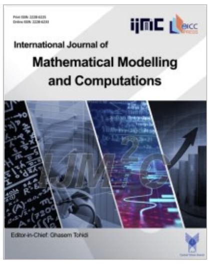

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Updating Most Frequently Used Degradation Models Under the Influence of Random Shocks

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Abstract

In modern maintenance planning, leveraging advanced capabilities necessitates accurate modeling of system degradation under catastrophic shocks with escalating effects. Reliable estimation of failure time is crucial for optimizing maintenance strategies. This study aims to analyze the impact of exponential and Weibull catastrophic shocks on the most common degradation models Wiener, Gamma, and Inverse Gaussian-without requiring model reconstruction. A rigorous mathematical analysis is conducted to assess the influence of these shocks on degradation patterns. The necessary conditions for applying the proposed analytical approach are outlined, ensuring its practical applicability. Furthermore, a case study is presented to validate the theoretical findings. Comparative analysis between the proposed analytical method and simulation results demonstrates no significant difference, confirming the robustness of the proposed approach. These findings contribute to enhancing predictive maintenance strategies by providing a refined understanding of catastrophic shock effects on degradation processes.

Keywords: Degradation Modeling, Catastrophic Shocks, Wiener process, Gamma process, Inverse Gaussian Process

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1. Introduction

Degradation in industrial systems and infrastructure is an inevitable process driven by operational and environmental factors. Accurate modeling of degradation is critical for predicting system reliability, optimizing maintenance, and reducing unexpected failures. Effective models quantify wear mechanisms, assess risks, and support proactive maintenance, minimizing downtime and costs. Researchers have developed various models to study competing failure processes in complex systems, focusing on degradation-shock interactions. Early studies established foundational reliability models, evolving into advanced frameworks that account for degradation-shock dependencies. Key advancements include modeling multi-component systems with shared shock loads, classifying shocks by severity, and using Markov processes for non-Poisson shock behaviors. Some models also address state-varying degradation, multi-stage processes, and recursive equations for dynamic failure conditions. Cascading shock effects in circuit systems further highlight how degradation-shock interactions can cause unpredictable failures. Shocks not only accelerate degradation but can also alter its patterns, shifting from linear to non-linear behavior. For example, a sudden shock in mechanical or electrical components may trigger complex failure mechanisms, such as accelerated fatigue or dielectric breakdown. In rare cases, shocks may simplify degradation by suppressing certain failure mechanisms, as observed in corrosion and wear scenarios. These dynamics necessitate models that capture both immediate and long-term system responses to shocks. Advanced modeling techniques enhance data-driven decision-making, extend asset lifespan, and improve system efficiency in industries like manufacturing, transportation, and energy. Degradation modeling methodologies fall into four categories:

a) Stochastic Processes: Wiener (Brownian motion with drift), Gamma (monotonic degradation), Inverse Gaussian, and Compound Poisson processes.
b) Deterministic Models: Linear, exponential, and power-law models.
c) Statistical and Machine Learning Techniques: Regression-based models, Bayesian methods, neural networks, and deep learning.
d) Physical and Empirical Models: Physics-based frameworks (e.g., Arrhenius model) and reliability-based approaches [1]

Stochastic models excel at capturing probabilistic degradation. The Wiener process suits systems with random fluctuations, ideal for predictive maintenance and remaining useful life (RUL) estimation. The Gamma process models irreversible degradation, such as crack propagation or corrosion. The Inverse Gaussian process handles variable-rate monotonic degradation, applicable in biomedical implants and semiconductors. The Compound Poisson Process (CPP) is effective for shock-driven degradation in mechanical systems, electronics, and infrastructure. This study analyzes the impact of random shocks on conventional degradation models, focusing on Wiener, Gamma, and Inverse Gaussian processes. It introduces a novel approach to update model parameters to account for exponential and Weibull random shocks, eliminating the need for new patterns or remodeling. This method simplifies the modeling process, offering practical benefits and significant time and cost savings. The paper is organized into five sections. Following the current introduction, Section 2 provides a concise and explicit review of the most significant prior research on the study of catastrophic shocks in degradation modeling. Then Section 3 presents a mathematical formulation and analysis of the impact of Poisson process shocks, on the intensification of continuous degradation processes. In this section, two types of random step increases -Gamma and Weibull-are considered for representing the effects of shocks on degradation process. Section 4 explicitly discusses the research findings and the contributions of the present study. Finally, the paper concludes with a list of references.

2. Literature Survey

Recent studies highlight the critical interplay between degradation and shocks in system reliability. This paper examines catastrophic shocks within continuous degradation processes, impacting reliability, maintenance, and planning in engineering. Early work [2] established models for degradation under shock effects. [3] introduced a framework for dependent failure processes, integrating degradation and shocks. [4] proposed a dynamic failure threshold deteriorating with shock accumulation, while [5] developed reliability models for multi-component systems with interdependent failures. [3]classified shocks as damage-inducing, catastrophic, or safety-zone. [6] used a Markov framework to model non-Poisson shocks. [7] analyzed random shocks in multi-state systems, and [8] developed a predictive model for remaining useful life (RUL`) under degradation-shock dependencies. [9] formulated recursive equations for system reliability, addressing both gradual and catastrophic failures. [10] proposed a degradation model for soft failures, using a Wiener process to capture random degradation and an exponential function for recoverable shock damages. Their model generalizes existing frameworks and extends to other stochastic processes, improving reliability and lifetime predictions. [11] modeled discrete degradation with random fatal shocks across four system life stages. Their competing risks model incorporates age- and state-dependent shocks, using a time-scaled acceleration factor and state-modulated functions. An extended Kalman filter enables state estimation, yielding a closed-form reliability function that adapts to new data.

[12]developed a hybrid model incorporating cumulative damage and degradation rate acceleration. By dividing the system lifecycle into four stages, their model uses adjustment coefficients to derive a closed-form reliability function, validated through a case study. [13] explored a mixed shock model where degradation and shocks interact, leading to soft or hard failures. Their model integrates state-dependent extreme and δ shock mechanisms, deriving reliability expressions via stochastic process theory. A case study on sea bridge pier columns demonstrates its applicability. [14] investigated multi-state reliability models capturing degradation-shock dependencies using a time-transform Wiener process. Their approach includes sensitivity analysis, parameter estimation via the ABC method, and validation through a case study. Figure 1, cited from [14] , illustrates a degradation-shock competing failure process, highlighting gradual degradation and sudden shocks as key failure drivers.

Recent advancements in statistical distributions have enhanced the modeling of degradation processes under uncertainty. For instance, [15] introduced a new five-parameter distribution, highlighting its properties and applications in reliability analysis, which can be adapted to model complex degradation patterns influenced by random shocks. Similarly, [16] compared estimators for the PDF and CDF of the three-parameter inverse Weibull distribution, providing insights into parameter estimation techniques that are crucial for accurately capturing shock-induced failures in stochastic degradation models like the Inverse Gaussian process. In the context of stochastic modeling for reliability and efficiency, [17] proposed a new stochastic model for classifying flexible measures in data envelopment analysis, which offers a framework for evaluating system performance under variable conditions, analogous to degradation-shock interactions in multi-component systems. These approaches collectively underscore the importance of robust parameter estimation and flexible stochastic models in enhancing predictive accuracy for degradation processes affected by catastrophic shocks. The dual-failure mechanism in Figure 1 distinguishes degradation-driven (continuous) and shock-induced (discrete) failures. This framework supports reliability engineering by analyzing failure mode interactions, enabling predictive maintenance, risk assessment, and optimal intervention strategies to enhance system reliability and lifespan.

