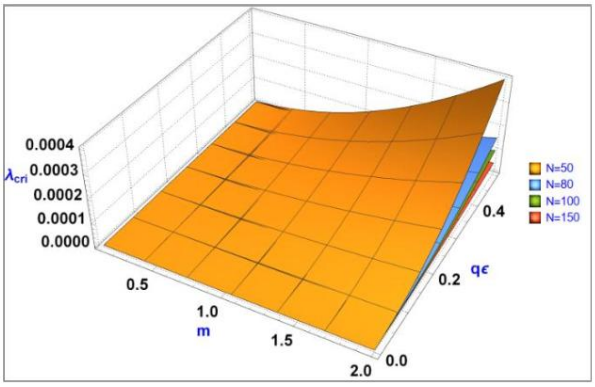

Figure 3. This figure shows the variation of critical value with respect to m and q Ɛ, for different values of N. Here, m ranges from 0.5 to 2, and q Ɛ ranges from 0 to 0.4.

$$E_n = \omega\left(n + \frac{1}{2}\right) + \frac{3\lambda}{4m^2\omega^2}[2n^2 + 2n + 1] - 2m\frac{\omega_e^2}{\omega^2}$$

where n is a positive integer or zero, and λ is a small positive real parameter.

The expected energy value in the case of a quartic anharmonic oscillator is written as follows:

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$$\boldsymbol{E} = \langle\psi|H|\psi\rangle =$$

$$\sum_{n=0}^{N-1}|c_n|^2[\omega\left[n+\frac{1}{2}\right]+\frac{3\lambda}{4m^2\omega^2}\times$$

$$[2n^2+2n+1]-2m\frac{\omega_e^2}{\omega^2}]$$

And the eigenfunction is given by the following expression:

$$|\psi\rangle = \sum_{n=0}^{N-1} c_n|n\rangle.$$

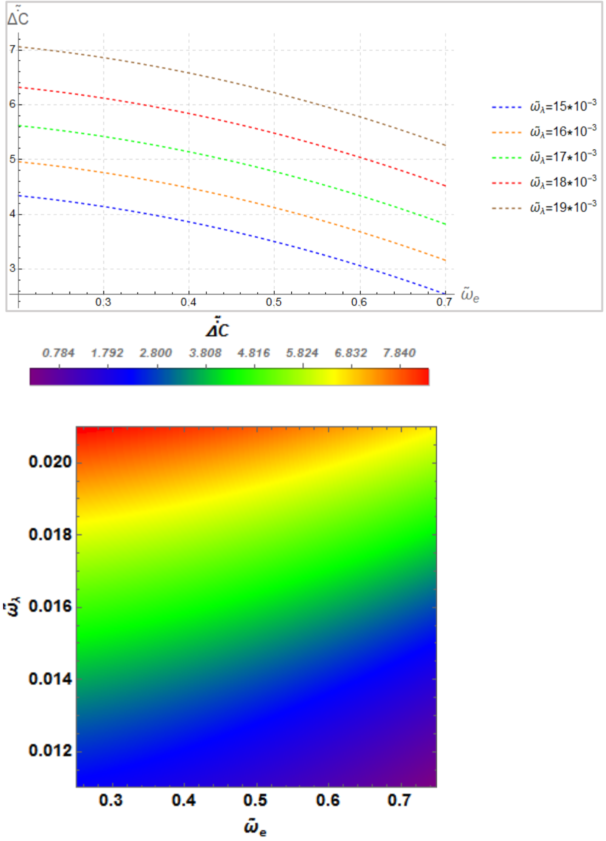

Figure 4. The effect of anharmonic parameter and electric field on the rate of complexity ($\widetilde{\Delta\dot{C}} = \frac{\pi\dot{C}-N\omega}{m}$). The rate of complexity increases by $\widetilde{\omega}_\lambda$ while it decreases by increasing $\widetilde{\omega}_e$. Note that we set N=100, $\widetilde{\omega}_\lambda \equiv \frac{\omega_\lambda}{\omega}$ and $\widetilde{\omega}_e \equiv \frac{\omega_e}{\omega}$.

In the case of a quartic anharmonic oscillator, the expected value of the energy is expressed as:

$$\boldsymbol{E} = \frac{1}{N}[\sum_{n=0}^{N-1}(\omega\left[n+\frac{1}{2}\right]+\frac{3\lambda}{4m^2\omega^2}\times$$

$$[2n^2+2n+1]-2m\frac{\omega_e^2}{\omega^2})]$$

$$=\frac{N\omega}{2}+\frac{\lambda}{4m^2\omega^2}+\frac{N^2\lambda}{2m^2\omega^2}-2m\frac{\omega_e^2}{\omega^2}$$

The above relation has been examined in the regime of large N.

