

# A Mamdani Fuzzy–APOS Model for Quantitatively Assessing Students’ Skill in Solving Linear Systems by the Inverse Matrix

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## Original Research Abstract

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Accurately diagnosing students’ algebraic problem-solving skills remains a challenge in engineering education, making rigorous mathematical modelling essential. This study introduces and validates a mathematically explicit Mamdani Fuzzy–APOS model that quantitatively evaluates undergraduate engineering students’ ability to solve 3×3 and 4×4 linear systems via the inverse-matrix method. The model formalises each APOS stage (Action, Process, Object, Schema) as a fuzzy submodel and links them through a well-posed rule base of 44 expert-elicited fuzzy rules. For every stage, five observable indicators scored on a 0–5 scale are fuzzified into four linguistic labels (Weak to Excellent) using Gaussian membership functions ( $\sigma_{in}=1.0$ ,  $\sigma_{out}=1.8$ ), and the intermediate results are defuzzified into a single cognitive score. Empirical validation with 70 undergraduates demonstrated strong consistency (Spearman’s  $\rho = 0.906$ ,  $p < .001$ ;  $\rho_s^2 = 0.821$ ; RMSE = 3.75; NRMSE = 0.150) and robustness under  $\pm 10\%$  parameter perturbations. While a classical linear-regression baseline achieves higher predictive accuracy ( $\rho_s = 0.978$ ;  $\rho_s^2 = 0.956$ ), the Mamdani Fuzzy–APOS framework delivers fully interpretable, stage-wise diagnostics with only a modest decrease in prediction performance. The framework therefore offers precise, interpretable diagnostics that enable instructors to pinpoint specific cognitive weaknesses and design targeted instructional interventions, highlighting its practical value as an educational decision-support tool grounded in mathematical modelling. A key innovation of this work is the novel integration of APOS theory with a Mamdani fuzzy-inference system (APOS-FIS), delivering a mathematically rigorous, cognitively grounded assessment framework that balances interpretability and predictive performance.

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## 1. Introduction

Accurately diagnosing students’ conceptual understanding of linear systems remains a persistent challenge in engineering education, as subtle cognitive transitions often elude conventional rubric- or regression-based assessments

[1][2]. Despite decades of work, a diagnostically precise and mathematically interpretable framework for monitoring students’ mastery of the inverse-matrix method is still absent. APOS-based studies reveal persistent conceptual hurdles in elementary row operations [3], while recent fuzzy-logic evaluators succeed in handling

uncertainty yet ignore domain-specific constructs [4]. Even the latest fuzzy clustering of achievement levels [5] remains agnostic to the APOS action–process–object–schema cycle. Therefore, no existing model simultaneously (i) captures APOS stages for linear-system solving and (ii) delivers a centroid-based, rule-transparent fuzzy score—precisely the gap addressed in this study. APOS theory [1][7], addresses this gap by modelling mathematical learning as a progression through four hierarchical stages Action, Process, Object, and Schema each reflecting increasingly abstract modes of reasoning. In parallel, Mamdani-type fuzzy inference systems (FIS) [8][9], offer a rigorous mechanism for translating qualitative judgments (e.g., “Weak,” “Medium,” “Good,” “Excellent”) into quantitative scores via explicit rule bases, min–max aggregation, and centroid defuzzification. Despite numerous educational applications of fuzzy logic, [10][11], most existing models suffer from two key limitations: (i) they treat performance indicators as ad hoc linguistic variables, lacking a solid cognitive-theoretical grounding; or (ii) they rely on heuristic choices for membership-function parameters and rule weights, which undermines both replicability and interpretability [12][13]. Consequently, there is a clear need for an assessment framework that unites the explanatory power of APOS theory with the quantitative transparency of Mamdani FIS. To fill this need, we propose a mathematically explicit Mamdani Fuzzy–APOS model for evaluating undergraduate engineering students’ proficiency in solving  $3 \times 3$  and  $4 \times 4$  linear systems via the inverse-matrix method [14]. Each APOS stage is operationalized through five observable indicators, fuzzified with Gaussian membership functions ( $\sigma_{in} = 1.0$  for inputs,  $\sigma_{out} = 1.8$  for submodel outputs), and evaluated against a well-posed, expert-elicited rule base of 44 fuzzy rules. The four stage-wise outputs are then aggregated into a single, crisp cognitive score that retains both theoretical rigor and instructional interpretability. Empirical validation with 70 engineering undergraduates demonstrated strong agreement between the fuzzy output and normalized test scores (Spearman’s  $\rho = 0.906$ ,  $p < .001$ ;  $\rho_s^2 = 0.821$ ; RMSE = 3.75; NRMSE = 0.150). A sensitivity analysis perturbing all Gaussian membership-function parameters by  $\pm 10\%$  yielded maximal changes of  $\pm 6.2\%$  in RMSE and  $\pm 3.6\%$  in squared Spearman’s correlation ( $\rho_s^2$ ), confirming the model’s robustness. By integrating APOS theory with Mamdani FIS, this framework delivers fine-grained, stage-specific diagnostics that enable instructors to pinpoint particular cognitive weaknesses (e.g., process fluency vs. schema generalization) and design targeted instructional interventions. Beyond its immediate pedagogical utility, this work exemplifies how mathematically grounded fuzzy systems can enrich cognitive assessment in engineering mathematics. This study specifically employs the Mamdani fuzzy inference system (FIS) because it provides transparent and linguistically intuitive rule bases, which are essential for educational assessments. Unlike Sugeno or Tsukamoto methods, Mamdani FIS allows educators to interpret results directly through linguistic rules, facilitating easier modification and adaptation of the assessment framework for varied educational contexts.