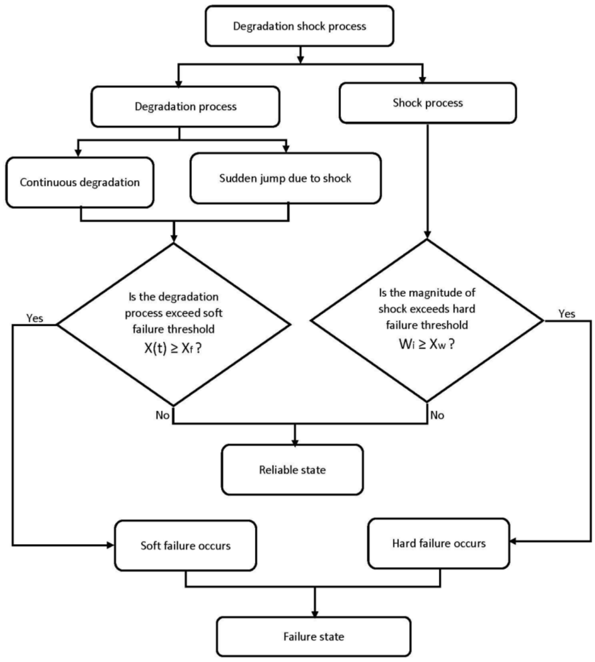

Figure 1. Degradation-shock competing failure process [14]

3. The Proposed Methodology

In continuous degradation processes, shocks occur as random, discrete events that cause abrupt increases in damage. Unlike purely jump-based models like the Compound Poisson Process, continuous degradation is typically modeled using stochastic processes such as the Wiener process, Gamma process, or Inverse Gaussian process, while shocks are superimposed on these processes. In the literature, there are three types of shock arrivals. a) Random Arrival of Shocks: Modeled using a Poisson process $N(t)$ with rate λ (shock occurrences per unit time). Can be generalized using Non-homogeneous Poisson processes (NHPP) for time-dependent shock rates. b) Shock Impact on Degradation: Each shock induces an instantaneous jump in the degradation level. The jump size J_i is often modeled by Weibull, Gamma, or Exponential distributions, depending on the nature of the damage. c) Combination with Continuous Degradation: The baseline degradation follows a continuous process (e.g., Wiener, Gamma, or Inverse Gaussian). Shocks introduce random jumps, leading to a mixed degradation model:

$$X(t) = D(t) + \sum_{i=1}^{N(t)} J_i \tag{1}$$

where:
$D(t)$ is the continuous degradation process (e.g., Wiener, Gamma or inverse Gaussian process).
$\sum J_i$ represents the cumulative damage from shocks occurring at times governed by $N(t)$.
From this point onward, the focus of the article will be on three common and widely used degradation models and two types of shock jump effects. It will analyze the impact of Gamma and Weibull shock effects on continuous degradation processes, including Wiener, Gamma, and inverse Gaussian processes. Additionally, the study will address the research question of how the final degradation model will be shaped if the degradation process is subjected to the mentioned catastrophic shocks. To enhance clarity, it is useful to provide an intuitive view of the mathematical derivations. In simple terms, the proposed approach demonstrates how the cumulative degradation process can still be approximated by the same type of stochastic model (Wiener, Gamma, or Inverse Gaussian) when sudden shocks are introduced. Instead of fully reconstructing the model, the effect of shocks is captured by updating only the drift and variance parameters. This allows practitioners to interpret the results as “an accelerated version of the original degradation process,” where shocks increase the mean degradation rate and introduce additional uncertainty.

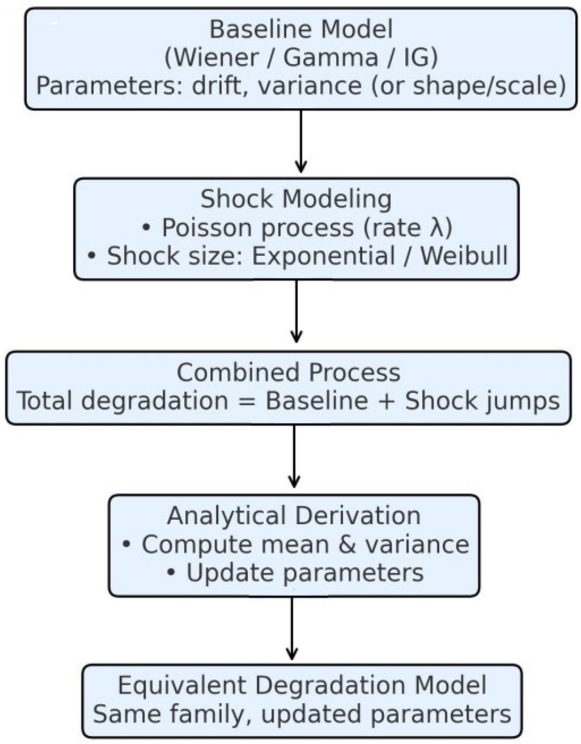

Figure 2. A Graphical survey to the proposed methodology

The sequence of derivations follows a systematic structure. First, the baseline degradation process is defined, characterized by its mean and variance. Second, the shock process is introduced as a Poisson arrival stream with random jump sizes. Third, the mean and variance of the combined process are derived by superimposing shock contributions. Finally, an equivalent stochastic process with updated parameters is identified, which preserves the statistical structure of the original model. This structured approach ensures that the impact of shocks can be analyzed without losing mathematical tractability, while also providing a clear link between theoretical results and practical parameter estimation.

To guide readers through the analytical framework, a schematic overview of the proposed methodology is presented in Figure 2. This diagram illustrates the logical flow from the baseline degradation model, through the introduction of stochastic shocks, to the derivation of an equivalent process with updated parameters. By following this stepwise representation, readers can more easily connect the mathematical derivations to the underlying intuition, ensuring that the complex proofs remain accessible and transparent.

To improve clarity and provide a concise reference, Table 1 summarizes the key parameter update equations derived for the Wiener, Gamma, and Inverse Gaussian degradation models under exponential and Weibull shocks. This table highlights how the drift, variance, or shape parameters of the baseline processes are adjusted when stochastic shocks are introduced. By presenting these results in a compact form, the reader can easily follow the logical flow of the mathematical derivations without the need to trace each equation in detail.

3.1. Modeling the Wiener Degradation Process Under the Influence of Poisson Shocks

Suppose the system has a degradation modeled by a random process with added Poisson shocks, where each shock induces a random jump from a Weibull distribution. The goal is to derive an equivalent Wiener process that models the total degradation, incorporating both sources of variation due to two types of shock jump.

a) Exponential shock jumps:

Let:

X be the original Wiener process degradation:

$$X(t) = \mu t + \sigma B(t) \tag{2}$$

whereB_t, is a standard Brownian motion, μ is the drift, and σ is the diffusion parameter.

Table 1. Glossary of terms before and after updating

Degradation	Basis Degradation	Shock Severity	Updated Degradation
Wiener	$X(t) \sim Wiener(\mu,\sigma)$	$Exponential$	$Y(t) \sim Wiener(\mu,\sigma)$
		$Weibull$	$Y(t) \sim Wiener(\mu,\sigma)$
Gamma	$G(t) \sim Gamma(\alpha,\beta)$	$Exponential$	$G(t) \sim Gamma(\tilde{\alpha},\tilde{\beta})$
		$Weibull$	$Y(t) \sim Gamma(\alpha,\beta)$
Inverse Gaussian	$X(t) \sim IG(\mu,\gamma)$	$Exponential$	$Y(t) \sim IG(\tilde{\mu},\tilde{\gamma})$
		$Weibull$	$Y(t) \sim IG(\mu,\gamma)$

$N(t)$ be a Poisson process with rate, λ (shock occurrence rate).

$Y_i \sim Exp(\beta)$ be the independent jump sizes from an exponential distribution.

Thus, the total degradation is:

$$D(t) = X(t) + \sum_{i=1}^{N(t)} Y_i \tag{3}$$

We approximate $D(t)$ with a new Wiener process:

$$D(t) \approx \tilde{\mu} t + \tilde{\sigma} B(t) \tag{4}$$

Here the expected can be calculated by Eq. 5.

$$\mathbb{E}[D(t)] = \mathbb{E}[X(t)] + \mathbb{E}\left[\sum_{i=1}^{N(t)} Y_i\right] \tag{5}$$

Since:

$$\mathbb{E}[N(t)] = \lambda t \tag{6}$$

$$\mathbb{E}[Y_i] = \frac{1}{\beta} \tag{7}$$

$$\mathbb{E}[X(t)] = \mu t \tag{8}$$

We get:

$$\mathbb{E}[D(t)] = \mu t + \lambda t \frac{1}{\beta} = \left(\mu + \frac{\lambda}{\beta}\right) t \tag{9}$$

So, the new drift is:

$$\tilde{\mu} = \mu + \frac{\lambda}{\beta} \tag{10}$$

The degradation variance can also be easily calculated from Eq. 11.

$$Var[D(t)] = Var[X(t)] + Var[\sum_{i=1}^{N(t)} Y_i] \tag{11}$$

$$Var[X(t)] = \sigma^2 t \tag{12}$$

$$Var\left[\sum_{i=1}^{N(t)} Y_i\right] = \mathbb{E}[N(t)]Var[Y] + Var[N(t)][\mathbb{E}[Y]]^2 \tag{13}$$