According to Eq. (2), the orthogonalization time is determined as follows:

$$\tau_\perp \geq \frac{\pi}{N\omega+m\frac{\omega_\lambda^2}{\omega^2}+2mN^2\frac{\omega_\lambda^2}{\omega^2}-4m\frac{\omega_e^2}{\omega^2}}$$

Here, ω_λ^2 is defined as $\frac{\lambda}{2m^3}$, and the following limit is assumed to hold:

$$\omega_e < \frac{\sqrt{\lambda\left(\frac{1}{2}+N^2\right)+m^2N\omega^3}}{2m^{\frac{3}{2}}}$$

which results in non-negative energy. Similar to the harmonic oscillator case, the complexity boundary will be considered as follows:

$$\dot{C} \leq \frac{1}{\tau_\perp}$$

Note that we consider that the gates are orthogonal. Consequently, the complexity rate in this scenario can be expressed as follows:

$$\dot{C} \leq \frac{1}{\pi}\left(N\omega+m\frac{\omega_\lambda^2}{\omega^2}+2mN^2\frac{\omega_\lambda^2}{\omega^2}-4m\frac{\omega_e^2}{\omega^2}\right)$$

Beyond that, there is a significant behavioral shift in the rate of complexity change, which is typified by:

$$\omega_\lambda^{cri} = \frac{2\omega_e}{\sqrt{1+2N^2}}.$$

We examine the perturbation operator $q\mathcal{E}x+\lambda x^4$. There exists a critical value of the anharmonic parameter at which the complexity level undergoes a significant qualitative change on either side, as detailed below.

$$\lambda = \frac{2mq^2}{1+2N^2}\mathcal{E}^2$$

The diagram depicting the relationships described above is shown in Fig 3.

In this context, an anharmonic parameter, such as a magnetic field applied to a charged harmonic oscillator, effectively enhances the rate of complexity, whereas the electric field can entirely counteract the additive effect of λ.

As can be clearly seen in Fig 4, The rate of complexity increases with λ whereas it decreases as ℰ increases. Furthermore, calculations demonstrate that, across all sequences, perturbations involving odd powers of x lead to a decrease in the complexity rate, while those involving even powers of x result in an increase in the complexity rate.

5. Conclusions

In this work, we systematically examined the fundamental limitations on the rate of quantum information processing by analyzing the bounds on complexity growth in systems modeled as charged quantum oscillators. Utilizing both the Margolus–Levitin theorem and Lloyd's bound, we investigated the minimal orthogonalization time—a quantum speed limit that directly correlates with the system's average energy—and its role as a key constraint on computational evolution. In particular, Lloyd's bound imposes an upper limit on the rate at which quantum complexity can increase, given by $\frac{d}{dt}C \leq \frac{2E}{\pi\hbar}$ where $\boldsymbol{E} = \langle H \rangle$ denotes the expectation value of the system's Hamiltonian. This relation encapsulates the energetic constraints governing the maximal rate of quantum computational growth.

The first part of this study focused on the dynamical evolution of a charged quantum harmonic oscillator in the presence of external electric and magnetic fields. Our analysis revealed that the magnetic field acts to enhance the rate of complexity growth, whereas the application of an electric field exerts a suppressive effect.

Additionally, depending on the magnetic field regime, the time required for orthogonalization-the shortest interval for a system to change from an initial quantum state to an orthogonal state—behaves qualitatively differently.

In particular, the system demonstrates markedly different evolution characteristics in the limits of weak and strong magnetic fields, indicating a nontrivial dependence of quantum complexity dynamics on field strength.

Extending our investigation to an anharmonic oscillator subjected to perturbations of the form $q\mathcal{E}x + \lambda x^4$, we examined the influence of these parameters on the complexity growth rate. Numerical analysis identified critical points within the parameter space, specifically in λ and energy $\boldsymbol{E}$-beyond which the dynamics of complexity evolution exhibit a phase transition. Notably, an increase in λ generally enhances the complexity growth rate, whereas an increase in E leads to a decrease in this rate once the critical threshold, defined by:

$$\lambda = \frac{2mq^2}{1+2N^2}\mathcal{E}^2$$

These findings indicate a nontrivial interplay among the system's physical parameters, reflecting the sensitivity of complexity dynamics to the structure of external perturbations. Specifically, perturbative terms involving odd powers of the position operator x were observed to suppress the rate of complexity growth, while even-powered perturbations contribute constructively, leading to an enhancement in the overall complexification of the quantum state.