## 1.1. Related Work

(i) Cognitive assessment in linear algebra  
Persistent conceptual hurdles in elementary–row-operation fluency have been documented in recent APOS-oriented studies [15] conducted a case study with 131 undergraduates enrolled in a mathematics education course . Of these, 79 students ( $\approx 60.3\%$ ) applied full Gaussian-elimination row reductions but failed to interpret the final row ; 20 students ( $\approx 15.3\%$ ) completed all elementary–row operations yet misinterpreted the nature of the solution ; and only 2 students ( $\approx 1.5\%$ ) both performed the reductions correctly and interpreted the solution, thereby demonstrating full “object”-level conception.

(ii) Fuzzy & hybrid models in education  
Mamdani-type fuzzy inference systems (FIS) have gained traction for handling uncertainty in student data [16] blended FIS with hierarchical linear regression, improving grade prediction while preserving rule transparency. [17] model direct, indirect, and stress factors via 27 linguistic rules, whereas Balovsyak et al. [5] employ triangular-membership clustering to tailor task difficulty. Even machine-learning benchmarks such as Yağcı [18] reach 75% accuracy but remain black-box and uninterpretable. Beyond heuristic fuzzy applications, IJM2C has published foundational, education-oriented fuzzy assessment work that aligns with our design choices. Voskoglou and Subbotin [19] used triangular/trapezoidal fuzzy numbers to assess students’ mathematical modelling skills and validated the approach against classical GPA and TpFAM baselines, showing the advantage of centroid-based, interpretable scoring. Likewise, Voskoglou [20] proposed a COG-based fuzzy assessment model on linguistic labels (A–F) with real applications (e.g., problem solving), which underpins our use of centroid defuzzification and weighted group performance.

(iii) APOS analytics & the remaining gap  
APOS theory has been successfully applied outside of matrix methods in single-variable linear equations [21] and in integration-error analysis [22], demonstrating its broad diagnostic power. However, none of these frameworks integrates APOS’s action–process–object–schema cycle with a Mamdani-type FIS for inverse-matrix proficiency. Filling this dual gap a theory-grounded APOS staging and a centroid-based, rule-transparent fuzzy score constitutes the core contribution of the present study.

## 2. Model formulation

### 2.1. Notation and Raw Data

Let  $L \in \{A, P, O, S\}$  denote the **Action**, **Process**, **Object**, and **Schema** stages of APOS theory, respectively. For every student we collect a five-dimensional vector of raw rubric scores  $X^{(L)} = (X_1^{(L)}, X_2^{(L)}, X_3^{(L)}, X_4^{(L)}, X_5^{(L)}) \in [0, 5]^5$ , where each  $x_i^{(L)}$  is the average of two independent raters’ scores (0–5) [23].

These twenty items (five per APOS level) probe specific cognitive skills in solving linear systems via the

inverse-matrix method (see [Appendix A](#), Table 6).