Using $Var(Y) = \frac{1}{\beta^2}$, we get:

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$$Var[\sum_{i=1}^{N(t)} Y_i] = \lambda t \frac{1}{\beta^2} + \lambda t (\frac{1}{\beta})^2 = \lambda t \frac{2}{\beta^2} \quad (14)$$

So, the total variance is:

$$Var[D(t)] = \sigma^2 t + \frac{2\lambda}{\beta^2} t \quad (15)$$

The new diffusion coefficient $\tilde{\sigma}$ satisfies Eq. (16).

$$\tilde{\sigma}^2 = \sigma^2 + \frac{2\lambda}{\beta^2} \quad (16)$$

Thus, the degradation process can be approximated by a new Wiener Process such as (17).

$$D(t) \approx \left(\mu + \frac{\lambda}{\beta}\right) t + \tilde{\sigma} B(t) \quad (17)$$

Where:

$$\tilde{\mu} = \mu + \frac{\lambda}{\beta} \ , \ \tilde{\sigma}^2 = \sigma^2 + \frac{2\lambda}{\beta^2} \quad (18)$$

It was thus demonstrated that if the Wiener degradation process is subjected to random shocks following a Poisson process with rate λ and a Gamma catastrophic effect, the resulting impact can be modeled as a new Wiener degradation process. In this new model, the drift and diffusion parameters of the new Wiener process become functions of the parameters of the original degradation model and those of the applied shocks. In brief to estimate the new parameters ($\tilde{\mu}$ and $\tilde{\sigma}$), you can estimate μ and σ from normal degradation data (without shocks) as well estimate λ and β from observed jumps (λ is the observed jump rate and β is estimated from the mean of jump sizes Then use as equation set (19).

$$\hat{\tilde{\mu}} = \hat{\mu} + \frac{\hat{\lambda}}{\hat{\beta}} \ , \ \hat{\tilde{\sigma}}^2 = \hat{\sigma}^2 + \frac{2\hat{\lambda}}{\hat{\beta}^2} \quad (19)$$

This new Wiener process acts as approximation for total degradation. Note that in general speaking, if exponential shocks are added as independent jumps to the Wiener process, the new process will become a Lévy process instead of a Wiener motion (a combination of Brownian motion and discrete jumps). To preserve the Wiener property, the jump sizes should not have a strong dependency on time. Therefore, the mean and variance of the jumps must be appropriately scaled. To ensure that the exponential shocks do not alter the Wiener process structure, the parameter β of the exponential distribution Exp (β) must be large enough. Consequently, the following condition for preserving the Wiener process should be justified. To prevent the shocks from affecting the drift of the Wiener process, this mean should be much smaller than the drift rate μ, i.e., $\gg \frac{1}{\beta}$. Also, to ensure that the shocks do not disrupt the Gaussian nature of the Wiener process, their variance should be much smaller than the inherent variance of the process, i.e., $\frac{1}{\beta^2} \ll \sigma^2$. Specifically, the following conditions should hold:

$$\beta \gg max\left(\frac{1}{\mu}, \frac{1}{\sigma}\right) \quad (20)$$

For the Wiener process to remain unaffected by exponential shocks, the parameter β should be sufficiently large. This ensures that both the mean and variance of the shocks are negligible compared to the drift and diffusion components of the Wiener process.

b) Weibull shock jumps:

$$X(t) = W(t) + \sum_{i=1}^{N(t)} J_i \quad (21)$$

where:
$W(t)$is a standard Wiener process with drift μ and variance σ^2, i.e.,

$$W(t) = \mu t + \sigma B(t) \quad (22)$$

where $B(t)$ is a standard Brownian motion.
$N(t)$is a Poisson process with rate λ, representing the occurrence of shocks.
J_i are independent and identically distributed (i.i.d.) jumps drawn from a Weibull distribution with shape parameter k and scale parameter θ.
Since $J_i \sim Weibull(k, \theta)$,its mean and variance are:

$$\mathbb{E}[J] = \theta \Gamma(1 + \frac{1}{k}) \quad (23)$$

$$Var[J] = \theta^2 \left[\Gamma\left(1 + \frac{2}{k}\right) - \left(\Gamma\left(1 + \frac{1}{k}\right)\right)^2\right] \quad (24)$$

Since $N(t) \sim Poisson(\lambda t)$, the expected number of jumps in time t is λt, and the total jump contribution has expected value and variance describe by Eq. 25 and 26.

$$\mathbb{E}\left[\sum_{i=1}^{N(t)} J_i\right] = \mathbb{E}[N(t)]\mathbb{E}[J] = \lambda t \theta \Gamma(1 + \frac{1}{k}) \quad (25)$$

$$Var\left[\sum_{i=1}^{N(t)} J_i\right] = \mathbb{E}[N(t)]Var[J]$$
$$= \lambda t \theta^2 \left[\Gamma\left(1 + \frac{2}{k}\right) - \left(\Gamma\left(1 + \frac{1}{k}\right)\right)^2\right] \quad (26)$$

Consequently, a general Wiener process with drift and variance can be written as:

$$Y(t) = \tilde{\mu}t + \tilde{\sigma}B(t) \quad (27)$$

To match the mean and variance with the original degradation process.
Equivalent drift:

$$\tilde{\mu} = \mu t + \lambda\theta\Gamma\left(1 + \frac{1}{k}\right) \quad (28)$$

Equivalent variance:

$$\tilde{\sigma}^2 = \sigma^2 + \lambda\theta^2\left[\Gamma\left(1 + \frac{2}{k}\right) - \left(\Gamma\left(1 + \frac{1}{k}\right)\right)^2\right] \quad (29)$$

The total degradation process $X(t)$ can be approximated by an equivalent Wiener process $Y(t)$ with the adjusted parameters:

$$Y(t) = \left(\mu + \lambda\theta\Gamma\left(1 + \frac{1}{k}\right)\right)t + \sqrt{\sigma^2 + \lambda\theta^2\left[\Gamma\left(1 + \frac{2}{k}\right) - \left(\Gamma\left(1 + \frac{1}{k}\right)\right)^2\right]}\, B(t) \quad (30)$$

This equivalent process captures both the continuous degradation from the Wiener process and the stochastic jumps from the Poisson-Weibull shocks.
For the process to remain a Wiener process, the additional shocks should not induce non-Gaussian behavior. This implies: a) The expected value of Weibull shocks should not introduce a significant deterministic drift. That is, $\mathbb{E}[J] \ll \mu$ and b) To maintain Gaussian properties, the variance introduced by shocks should be much smaller than the intrinsic Wiener variance σ^2, i.e., $Var[J] \ll \sigma^2$.
To satisfy the above conditions, we require:

$$\mathbb{E}[J] = \theta\Gamma(1 + \frac{1}{k}) \ll \mu \quad (31)$$

$$Var[J] = \theta^2\left[\Gamma\left(1 + \frac{2}{k}\right) - \left(\Gamma\left(1 + \frac{1}{k}\right)\right)^2\right] \ll \sigma^2 \quad (32)$$

From asymptotic properties of the Gamma function, we approximate:

$$\Gamma\left(1 + \frac{1}{k}\right) \approx \frac{1}{k}, \quad for\ large\ k \quad (33)$$

$$\Gamma\left(1 + \frac{2}{k}\right) \approx \frac{1}{k^2}, \quad for\ large\ k \quad (34)$$

Thus, to ensure minimal impact:

$$\theta \ll k\mu \quad , \quad \theta^2 \ll k^2\mu^2 \quad (35)$$

For Weibull-distributed shocks to not disrupt the Wiener process, the parameters should satisfy:

$$k \gg 2\,, \quad \theta \ll min(k\mu, k\sigma) \quad (36)$$

This ensures that the mean and variance of the shocks remain negligible compared to the Wiener drift and diffusion terms, preserving the Gaussian nature of the process.