These findings provide important insights into the underlying quantum constraints governing computational speed and complexity growth, and serve as a conceptual link between quantum mechanics, information theory, and holographic frameworks like the AdS/CFT correspondence. The application of these ideas to charged thermofield double states (cTFD) in free scalar quantum field theories exposed to external electromagnetic fields is one of the fascinating directions the study suggests for further investigation.

Additionally, the findings motivate further exploration into the holographic characterization of complexity in charged black hole geometries, potentially deepening our understanding of the interplay between quantum information and gravitational dynamics.

This study advances the understanding of the intricate relationship between system energy, external perturbations, and the computational complexity in quantum systems. It elucidates the existence of critical thresholds that fundamentally regulate the dynamical behavior of complexity growth, thereby providing valuable insights into the mechanisms governing complexification processes within quantum frameworks.

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Authors Contribution
All the authors have participated sufficiently in the intellectual content, conception and design of this work or the analysis and interpretation of the data (when applicable), as well as the writing of the manuscript.

Availability of data and materials
The data that support the findings of this study are available from the corresponding author, upon reasonable request.

Conflict of interests
The author states that there is no conflict of interest.

References

[1] Aaronson S. The complexity of quantum states and transformations: From quantum money to black holes. arXiv:1607.05256 [quant-ph]. 2016. Available from: https://arxiv.org/abs/1607.05256

[2] Alishahiha M, Babaei Vekni K, Tanhayi MR. Complexity and near extremal charged black branes. arXiv:1901.00689 [hep-th]. 2019. Available from: https://arxiv.org/abs/1901.00689

[3] Arora S, Barak B. Computational complexity, a modern approach. Cambridge: Cambridge University Press; 2009.

[4] Bekenstein JD. Black holes and entropy. *Phys Rev D*. 1973;7(8):2333-46. https://doi.org/10.1103/PhysRevD.7.2333

[5] Brown AR, Roberts DA, Susskind L, Swingle B, Zhao Y. Complexity, action, and black holes. *Phys Rev D*. 2016;93(8):086006. https://doi.org/10.1103/PhysRevD.93.086006

[6] Cottrell W, Montero M. Complexity is simple! *J High Energy Phys*. 2018;2018(2):39. https://doi.org/10.1007/JHEP02(2018)039

[7] Dippel O, Schmelcher P, Cederbaum LS. Charged anisotropic harmonic oscillator and the hydrogen atom in crossed fields. *Phys Rev A*. 1994;49(6):4415-22. https://doi.org/10.1103/PhysRevA.49.4415

[8] Hawking SW. Black hole explosions. *Nature*. 1974;248(5443):30-1. https://doi.org/10.1038/248030a0

[9] Khorasani F, Pirmoradian R, Tanhayi MR. Position dependence of Nielsen complexity for the thermofield double state. *Phys Lett B*. 2024; 851:138585. https://doi.org/10.1016/j.physletb.2024.138585

[10] Levitin LB. Physical limitations of rate, depth, and minimum energy in information processing. *Int J Theor Phys.* 1982;21(4):299-309. https://doi.org/10.1007/BF01857726

[11] Lloyd S. Ultimate physical limits to computation. *Nature.* 2000;406(6799):1047-54. https://doi.org/10.1038/35023282

[12] Lloyd S, Ng YJ. Black hole computers. Sci Am. 2004;291(5):52-61. https://doi.org/10.1038/scientificamerican1104-52

[13] Nishioka T, Ryu S, Takayanagi T. Holographic entanglement entropy: An overview. *J Phys A: Math Theor*. 2009;42(50):504008. https://doi.org/10.1088/1751-8113/42/50/504008

[14] Pirmoradian R, Tanhayi MR. Non-local probes of entanglement in the scale-invariant gravity. *Int J Geom Methods Mod Phys*. 2021;18(12):2150197. https://doi.org/10.1142/S0219887821501975

[15] Pirmoradian R, Tanhayi MR. Symmetry-resolved entanglement entropy for local and non-local QFTs. *Eur Phys J C*. 2024;84(9):849. https://doi.org/10.1140/epjc/s10052-024-12845-z

[16] Pirmoradian R, Tanhayi MR. Information dynamics in quantum harmonic systems: Insights from toy models. arXiv:2501.14359 [quant-ph]. 2025. Available from: https://arxiv.org/abs/2501.14359

[17] Reza Tanhayi M, Vazirian R, Khoeini-Moghaddam S. Complexity growth following multiple shocks. *Phys Lett B*. 2019;790:49-55. https://doi.org/10.1016/j.physletb.2019.01.003

[18] Susskind L. Entanglement is not enough. Fortschr Phys. 2016;64(1):49-71. https://doi.org/10.1002/prop.201500095