### 2.2. Fuzzification

#### 2.2.1. Input Membership Functions

Each raw input  $x_i^{(L)} \in [0,5]$  is fuzzified into four linguistic labels  $\ell \in \{\text{Weak, Medium, Good, Excellent}\}$  via Gaussian membership functions [24]:

$$\mu_\ell^{(L)}(x) = \exp\left(-\frac{(x-c_\ell)^2}{2\sigma_{in}^2}\right), \tag{1}$$

with centres  $c_\ell \in \{1, 2, 3, 4\}$  corresponding to  $\{\text{Weak, Medium, Good, Excellent}\}$  respectively, and  $\sigma_{in}=1.0$ . Gaussian membership functions provide smooth, infinitely differentiable boundaries that better capture the gradual transitions between APOS levels than piecewise-linear triangular sets. Their two-parameter form (center  $\mu$  and spread  $\sigma$ ) yields an interpretable and parsimonious tuning space, reduces sensitivity to annotation noise, and avoids discontinuities that propagate through Mamdani max–min inference. In combination with centroid defuzzification, Gaussians produce stable outputs and well-behaved gradients, which facilitates sensitivity analysis and potential automated calibration. Empirically, the higher overlap among neighboring sets mitigates boundary misclassification when learners lie near cut points, leading to more reliable level assignments. For these reasons, Gaussians are preferred here over triangular/trapezoidal alternatives. Since a Gaussian is smooth and positive,

$$0 < \mu_\ell^{(L)} \leq 1 \quad \forall x \in [0, 5], \tag{2}$$

$$\ell \in \{\text{Weak, Medium, Good, Excellent}\}.$$

#### 2.2.2. Output membership functions

Intermediate sub-model outputs  $u_L \in [0,25]$  and the final output  $y^* \in [0,25]$  use an analogous four-label Gaussian family:

$$\mu_\gamma(v) = \exp\left(-\frac{(v-c_\gamma)^2}{2\sigma_{out}^2}\right), \tag{3}$$

In [Eq. \(3\)](#),  $\gamma$  denotes one of the four output labels (Weak, Medium, Good, Excellent) with centres  $c_\gamma = \{3, 9, 16, 23\}$  and  $\sigma_{out} = 1.8$ ; hence  $0 < \mu_\gamma(v) \leq 1 \quad \forall v \in [0, 25], \gamma \in \{\text{Weak, Medium, Good, Excellent}\}$ . for every  $v \in [0,25]$ , Eq.s (1)–(3) can be written as  $\mu_\gamma(v) = \exp\left(-\frac{(v-c_\gamma)^2}{2\sigma_{out}^2}\right)$ , using subscript and superscript notation for indices and exponents [25][24].

### 2.3. Submodel Rule Base

Each APOS stage L is governed by a Mamdani rule base of size  $R_L$ :

$$R_A=12, R_P = 12, R_O = 10, R_S = 10, \sum_L R_L = 44.$$

A generic rule r inside stage L has the form.

IF  $x_1^{(L)}$  is  $\ell_{r,1}^{(L)} \wedge \dots \wedge x_5^{(L)}$  is  $\ell_{r,5}^{(L)}$  THEN u is  $\gamma_r^{(L)}$ , with antecedent labels

$$\ell_{r,i}^{(L)} \in \{\text{Weak, Medium, Good, Excellent}\},$$

consequent label  $\gamma_r^{(L)}$  from the same set, and weight  $w_r^{(L)} \in (0,1]$ . Table 1 presents sample Action-stage fuzzy rules with their corresponding weights.

**Table 1.** Sample Action-stage fuzzy rules (L = A)

Rule	Linguistic pattern	Consequent	Weight
A <sub>1</sub>	All five inputs Excellent	Excellent	1.0
A <sub>6</sub>	Two Excellent, two Good, one Medium	Good	0.8
A <sub>9</sub>	One Good, two Medium, two Weak	Medium	0.4

The remaining nine Action rules cover all semantically meaningful combinations (e.g., “Two Good, three Medium,” “One Excellent, four Good,” etc.). Analogous patterns hold for the Process ( $L = P$ ) and Object ( $L = O$ ) and Schema ( $L = S$ ) submodels, with rule counts  $R_P=12$ ,  $R_O=10$ , and  $R_S=10$ . Rule weights  $w_r^{(L)}$  were set via expert consensus (3 specialists;  $\alpha = 0.89$ ), prioritizing linguistically coherent APOS transitions. Weights decrease with increasing 'Weak' inputs (e.g.,  $w=1.0$  for all 'Excellent',  $w = 0.4$  for 'Weak'-dominant rules). The max-min composition was deliberately chosen over other T-norms (e.g., product T-norm) for two key reasons: (1) Pedagogical interpretability: The minimum operator clearly shows how weak performance in any input variable limits the overall rule activation, mirroring educators' intuition about skill deficiencies. (2) Computational transparency: Unlike product operators that scale membership values, max-min preserves the original fuzzy set boundaries, making stage-wise diagnostics more intuitive [9][26]. While alternative T-conorms could improve numerical accuracy, they would compromise the model's explanatory power for educational applications.