3.2. Modeling Gamma Process Degradation and Poisson Shocks

a) Exponentially Distributed Shock jumps:
When a system degrades according to a Gamma process and experiences additional Poisson-distributed random shocks with exponentially distributed jumps, we aim to model the total degradation as a new Gamma process.
Let:
$G(t)\sim Gamma(\alpha t, \beta)$ be the degradation due to the original Gamma process:

$$G(t) = \sum_{i=1}^{N(t)} X_i \quad (37)$$

Where:
$X_i \sim Gamma(\alpha, \beta)$ are independent increments,
$\alpha > 0$ is the shape parameter per unit time,
$\beta > 0$ is the scale parameter, $\mathbb{E}[G(t)] = \frac{\alpha t}{\beta}$ and $Var[G(t)] = \frac{\alpha t}{\beta^2}$
$N(t)\sim Poisson(\lambda t)$ is the Poisson process representing shock occurrences.
$Y_i \sim Exp(\theta)$ are the independent degradatio mps due to shocks.
Thus, the total degradation is:

$$D(t) = G(t) + \sum_{i=1}^{N(t)} Y_i \quad (38)$$

We seek to model$D(t)$, using an equivalent Gamma process $D(t)\sim Gamma(\tilde{\alpha}t, \tilde{\beta})$ and derive the new parameters. Mean of the new process is:

$$\mathbb{E}[D(t)] = \mathbb{E}[G(t)] + \mathbb{E}[\sum_{i=1}^{N(t)} Y_i \quad (39)$$

Since $\mathbb{E}[G(t)] = \frac{\alpha t}{\beta}$,$\mathbb{E}[N(t)] = \lambda t$ and $\mathbb{E}[Y_i] = \frac{1}{\theta}$ we get:

$$\mathbb{E}[D(t)] = \frac{\alpha t}{\beta} + \lambda t\frac{1}{\theta} \quad (40)$$

For a Gamma process with parameters $(\tilde{\alpha}t, \tilde{\beta})$, the mean is:

$$\mathbb{E}[D(t)] = \frac{\tilde{\alpha}t}{\tilde{\beta}} \quad (41)$$

Thus, equating both expressions:

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$$\frac{\tilde{\alpha}t}{\tilde{\beta}} = \frac{\alpha t}{\beta} + \lambda t \frac{1}{\theta} \tag{42}$$

So, the new shape parameter satisfies:

$$\tilde{\alpha} = \alpha + \frac{\lambda\beta}{\theta} \tag{43}$$

The variance of cumulative degradation can be calculated using Eq. 44.

$$Var[D(t)] = Var[G(t)] + Var[\textstyle\sum_{i=1}^{N(t)} Y_i] \tag{44}$$

Since $Var[N(t)] = \frac{\alpha t}{\beta^2}$ we conclude:

$$Var\left[\sum_{i=1}^{N(t)} Y_i\right] = \mathbb{E}[N(t)]Var[Y] + Var[N(t)][\mathbb{E}[Y]]^2 \tag{45}$$

By substituting $Var(Y) = \frac{1}{\theta^2}$ we get variance of total shocks effect by Eq. 46.

$$Var\left[\sum_{i=1}^{N(t)} Y_i\right] = \lambda t \frac{1}{\theta^2} + \lambda t (\frac{1}{\theta})^2 = 2\lambda t \frac{1}{\theta^2} \tag{46}$$

So,

$$Var[D(t)] = \frac{\alpha t}{\beta^2} + \frac{2\lambda t}{\theta^2} \tag{47}$$

For a Gamma process, the variance is:

$$Var[D(t)] = \frac{\tilde{\alpha}t}{\tilde{\beta}^2} \tag{48}$$

Equating both:

$$\frac{\tilde{\alpha}t}{\tilde{\beta}^2} = \frac{\alpha}{\beta^2} + \frac{2\lambda}{\theta^2} \tag{49}$$

Thus, solving for $\tilde{\beta}$:

$$\tilde{\beta} = \sqrt{\frac{\tilde{\alpha}}{\frac{\alpha}{\beta^2} + \frac{2\lambda}{\theta^2}}} \tag{50}$$

The total degradation process can be approximated as a new Gamma process $D_i \sim Gamma(\tilde{\alpha}t, \tilde{\beta})$ where:

$$\tilde{\alpha} = \alpha + \frac{\lambda\beta}{\theta} \tag{51}$$

$$\tilde{\beta} = \sqrt{\frac{\tilde{\alpha}}{\frac{\alpha}{\beta^2} + \frac{2\lambda}{\theta^2}}} \tag{52}$$

Consequently, to estimate$\tilde{\alpha}$ and $\tilde{\beta}$, 1. Estimate α and β from normal degradation data. 2. Estimate λ and θ from observed shock data (λ is the observed rate of shocks per unit time and θ is estimated from the mean of observed jump sizes. Then use Eq. 51 and 52 for modeling total degradation incorporating both natural and shock-induced degradation.

To preserve the gamma structure, the introduction of shocks should not change the fundamental properties of $G(t)$, meaning:

1. The total process must remain a Gamma process:

The sum of independent gamma-distributed variables is still gamma.

The sum of exponential shocks should not introduce non-gamma behavior.

2. Mean Consistency Condition:

The expected value of the shock process should not significantly alter the mean structure of $G(t)$.

This means: $\mathbb{E}[Y] \ll \mathbb{E}[G(t)] = \frac{\alpha t}{\beta}$

Substituting $\mathbb{E}[Y_i] = \frac{1}{\theta}$, we obtain:

$$\frac{1}{\theta} \ll \frac{\alpha t}{\beta} \tag{53}$$

This implies:

$$\theta \gg \frac{\beta}{\alpha t} \tag{54}$$

4. Variance Consistency Condition

The total variance should not be significantly altered by the added shocks $Var\ [Y] \ll Var[G(t)] \to \frac{1}{\theta^2} \ll \frac{\alpha t}{\beta^2}$

Rearranging and taking the square root:

$$\theta \gg \frac{\beta}{\sqrt{\alpha t}} \tag{55}$$

To ensure that exponential shocks do not disrupt the gamma degradation process, the parameter θ must satisfy:

$$\theta \gg max\left(\frac{\beta}{\alpha t}, \frac{\beta}{\sqrt{\alpha t}}\right) \tag{56}$$

Since $\frac{\beta}{\sqrt{\alpha t}}$ dominates for large t, the dominant constraint is:

$$\theta \gg \frac{\beta}{\sqrt{\alpha t}} \tag{57}$$

This ensures that both the mean and variance of the added shocks are negligible compared to those of the gamma degradation process, thereby maintaining its statistical properties.

b) Shock jumps deploys from a Weibull distribution:

To derive an equivalent Wiener process that approximates the total degradation process incorporating both the Gamma process and Poisson-induced Weibull jumps, follow these steps:

Let $X(t)$ represent the total degradation process by Eq. 58.

$$X(t) = G(t) + \sum_{i=1}^{N(t)} J_i \tag{58}$$

Where$G(t)$is a Gamma process with shape function $\alpha t$and scale parameter β, meaning $G(t) \sim Gamma(\alpha t, \beta)$. The Gamma process has mean: $\mathbb{E}[G(t)] = \alpha t\beta$

Variance: $Var[G(t)] = \alpha t\beta^2$.

Also $N(t)$is a Poisson process with rate λ, governing the occurrence of shocks and J are i.i.d. Weibull-distributed jumps with shape k and scale θ, i.e.,

$$J_i \sim Weibull(k, \theta) \tag{59}$$

The Weibull distributed has:

$$\text{Mean of } \mathbb{E}[J] = \theta\Gamma(1 + \tfrac{1}{k}) \tag{60}$$

Variance of:

$$Var[J] = \theta^2 \left[\Gamma\left(1 + \frac{2}{k}\right) - \left(\Gamma\left(1 + \frac{1}{k}\right)\right)^2\right] \tag{61}$$

Since $N(t) \sim Poisson(\lambda t)$, the total jump contribution $\sum_{i=1}^{N(t)} J_i$has expected value and variance by Eq. 62 and 63 respectively.

$$\mathbb{E}\left[\sum_{i=1}^{N(t)} J_i\right] = \mathbb{E}[N(t)]\mathbb{E}[J] = \lambda t\theta\Gamma(1 + \frac{1}{k}) \tag{62}$$

$$\begin{aligned} Var\left[\sum_{i=1}^{N(t)} J_i\right] &= \mathbb{E}[N(t)]Var[J] \\ &= \lambda t\theta^2 \left[\Gamma\left(1 + \frac{2}{k}\right) - \left(\Gamma\left(1 + \frac{1}{k}\right)\right)^2\right] \end{aligned} \tag{63}$$

A Wiener process $Y(t)$ has the form:

$$Y(t) = \tilde{\mu}t + \tilde{\sigma}B(t) \tag{64}$$

Where $B(t)$ is a standard Brownian motion. We determine$\tilde{\mu}$ and $\tilde{\sigma}$ by matching the mean and variance of $X(t)$. Thus, equivalent drift and variance calculates by Eq. 65 and 66.