[19] Van Raamsdonk M. Evaporating firewalls. *J High Energy Phys*. 2014;2014(11):38. https://doi.org/10.1007/JHEP11(2014)038

Appendix A: Eigenvalue for perturbation theory

In this appendix, we determine the energy variation of a simple anharmonic oscillator, characterized by the following Hamiltonian

$$H = \frac{1}{2m} p^2 + \frac{1}{2} m\omega^2 x^2 + \lambda_o x^{2k-1} + \lambda_e x^{2k}$$

we examine the energy variation of a simple anharmonic oscillator characterized by the following Hamiltonian, where $0 \leq \lambda_o, \lambda_e \ll 1$ and k is a positive integer. It is demonstrated that the growth rate of complexity for an anharmonic oscillator increases under even-order perturbations, whereas it decreases under odd-order perturbations. In this study, we analyze the energy variation of a simple anharmonic oscillator described by the following Hamiltonian, where $0 \leq \lambda_o, \lambda_e \ll 1$, and k is a positive integer. It is shown that the complexity growth rate increases in response to even-order perturbations, whereas it decreases under odd-order perturbations. Interestingly, the first-order correction arising from the odd-order perturbation is zero, while the second-order correction to the energy is given by:

$$E^2_{n_{odd}} = -{\lambda_o}^2 \sum_{m \neq n} \frac{|\langle m|x^{2k-1}|n\rangle|^2}{E_{mn}}$$

which yields a negative value, whereas a positive value is obtained in the case of even-order perturbations.

$$E^1_{n_{even}} = \lambda_e \langle n|x^{2k-1}|n\rangle$$

where the position operator x is defined as:

$$x = \sqrt{\frac{1}{2m\omega}} (a \; + a^\dagger).$$

The first-order corrections for a harmonic oscillator under perturbations of the $\lambda_1 x, \lambda_2 x^2, \lambda_3 x^3, \ldots$ are now calculated. which leads to:

$$\left\langle m\middle|(a + a^\dagger)\middle|n\right\rangle = \sqrt{n}\delta_{m,n-1} + \sqrt{n+1}\delta_{m,n+1}$$

$$\left\langle m\middle|(a + a^\dagger)^2\middle|n\right\rangle = 2n\delta_{m,n} + \delta_{m,n} + \sqrt{n-1}\sqrt{n}\delta_{m,n-2} + \sqrt{n+1}\sqrt{n+2}\delta_{m,n+2}$$

$$\left\langle m\middle|(a + a^\dagger)^3\middle|n\right\rangle = 3n^{3/2}\,\delta_{m,n-1} + 3\sqrt{n+1}\,n\,\delta_{m,n+1} + \sqrt{n-2}\sqrt{n-1}\sqrt{n}\,\delta_{m,n-3} + \; 3\sqrt{n+1}\,\delta_{m,n+1} + \sqrt{n+1}\sqrt{n+2}\sqrt{n+3}\,\delta_{m,n+3}$$

Following some algebraic manipulation, one also finds that

$$\langle n|x^2|n\rangle = \frac{2n+1}{2m\omega}$$

$$\langle n|x^4|n\rangle = \frac{3(2n^2+2n+1)}{4m^2\omega^2}$$

$$\langle n|x^6|n\rangle = \frac{5(2n+1)(2n^2+2n+3)}{8m^3\omega^3}$$

$$\langle n|x^8|n\rangle = \frac{35(2n^4+4n^3+10n^2+8n+3)}{16m^4\omega^4}$$

$$\langle n|x^{10}|n\rangle = \frac{63(2n+1)(2n^4+4n^3+18n^2+16n+15)}{32m^5\omega^5}$$

Energy corrections can also be obtained up to the second order.

$$\sum_{m \neq n} \frac{|\langle m|x|n\rangle|^2}{E_{mn}} = \frac{1}{(2m\omega)\omega}$$

$$\sum_{m \neq n} \frac{|\langle m|x^3|n\rangle|^2}{E_{mn}} = \frac{30n^2+30n+11}{(2m\omega)^3\omega}$$

$$\sum_{m \neq n} \frac{|\langle m|x^5|n\rangle|^2}{E_{mn}} = \frac{630n^4+1260n^3+2030n^2+1400n+449}{(2m\omega)^5\omega}$$

$$\sum\nolimits_{m \neq n} \frac{\left|\left\langle m\middle|x^7\middle|n\right\rangle\right|^2}{E_{mn}} = \frac{3(4004n^6+12012n^5+4235n^4+64680n^3+81788n^2+51450n+14793)}{(2m\omega)^7\omega}$$

These calculations suggest that the complexity growth rate increases with even-order perturbations and decreases with odd-order perturbations.