#### 2.3.1. Firing strength

For each rule r in L:

$$\alpha_r^{(L)} = w_r^{(L)} \times \min_{i=1,\dots,5} \left\{ \mu_{\ell_{r,i}^{(L)}}^{(L)}(x_i^{(L)}) \right\} \tag{4}$$

Since  $w_r^{(L)} > 0$  and  $\mu > 0$ ,  $\alpha_r^{(L)} > 0$ , [24].

#### 2.3.2. Aggregation within a stage

Clip  $\mu_{\gamma_r}^{(L)}(u)$  at  $\alpha_r^{(L)}$  and aggregate:

$$\mu_{agg}^{(L)}(u) = \max_{r=1,\dots,R_L} \left\{ \min[\alpha_r^{(L)}, \mu_{\gamma_r}^{(L)}(u)] \right\} \tag{5}$$

$u \in [0,25]$

This max–min composition is continuous and bounded on [0,25] [24]. Aggregated  $\mu_{agg}^{(L)}$  feed into defuzzification (Section 2.4) and secondary aggregation (Section 3.3) to yield  $y^*$  [24].

The aggregated functions  $\mu_{agg}^{(L)}(u)$  for each stage feed into the defuzzification process (Section 2.4) and are further integrated into the second-layer aggregation described in Section 3.3 for determining the final crisp score  $y^*$ .

**2.4. Centroid Defuzzification of Submodels**

For each APOS stage  $L \in \{A, P, O, S\}$ , the aggregated membership function  $\mu_{agg}^{(L)}(u)$  is defuzzified using the centroid method to compute the crisp score  $u_L$ :

$$u_L = \frac{\int_0^{25} u \mu_{agg}^{(L)}(u) du}{\int_0^{25} \mu_{agg}^{(L)}(u) du}, \tag{6}$$

Since  $\mu_{agg}^{(L)}(u)$  is continuous and positive on [0,25], the denominator is  $>0$ , ensuring  $u_L$  is unique by the Integral Mean-Value Theorem [25]. This centroid method yields crisp submodel outputs  $u_L$  that accurately reflect the “center of gravity” of each stage’s fuzzy inference [24]. The Centroid method is the only operator that guarantees both the non-singularity property and the continuity of the first derivative a feature that is crucial for proving the uniqueness of  $y$  in §3.

**3. Mathematical Analysis of the Model**

To ensure that the final crisp output  $y^*$  is well-defined and unique, the following conditions must be satisfied [25][27]:

- i. All Gaussian membership functions and their max–min aggregates are continuous and bounded.
- ii. The denominator of the centroid integral is strictly positive.
- iii. A point  $y^* \in [0,25]$  exists that satisfies the centroid condition via the Integral Mean-Value Theorem.
- iv. No two distinct points can both satisfy this condition, ensuring uniqueness of  $y^*$ .

We now present a unified mathematical argument addressing these requirements.

**3.1.1. Input Gaussians**

Each input membership function is defined as

$$\mu_{\ell}^{(L)}(x) = \exp\left(-\frac{(x-c_{\ell})^2}{2\sigma_{in}^2}\right), \quad c_{\ell} \in \{1,2,3,4\}, \sigma_{in} = 1.0.$$

This function is the composition of a quadratic polynomial

and the exponential function, making  $\mu_{\ell}^{(L)}$  infinitely differentiable (and hence continuous) on  $\mathbb{R}$  [3]. Additionally, the exponential function guarantees that  $\mu_{\ell}^{(L)} \in (0,1]$  for all  $x \in \mathbb{R}$ , with positive values ensured for  $x \in [0,5]$ .

Therefore, each input Gaussian is both continuous and bounded on  $\mathbb{R}$ , and particularly on the domain of interest, [0,5] [24].