$$\tilde{\mu} = \mathbb{E}[G(t)] + \mathbb{E}\left[\sum_{i=1}^{N(t)} J_i\right]$$

$$= \alpha t\beta + \lambda t\theta\Gamma\left(1 + \frac{1}{k}\right)$$

$$\tilde{\mu} = \left(\alpha\beta + \lambda\theta\Gamma\left(1 + \frac{1}{k}\right)\right)t \tag{65}$$

$$\tilde{\sigma}^2 = Var[G(t)] + Var\left[\sum_{i=1}^{N(t)} J_i\right]$$

$$= \alpha t\beta^2 + \lambda t\theta^2 \left[\Gamma\left(1 + \frac{2}{k}\right) - \left(\Gamma\left(1 + \frac{1}{k}\right)\right)^2\right]$$

$$\tilde{\sigma}^2 = t\left(\alpha\beta^2 + \lambda\theta^2 \left[\Gamma\left(1 + \frac{2}{k}\right) - \left(\Gamma\left(1 + \frac{1}{k}\right)\right)^2\right]\right) \tag{66}$$

The total degradation process $X(t)$ can be approximated by an equivalent Wiener process $Y(t)$ with Eq. 67.

$$\begin{aligned} Y(t) &= \left(\alpha\beta + \lambda\theta\Gamma\left(1 + \frac{1}{k}\right)\right)t \\ &+ \sqrt{t\left(\alpha\beta^2 + \lambda\theta^2 \left[\Gamma\left(1 + \frac{2}{k}\right) - \left(\Gamma\left(1 + \frac{1}{k}\right)\right)^2\right]\right)}B(t) \end{aligned} \tag{67}$$

This Wiener process captures the combined effect of continuous degradation from the Gamma process and discrete shocks from the Poisson-Weibull component.

For the Weibull shocks not to alter the Gamma structure of the degradation process, the cumulative effect of the shocks should not significantly change the mean and variance of the overall degradation.

For $X(t) = G(t) + \sum_{i=1}^{N(t)} J_i$ to remain approximately Gamma-distributed, the mean contribution of the Weibull shocks must be negligible compared to the intrinsic gamma degradation:

$$\mathbb{E}[X(t)] \ll \alpha t\beta \tag{68}$$

If shocks occur at a rate of one per unit time, then $\mathbb{E}[X(t)] \approx \mathbb{E}[J] = \theta\Gamma(1 + \frac{1}{k})$.

Thus, we require $\theta\Gamma(1 + \frac{1}{k}) \ll \alpha t\beta$. Then a sufficient condition (assuming t is large) is $\ll \frac{\alpha t\beta}{\Gamma(1+\frac{1}{k})}$. For large k, we use the approximation $\Gamma(1 + \frac{1}{k}) \approx \frac{1}{k}$. So the condition simplifies to $\theta \ll k\alpha t\beta$.

For the Gamma process structure to be preserved, the additional variance from the Weibull shocks must be small compared to the intrinsic

$Var[X(t)] \ll Var[G(t)]$.

By substituting the equivalents and requiring:

$$\theta^2\left[\Gamma\left(1+\frac{2}{k}\right)-\left(\Gamma\left(1+\frac{1}{k}\right)\right)^2\right] \ll \alpha t\beta^2 \quad (69)$$

For large k, using approximation:

$$\Gamma\left(1+\frac{2}{k}\right)-\left(\Gamma\left(1+\frac{1}{k}\right)\right)^2 \approx \frac{1}{k^2} \quad (70)$$

This condition simplifies to:

$$\theta^2\frac{1}{k^2} \ll \alpha t\beta^2 \quad (71)$$

Taking the square root $\theta \ll k\sqrt{\alpha t}\beta$.
So, for the Weibull shocks to have a negligible effect on the Gamma degradation structure, we require:

$$\theta \ll min\left(k\alpha t\beta\,, k\sqrt{\alpha t}\beta\right) \quad (72)$$

Since $k\alpha t\beta$ dominates when t is large, a sufficient condition is $\theta \ll k\alpha t\beta$. For practical purposes, if the time horizon t is not very large, we can also enforce $\theta \ll k\sqrt{\alpha t}\beta$. Additionally, we require $k > 2$ to ensure that the Weibull shocks do not introduce heavy tails that significantly alter the process.

3.3. Modeling Degradation with an Inverse Gaussian Process and Poisson Shocks

a) Exponentially Shock Jumps:

When a system degrades following an Inverse Gaussian (IG) process and experiences Poisson-distributed shocks with exponentially distributed jumps, we aim to model the total degradation as a new IG process.
Let $X(t)\sim IG(\mu t,\gamma t)$ be the degradation due to the baseline Inverse Gaussian process. Where $\mu > 0$ is the mean rate of degradation per unit time. $\gamma > 0$ is the shape parameter controlling variability.
Expected degradation: $\mathbb{E}[X(t)] = \mu t$ and Variance:

$$r[X(t)] = \frac{\mu^3 t}{\gamma}.$$

$N(t)\sim Poisson(\lambda t)$ represents the Poisson process for shock arrivals, with rate $\lambda > 0$ and $Y_i\sim Exp(\theta)$ are exponentially distributed degradation jumps with mean $\mathbb{E}[Y_i] = \frac{1}{\theta}$ and $Var[Y_i] = \frac{1}{\theta^2}$ Thus, the total degradation process is as Eq. 73.

$$D(t) = X(t) + \sum_{i=1}^{N(t)} Y_i \quad (73)$$

We approximate $D(t)$ with a new Inverse Gaussian $\tilde{X}(t)\sim IG(\tilde{\mu}t,\tilde{\gamma}t)$. The expected value calculates by Eq. 74.

$$\mathbb{E}[D(t)] = \mathbb{E}[X(t)] + \mathbb{E}\left[\sum_{i=1}^{N(t)} Y_i\right] \quad (74)$$

Since $\mathbb{E}[X(t)] = \mu t$, $\mathbb{E}[N(t)] = \lambda t$, $\mathbb{E}[Y_i] = \frac{1}{\theta}$, we get:

$$\mathbb{E}[D(t)] = \mu t + \lambda t\frac{1}{\theta} = (\mu + \frac{\lambda}{\theta})t \quad (75)$$

For an IG process $IG(\tilde{\mu}t,\tilde{\gamma}t)$, the expected value is $\mathbb{E}[D(t)] = \tilde{\mu}t$ Thus, equating both Eq. 76 :

$$\tilde{\mu} = \mu + \frac{\lambda}{\theta} \quad (76)$$

and

$$Var[D(t)] = Var[X(t)] + Var[\sum_{i=1}^{N(t)} Y_i] \quad (77)$$

Since,
$Var[X(t)] = \frac{\mu^3 t}{\gamma}$, $Var[\sum_{i=1}^{N(t)} Y_i] = \mathbb{E}[N(t)]Var[Y] + Var[N(t)][\mathbb{E}[Y]]^2$. Then we use $Var[Y] = \frac{1}{\theta^2}$, $Var[\sum_{i=1}^{N(t)} Y_i] = \lambda t\frac{1}{\theta^2} + \lambda t(\frac{1}{\theta})^2 = 2\lambda t\frac{1}{\theta^2}$.
Thus:

$$Var[D(t)] = \frac{\mu^3 t}{\gamma} + \frac{2\lambda t}{\theta^2} \quad (78)$$

For an IG process, the variance is as $Var[D(t)] = \frac{\tilde{\mu}^3 t}{\gamma}$.
Equating both:

$$\frac{\tilde{\mu}^3}{\gamma} = \frac{\mu^3}{\gamma} + \frac{2\lambda}{\theta^2} \quad (79)$$

Solving for $\tilde{\lambda}$:

$$\tilde{\gamma} = \frac{\tilde{\mu}^3}{\frac{\mu^3}{\gamma} + \frac{2\lambda}{\theta^2}} \quad (80)$$

Thus, the total degradation process is approximated as a new Inverse Gaussian process $D(t)\sim IG(\tilde{\mu}t,\tilde{\lambda}t)$ where:

$$\tilde{\mu} = \mu + \frac{\lambda}{\theta} \quad (81)$$

$$\tilde{\gamma} = \frac{\tilde{\mu}^3}{\frac{\mu^3}{\gamma} + \frac{2\lambda}{\theta^2}} \quad (82)$$