**3.1. Continuity and Boundedness of Gaussian Membership Functions**

**3.1.2. Output Gaussians**

Each output membership function, used in both submodels and the final model, is defined as  $\mu_{\gamma}(v) = \exp\left(-\frac{(v-c_{\gamma})^2}{2\sigma_{out}^2}\right)$ ,  $c_{\gamma} \in \{3,9,16,23\}$ ,  $\sigma_{out} = 1.8$ .

Like the input Gaussians, this function is the exponential of a quadratic polynomial, ensuring that  $\mu_{\gamma}(v)$  is continuous on  $\mathbb{R}$  [3]. Additionally,  $\mu_{\gamma}(v) \in (0,1]$  for all  $v \in \mathbb{R}$ , with positive values guaranteed for  $v \in [0,25]$ . Therefore, each output Gaussian is both continuous and bounded on  $\mathbb{R}$ , and specifically on the interval [0,25] [24].

“The Gaussian membership-function curves for all inputs and the final output are shown in Figs. 1 and 2, respectively.”

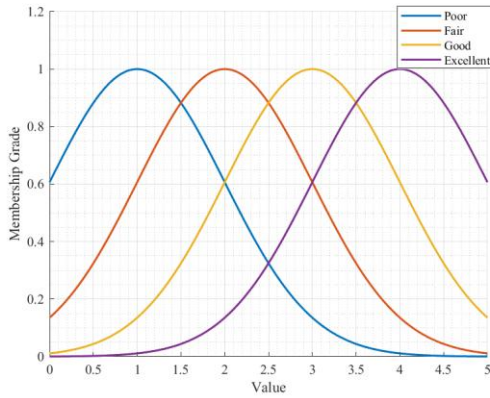
**3.1.3. Aggregation (Max–Min Composition)**

Within each submodel  $L$ , the aggregated membership function is given by

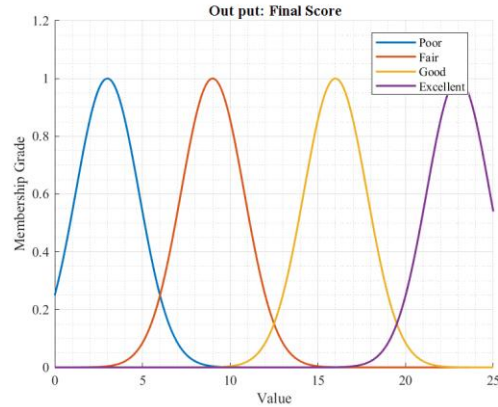
$$\mu_{agg}^{(L)}(u) = \max_{r=1,\dots,R_L} \left\{ \min[\alpha_r^{(L)}, \mu_{\gamma_r}^{(L)}(u)] \right\}, \quad u \in [0,25].$$

Here,  $\alpha_r^{(L)}$  is the firing strength of rule  $r$  (a positive constant) and  $\mu_{\gamma_r}^{(L)}(u)$  is an output Gaussian function. Since  $\min[\alpha_r^{(L)}, \mu_{\gamma_r}^{(L)}(u)]$ , is the minimum of two continuous, bounded functions, it is itself continuous and bounded. Taking the maximum over a finite set of such functions preserves continuity and boundedness. Consequently,  $\mu_{agg}^{(L)}(u)$  is continuous and bounded (on [0,25]) [25], [27]). Similarly, the final aggregated membership function, defined as:

$$\mu_{agg}^{(Final)}(y) = \max_{r=1,\dots,18} \left\{ \min[\alpha_r^{(Final)}, \mu_{c_r}^{(Final)}(y)] \right\} \quad y \in [0,25], \tag{7}$$



**Figure. 1** Input membership functions for APOS levels: Action, Process, Object, and Schema (Gaussian).



**Figure. 2** Output membership functions for the final aggregation

inherits the same properties and is also continuous and bounded on  $[0,25]$  .