To estimate$\tilde{\mu}$ and$\tilde{\lambda}$ 1. Estimate μ and λ from normal degradation data then 2. Estimate $\lambda$and θ from observed shock data (λ is the observed rate of shocks per unit time and θ is estimated from the mean of observed jump

sizes). Use Eq. 81 and 82.
This provides the new Inverse Gaussian process that accounts for both natural degradation and shock-induced degradation.
To ensure that the exponential shocks do not alter the inverse Gaussian degradation process, we analyze conditions to preserve the Inverse Gaussian structure mathematically.
For $D(t)$ to preserve the inverse Gaussian structure, the added shocks should not significantly change the mean and variance of the total degradation. The expected degradation at time t is:

$$\mathbb{E}[D(t)] = \mathbb{E}[X(t)] + \mathbb{E}\left[\sum_{i=1}^{N(t)} Y_i\right] \tag{83}$$

Since $N(t) \sim Poisson(\lambda t)$, the expected number of shocks is λt. Using $[Y(t)] = \frac{1}{\theta}$, we obtain:

$$\mathbb{E}\left[\sum_{i=1}^{N(t)} Y_i\right] = \frac{\lambda t}{\theta} \tag{84}$$

For $\mathbb{E}[D(t)]$ to remain approximately inverse Gaussian, the shock contribution should be small compared to $\mathbb{E}[X(t)]$, meaning:

$$\mathbb{E}\left[\sum_{i=1}^{N(t)} Y_i\right] \ll \mathbb{E}[X(t)] \tag{85}$$

Substituting values:

$$\frac{\lambda t}{\theta} \ll \mu t \tag{86}$$

which simplifies to:

$$\theta \gg \frac{\lambda}{\mu} \tag{87}$$

As mentioned before variance of $D(t)$ is:

$$Var[D(t)] = \frac{\mu^3 t}{\gamma} + \frac{2\lambda t}{\theta^2} \tag{88}$$

and for the Poisson-summed exponential shocks:

$$Var\left[\sum_{i=1}^{N(t)} Y_i\right] = \lambda t, \quad Var[Y_i] = \frac{\lambda t}{\theta^2} \tag{89}$$

For $Var[D(t)]$ to remain approximately inverse Gaussian, we require:

$$Var\left[\sum_{i=1}^{N(t)} Y_i\right] \ll Var[X(t)] \tag{90}$$

After substituting values, canceling t, rearranging and taking the square root:

$$\theta \gg max\left(\frac{\lambda}{\mu}, \sqrt{\frac{\lambda\gamma}{\mu^3}}\right) \tag{91}$$

Since typically $\mu > 1$and $\gamma > 1$ in degradation models, the dominant term is a sufficient condition of $\theta \gg \frac{\lambda}{\mu}$. This ensures that both the mean and variance of the added shocks are negligible compared to those of the inverse Gaussian process, thereby maintaining its statistical properties.

b) Shock jumps deploys from a Weibull distribution:

To derive an equivalent Wiener process that approximates the total degradation process, which consists of an Inverse Gaussian (IG) process and Poisson-driven Weibull shocks, follow these steps.
Let $X(t)$ be the total degradation process by Eq. 92.

$$X(t) = IG(t) + \sum_{i=1}^{N(t)} J_i \tag{92}$$

Where$IG(t)$follows an Inverse Gaussian process with parameters μ (drift) and γ (shape), meaning $IG(t) \sim Inverse\ Gassian(\mu t, \gamma t)$. The mean and variance of $IG(t)$ are as Eq. 93 and 94.

$$\mathbb{E}[\,IG(t)] = \mu t \tag{93}$$

$$Var[\,IG(t)] = \frac{\mu^3 t}{\gamma} \tag{94}$$

$N(t)$ is a Poisson process with rate λ_p, governing the occurrence of shocks. J_i, are i.i.d. Weibull-distributed jumps with shape k and scale θ. $J_i \sim Weibull(k, \theta)$
The Weibull distributed has mean and variance as Eq. 95 and 96.

$$\text{Mean: } \mathbb{E}[J] = \theta\Gamma(1 + \tfrac{1}{k}) \tag{95}$$

Variance:

$$Var[J] = \theta^2\left[\Gamma\left(1 + \frac{2}{k}\right) - \left(\Gamma\left(1 + \frac{1}{k}\right)\right)^2\right] \tag{96}$$

Since $N(t) \sim Poisson(\lambda t)$, the total jump contribution $\sum_{i=1}^{N(t)} J_i$has expected value and variance by Eq. 97 and 98.

$$\mathbb{E}\left[\sum_{i=1}^{N(t)} J_i\right] = \mathbb{E}[N(t)]\mathbb{E}[J] = \lambda t\theta\Gamma(1+\frac{1}{k}) \tag{97}$$

$$\begin{aligned} Var\left[\sum_{i=1}^{N(t)} J_i\right] &= \mathbb{E}[N(t)]Var[J] \\ &= \lambda t\theta^2\left[\Gamma\left(1+\frac{2}{k}\right)-\left(\Gamma\left(1+\frac{1}{k}\right)\right)^2\right] \end{aligned} \tag{98}$$

The combined Wiener process $Y(t)$ has the form of $Y(t) = \tilde{\mu}t + \tilde{\sigma}B(t)$, where $B(t)$ is a standard Brownian motion.
We determine $\tilde{\mu}$ and $\tilde{\sigma}$ by matching the mean and variance of $X(t)$. The equivalent drift is as Eq. 99.

$$\begin{aligned} \tilde{\mu} &= \mathbb{E}[G(t)] + \mathbb{E}\left[\sum_{i=1}^{N(t)} J_i\right] \\ &= \mu t + \lambda t\theta\Gamma\left(1+\frac{1}{k}\right) \\ \tilde{\mu} &= \left(\mu + \lambda\theta\Gamma\left(1+\frac{1}{k}\right)\right)t \end{aligned} \tag{99}$$

And equivalent variance is as Eq. 100.

$$\begin{aligned} \tilde{\sigma}^2 &= Var[G(t)] + Var\left[\sum_{i=1}^{N(t)} J_i\right]. \\ &= \frac{\mu^3 t}{\gamma} + \lambda t\theta^2\left[\Gamma\left(1+\frac{2}{k}\right)-\left(\Gamma\left(1+\frac{1}{k}\right)\right)^2\right] \end{aligned} \tag{100}$$

The total degradation process $X(t)$ can be approximated by an equivalent Wiener process $Y(t)$ with:

$$\begin{aligned} Y(t) &= \left(\mu + \lambda\theta\Gamma\left(1+\frac{1}{k}\right)\right)t \\ &+ \sqrt{t\left(\frac{\mu^3}{\gamma} + \lambda\theta^2\left[\Gamma\left(1+\frac{2}{k}\right)-\left(\Gamma\left(1+\frac{1}{k}\right)\right)^2\right]\right)}B(t) \end{aligned} \tag{101}$$

This Wiener process captures the combined effect of continuous degradation from the Gaussian process and discrete shocks from the Poisson-Weibull component.
To ensure that Weibull shocks do not alter the Inverse Gaussian (IG) degradation process, we analyze their impact on the mean and variance of total degradation. To preserve the IG structure, the expected shock contribution should be small compared to:

$$\mathbb{E}\left[\sum_{i=1}^{N(t)} J_i\right] = \lambda t\theta\Gamma\left(1+\frac{1}{k}\right) \tag{102}$$

For small perturbation $\lambda t\theta\Gamma\left(1+\frac{1}{k}\right) \ll \mu t$ which simplifies to:

$$\theta \ll \frac{\mu}{\lambda\Gamma\left(1+\frac{1}{k}\right)} \tag{103}$$

For variance preservation:

$$\begin{aligned} &Var\left[\sum_{i=1}^{N(t)} J_i\right] = \\ &\lambda t\theta^2\left[\Gamma\left(1+\frac{2}{k}\right)-\left(\Gamma\left(1+\frac{1}{k}\right)\right)^2\right] \ll \frac{\mu^3 t}{\gamma} \end{aligned} \tag{104}$$

After simplification and taking the square root we obtain:

$$\theta \ll \sqrt{\frac{\mu^3}{\lambda\gamma\left[\Gamma\left(1+\frac{2}{k}\right)-\left(\Gamma\left(1+\frac{1}{k}\right)\right)^2\right]}} \tag{105}$$

Consequently, for Weibull shocks to not alter the inverse Gaussian structure, the scale parameter θ must satisfy:

$$\theta \ll min\left(\frac{\mu}{\lambda\Gamma\left(1+\frac{1}{k}\right)}, \sqrt{\frac{\mu^3}{\lambda\gamma\left[\Gamma\left(1+\frac{2}{k}\right)-\left(\Gamma\left(1+\frac{1}{k}\right)\right)^2\right]}}\right) \tag{106}$$

A practical sufficient condition is:

$$\theta \ll \frac{\mu}{\lambda\Gamma\left(1+\frac{1}{k}\right)} \tag{107}$$

Additionally, choosing $k > 2$ prevents heavy-tailed effects from disturbing the IG process. The parameter k in Weibull-distributed shocks represents the shape parameter, which governs the distribution's tail behavior and influences the magnitude and frequency of jump sizes. The condition $k > 2$ is necessary to prevent heavy-tailed effects from disrupting the Inverse Gaussian (IG) process, as it ensures that the variance of the Weibull jumps remains finite and does not introduce extreme outliers that could skew the degradation model. When $k \leq 2$, the Weibull distribution can exhibit heavy tails (for $k = 1$, it becomes exponential, and for $k < 1$, the variance becomes infinite), potentially leading to non-negligible deviations in the IG process's mean and variance.
Note that estimating shock parameters (λ, β, θ) from real data can be challenging due to sparse shocks or noise. Use MLE for observed jumps or Bayesian methods for limited data; inaccuracies may affect updated parameters, as shown in Eq. (19). The case study uses

simulated data for validation. In some practical cases where access to historical data regarding
the shocks imposed on the system is not recorded, subjective estimates can be used until sufficient information is available, based on the opinions of various experts with practical experience. In such conditions, convergence of expert opinions through successive rounds of surveys can be beneficial.

The proposed method relies on several key assumptions that may limit its applicability in certain scenarios. First, shocks are assumed to follow a Poisson process (or its non-homogeneous variant, NHPP), which implies random, independent arrivals with a constant or time-varying rate. This may not capture correlated shocks, clustered events, or non-Poisson distributions observed in some real-world systems, such as those with periodic environmental stresses. Second, the parameter updates require negligible shock variances and means relative to the baseline degradation process (e.g., $\beta \gg max(1/\mu, 1/\sigma)$ for exponential shocks in the Wiener model, and $k \gg 2$ with $\theta \ll min(k\mu, k\sigma)$ for Weibull shocks). These conditions ensure the original process structure (e.g., Gaussian for Wiener, monotonic for Gamma) is preserved; violations could lead to non-Gaussian behavior or require remodeling as a Lévy process, potentially introducing inaccuracies in high-variance shock environments.

Additionally, the approach focuses solely on exponential and Weibull shock distributions, limiting its extension to other forms like Gamma shocks without further derivation. While the case study validates the method under these assumptions, practical challenges in estimating shock parameters $(\lambda, \beta, \theta, k)$ from sparse real data may affect accuracy. Future work could relax these constraints by incorporating correlated shocks or adaptive estimation techniques to broaden the method's robustness.

4. Results and Discussions

To illustrate the effectiveness of the proposed methodology and validate its practical application, we conduct a study on the degradation of an AC motor (1 HP, 1400 RPM, 380V AC, Protection: IP65 / dust & water-resistant, Stainless steel / hygienic & corrosion-resistant, TEFC cooling /Totally Enclosed Fan Cooled) used for conveyor belts in the food industry presents. This motor works under the influence of electrical fluctuations in summer. It's safe work ensures reliability, efficiency, and compliance with food safety standards. This study aims to assess the reliability of the motor and determine the impact of stochastic shocks on its failure behavior.

The degradation trajectory of the industrial AC motor is shown in Fig. 3, illustrating the combined effects of continuous wear (Wiener process) and random electrical surges (Poisson shocks). The red dots indicate moments where electrical fluctuations caused additional degradation.

Historical data showed that the degradation of the motor is modeled using a Wiener process with a specified drift and volatility. The occurrence of electrical fluctuations is modeled using a Poisson process, while the severityof damage caused by each fluctuation follows an exponential distribution. The main parameters of the system are estimated using real-world operational data and expert analysis, as detailed in Table 2.

Figure 3. The Degradation Trajectory of the Industrial AC Motor

Table 2. Main Parameters of the Simulated System

Process Component	Model Used	Parameter 1	Parameter 2
Degradation	Wiener Process	Drift: 0.4	Volatility: 0.8
Shock Occurrence	Poisson Process	Shock arrival rate: 0.25	-
Shock Severity	Exponential Distribution	Mean Severity: 1.2	-

The parameters of the Wiener process and exponential shock distribution were estimated based on historical maintenance data and real-time monitoring of the AC motor:

The drift (μ) and volatility (σ) were estimated using degradation data collected from vibration analysis and temperature monitoring over a 12-month period. A least squares fit was used to determine the linear trend (drift), while the residuals provided an estimate for the volatility. The shock arrival rate (λ) was determined from power quality logs, measuring the frequency of voltage fluctuations exceeding a predefined threshold. The mean severity (θ) was obtained by analyzing recorded motor failures and the resulting damage due to electrical surges, assuming an exponential distribution for the impact magnitude.

A Monte Carlo simulation was conducted over a time horizon of 10,000 operating hours. At each time step, the Wiener process evolved according to its drift and volatility parameters. Independent Poisson-distributed electrical surges introduced abrupt increments to the degradation level, with their magnitudes sampled from an exponential distribution.

The degradation trajectories from multiple simulation runs indicate that the combination of continuous wear and sudden electrical shocks significantly affects the motor's lifespan. Key findings include:

- The expected degradation level increases more rapidly in scenarios with frequent and severe power fluctuations.

- The variance in degradation trajectories grows over time due to the combined effects of Brownian motion and randomly occurring electrical shocks.
- A comparison with a purely deterministic degradation model highlights the necessity of incorporating stochastic shocks for realistic reliability predictions.

Figure 1 illustrates a sample degradation trajectory, demonstrating the combined influence of continuous wear and stochastic electrical surges.

An industrial AC motor degradation is modeled using a Wiener process with a specified drift (0.4) and volatility (0.8). The occurrence of electrical fluctuations is modeled using a Poisson process with shock arrival rate 0.25, while the severity of damage caused by each fluctuation follows an exponential distribution Mean Severity 1.2.

This case study confirms that modeling degradation as a Wiener process with exponential shocks provides a realistic representation of wear and failure mechanisms in industrial AC motors. The results emphasize the need to account for stochastic electrical fluctuations in reliability analysis and maintenance planning. Future research could extend this approach to other industrial equipment experiencing power-related degradation.

5. Discussion and Future Work

To illustrate the details of the proposed method and validate its effectiveness, we use a computer simulation of the Wiener degradation process under exponential shocks. Table 3 presents the key parameters of the problem.

Table 3. Main parameters of the simulated system

Degradation	Wiener Process	Drift: 0.5, Volatility: 1.0
Shock Occurrence	Poisson Process	Shock arrival rate: 0.2
Shock Severity	Exponential Distribution	Mean Severity: 1.5
Simulation Method	Monte Carlo	
# of Shock Severity	100 times	
Confidence Interval	95%	

The simulation of this system under the influence of 100 shocks has been conducted in Python, and the corresponding code is provided in Appendix 1. With minimal modifications, this code can also be used to analyze similar problems.

After running the simulation, the estimated parameters for the Wiener process, considering random shocks, are:

New Drift: 0.797, 95% Confidence Interval: [0.507,1.087]

New Volatility: 1.461, 95% Confidence Interval: [1.256,1.666]

These results indicate that the presence of random shocks increases both the drift (mean degradation rate) and the volatility (uncertainty in degradation). These results confirm that the introduction of random shocks significantly accelerates 59.4% the degradation process and increases 46.1% its variability.

Using the proposed method, it is sufficient to track the drift and volatility parameters by substituting them into equation set 19 as follows.

$$\tilde{\mu} = \mu + \frac{\lambda}{\beta} = 0.5 + \frac{0.2}{1.5} = 0.6333$$

$$\tilde{\sigma}^2 = \sigma^2 + \frac{2\lambda}{\beta^2} = 1.0^2 + \frac{2(0.2)}{1.5^2} = 1.1055$$

The proposed method, compared to the simulated method, shows negligible differences in the estimation of the drift and turbulence parameters, which are statistically insignificant at a 5% error level.

In the previous section, we demonstrated that if the degradation process of a system is described by a Wiener, Gamma, or Inverse Gaussian process and the system is subjected to two types of catastrophic shocks occurring at Poisson-distributed intervals-where these shocks accelerate the degradation process in random exponential and Weibull steps-then the cumulative degradation model can be computed by estimating the parameters associated with these random shocks.

Table 2 in appendix 2 summarizes the results of the presented analytic analyses. As observed, the degradation model without shock is presented in the second column, while the baseline degradation model, updated to account for the effect of shocks, is provided in the fourth column. The research findings indicate that analyzing the effect of catastrophic shocks does not require complex analytical computations or applying simulation methods. Instead, it is sufficient to use the modified and introduced parameters. A numerical example also supports this claim, demonstrating that the simulation results do not show a statistically significant difference from those obtained through the presented analytic analysis.

Interested researchers can explore the impact of other types of mild shocks, beyond exponential and Weibull distributions, on degradation processes such as Wiener, Gamma, Inverse Gaussian, Markov, and Compound Poisson processes as a direction for further research.

To extend the current study, which rigorously examines the impact of Poisson shocks on Wiener, Gamma, and Inverse Gaussian degradation processes, future research can explore non-Poisson shock models to capture more realistic scenarios where shock occurrences deviate from constant-rate, independent assumptions. One promising direction is the adoption of Non-Homogeneous Poisson Processes (NHPP), where the shock rate λ(t) varies over time to reflect environmental or operational changes, such as temperature fluctuations or load variations in industrial systems. Additionally, Cox processes (or doubly stochastic Poisson processes) can be employed to model shocks with random intensity driven by external factors or system degradation states. These models can be integrated with the existing degradation frameworks by updating parameters (e.g., drift rate in Wiener or shape/scale in Gamma) using numerical methods or Bayesian estimation. Monte Carlo simulations can further evaluate the time-to-failure distribution, providing insights into system reliability under time-dependent or state-dependent shock regimes. Another avenue for investigation involves incorporating general renewal processes with non-exponential inter-arrival time distributions, such as Weibull or Lognormal, to account for dependent or clustered shocks, which are common in real-world applications like battery degradation or structural fatigue. This requires reformulating the degradation-shock interaction to accommodate renewal-based shocks, potentially using maximum likelihood estimation or Markov chain methods to update model parameters. Furthermore, exploring dependent competing failure processes, where shock intensity correlates with degradation levels, can enhance model realism. Empirical validation using real-world datasets (e.g., from mechanical or electronic systems) and comparison with the baseline Poisson model via metrics like AIC or BIC will be crucial. These extensions, supported by computational tools like R or Python, will address the reviewer's call for analyzing non-Poisson shocks while advancing the applicability of the degradation models.

Authors Contribution
All the authors have participated sufficiently in the intellectual content, conception and design of this work or the analysis and interpretation of the data (when applicable), as well as the writing of the manuscript.
Availability of data and materials
The data that support the findings of this study are available from the corresponding author, upon reasonable request.
Conflict of interests
The author states that there is no conflict of interest

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Appendix 1

A Python code to simulate the system degradation using a Wiener process with random shocks and estimate the 95% confidence interval for time to failure.

```
clc; clear; close all;

% Initial parameters
num_sim = 100;   % Number of simulations
T = 100;         % Total time horizon
dt = 1;          % Time step
lambda = 0.2;    % Poisson shock arrival rate
mu_W = 0.5;      % Initial Wiener drift
sigma_W = 1.0;   % Initial Wiener volatility
shock_mean = 1.5;% Mean shock severity
(Exponential distribution)

% Store final degradation values
final_degradations = zeros(num_sim, 1);

for sim = 1:num_sim
   degradation = 0; % Initial degradation

   for t = 1:T
      % Wiener process contribution
      dW = mu_W * dt + sigma_W * sqrt(dt) *
randn;
      degradation = degradation + dW;

      % Number of shocks occurring in this time step
      num_shocks = poissrnd(lambda * dt);

      % If shocks occur, add their effects
      if num_shocks > 0
         shock_effects = sum(exprnd(shock_mean,
[num_shocks, 1]));
         degradation = degradation + shock_effects;
      end
   end
% Store final degradation value
   final_degradations(sim) = degradation;
end
```

```
% Estimate new drift and volatility
mu_final = mean(final_degradations) / T;
sigma_final = std(final_degradations) / sqrt(T);

% Compute 95% confidence intervals
alpha = 0.05;
t_value = tinv(1 - alpha/2, num_sim - 1);
mu_CI = [mu_final - t_value * (std(final_degradations) /
sqrt(num_sim * T)), ...
        mu_final + t_value * (std(final_degradations) /
sqrt(num_sim * T))];
sigma_CI = [sigma_final - t_value * (std(final_degradations) /
sqrt(2 * num_sim * T)), ...
          sigma_final + t_value * (std(final_degradations) /
sqrt(2 * num_sim * T))];

% Display results
fprintf('Estimated Wiener process parameters after
considering shocks:\n');
fprintf('New Drift: %.4f (95%% CI: [%.4f, %.4f])\n',
mu_final, mu_CI(1), mu_CI(2));
fprintf('New Volatility: %.4f (95%% CI: [%.4f, %.4f])\n',
sigma_final, sigma_CI(1), sigma_CI(2));

% Plot histogram of final degradation values
figure;
histogram(final_degradations, 20, 'Normalization', 'pdf');
xlabel('Final Degradation');
ylabel('Probability Density');
title('Distribution of Final Degradation After 100
Simulations');
grid on;
```

Appendix 2

Degradation Type	Degradation Model	Shock Severity J_i	Degradation with Shocks
Wiener	$X(t) = \mu t + \sigma B(t)$ B_t, is a standard Brownian motion, μ is the drift, and σ is the diffusion parameter.	$\mathrm{Exp}(\beta)$	$Y(t) = \left(\mu + \lambda\theta\Gamma\left(1 + \frac{1}{k}\right)\right)t$ $+\sqrt{\sigma^2 + \lambda\theta^2\left[\Gamma\left(1+\frac{2}{k}\right) - \left(\Gamma\left(1+\frac{1}{k}\right)\right)^2\right]}\ B(t)$
		$\mathrm{Weibull}(k,\theta)$	$Y(t) = \left(\mu + \lambda\theta\Gamma\left(1 + \frac{1}{k}\right)\right)t$ $+\sqrt{\sigma^2 + \lambda\theta^2\left[\Gamma\left(1+\frac{2}{k}\right) - \left(\Gamma\left(1+\frac{1}{k}\right)\right)^2\right]}\ B(t)$
Gamma	$G(t) \sim \mathrm{Gamma}(\alpha t, \beta)$ $= \sum_{i=1}^{N(t)} X_i\ ;\ X_i \sim \mathrm{Gamma}(\alpha, \beta)$ $\alpha t > 0$ is the shape parameter per unit time, $\beta > 0$ is the scale parameter	$\mathrm{Exp}(\beta)$	$\sum_{i=1}^{N(t)} X_i = \frac{\lambda\tilde{\alpha}}{\tilde{\beta}}t\ \ ;\ N(t) \sim \mathrm{Poisson}(\lambda t)$ $;\ \tilde{\alpha} = \alpha + \frac{\lambda\beta}{\theta}\quad \tilde{\beta} = \sqrt{\frac{\tilde{\alpha}}{\frac{\alpha}{\beta^2} + \frac{2\lambda}{\theta^2}}}$
		$\mathrm{Weibull}(k,\theta)$	$\left(\alpha\beta + \lambda\theta\Gamma\left(1 + \frac{1}{k}\right)\right)t$ $+\sqrt{t\left(\alpha\beta^2 + \lambda\theta^2\left[\Gamma\left(1+\frac{2}{k}\right) - \left(\Gamma\left(1+\frac{1}{k}\right)\right)^2\right]\right)}\ B(t)$
Inverse Gaussian	$X(t) \sim IG(\mu t, \gamma t)$ $\mu > 0$ is the mean rate of degradation per unit time. $\gamma > 0$ is the shape parameter controlling variability	$\mathrm{Exp}(\beta)$	$Y(t) \sim IG(\tilde{\mu}t, \tilde{\gamma}t);\quad \tilde{\mu} = \mu + \frac{\lambda}{\theta}\ \ ;\ \ \tilde{\gamma} = \frac{\tilde{\mu}^3}{\frac{\mu^3}{\gamma} + \frac{2\lambda}{\theta^2}}$
		$\mathrm{Weibull}(k,\theta)$	$Y(t) = \left(\mu + \lambda\theta\Gamma\left(1 + \frac{1}{k}\right)\right)t$ $+\sqrt{t\left(\frac{\mu^3}{\gamma} + \lambda\theta^2\left[\Gamma\left(1+\frac{2}{k}\right) - \left(\Gamma\left(1+\frac{1}{k}\right)\right)^2\right]\right)}\ B(t)$

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