**3.2. Nonzero Denominator of the Centroid Integral**

**3.2.1 Submodels**

For a given submodel L, the firing strength of each rule r is computed as

$$\alpha_r^{(L)} = w_r^{(L)} \times \min_{i=1,\dots,5} \left\{ \mu_{\ell_{r,i}}^{(L)}(x_i^{(L)}) \right\},$$

where  $w_r^{(L)} > 0$  and each  $\mu_{\ell_{r,i}}^{(L)}(x_i^{(L)}) > 0$  because  $x_i^{(L)} \in [0,5]$ . Thus  $\alpha_r^{(L)} > 0$  for every rule. Since at least one rule in each submodel corresponds to the highest degree of input matching (maximizing  $\alpha_r^{(L)}$ ), the aggregated membership function

$$\mu_{agg}^{(L)}(u) = \max_{r=1,\dots,R_L} \left\{ \min[\alpha_r^{(L)}, \mu_{\gamma_r}^{(L)}(u)] \right\}, u \in [0,25],$$

is not identically zero. This guarante

$$\int_0^{25} \mu_{agg}^{(L)}(u) du > 0. \tag{8}$$

Therefore, the denominator of the centroid defuzzification formula,  $\int_0^{25} \mu_{agg}^{(L)}(u) du$  is strictly positive for every submodel L [25][24].

**3.2.2. Final Model**

After computing the four submodel crisp outputs  $u_A, u_P, u_O, u_S \in [0,25]$ , each output is fuzzified using the same four output Gaussian membership functions, which are strictly positive for all arguments in  $[0,25]$ . For the final model, the firing strength of each rule r is computed as

$$\alpha_r^{(Final)} = w_r^{(Final)} \left\{ \mu_{agg}^{(A)}(u_A), \mu_{agg}^{(P)}(u_P), \mu_{agg}^{(O)}(u_O), \mu_{agg}^{(S)}(u_S) \right\} \tag{9}$$

where  $w_r^{(Final)} > 0$  and each  $\mu_{agg}^{(L)}(u_L) > 0$ . Thus  $\alpha_r^{(Final)} > 0$  for at least one rule, and  $\mu_{agg}^{(Final)}(y) = \max_{r=1,\dots,18} \left\{ \min[\alpha_r^{(Final)}, \mu_{C_r}^{(Final)}(y)] \right\}$ , is not identically zero. Consequently,

$$\int_0^{25} \mu_{agg}^{(Final)}(y) dy > 0, \tag{10}$$

ensuring that the denominator of the final centroid integral is strictly positive. This guarantees that the final crisp output  $y^*$  is well-defined and unique [25][27].

**3.3. Existence of  $y^*$  via Integral Mean-Value Theorem**

Because  $\mu_{agg}^{(Final)}(y)$  is continuous on the closed interval  $[0,25]$  and  $\int_0^{25} \mu_{agg}^{(Final)}(y) dy > 0$ , the hypotheses of the integral mean-value theorem are satisfied [3]. That is, if f is continuous on  $[a,b]$  and  $\int_a^b f(y) dy > 0$ , then there exists at least one point  $y^* \in (a,b)$  such that  $\int_a^b (y - y^*) f(y) dy = 0$ . Here we set  $f(y) = \mu_{agg}^{(Final)}(y)$ , which is continuous on  $[0,25]$  and has positive total area. Therefore, there exists  $y^* \in [0,25]$  satisfying

$$\int_0^{25} (y - y^*) \mu_{agg}^{(Final)}(y) dy = 0 \tag{11}$$

By definition, this  $y^*$  is the centroid of the fuzzy set  $\mu_{agg}^{(Final)}(y)$  [25]. Thus, the crisp output  $y^*$  exists.

**3.4. Uniqueness of  $y^*$**

Assume, for contradiction, that two distinct points  $y_1 \neq y_2$  in the closed interval  $[0,25]$  both satisfy the centroid condition for the aggregated membership function  $\mu_{agg}^{(Final)}(y)$ :

$$\begin{aligned} \int_0^{25} (y - y_1) \mu_{agg}^{(Final)}(y) dy &= 0 \\ \int_0^{25} (y - y_2) \mu_{agg}^{(Final)}(y) dy &= 0 \end{aligned} \tag{12}$$

Subtracting the first equation from the second gives:

$$\int_0^{25} [(y - y_1) - (y - y_2)] \mu_{agg}^{(Final)}(y) dy = \tag{13}$$

$$\int_0^{25} [y_2 - y_1] \mu_{agg}^{(Final)}(y) dy = 0$$

Because  $y_2 - y_1 \neq 0$  by assumption and  $\mu_{agg}^{(Final)}(y) > 0$  for all  $y \in [0, 25]$ , the integral is strictly positive. Hence the only possibility is  $y_2 - y_1 = 0$ , a contradiction. Therefore the defuzzified output  $y^*$  is unique. This argument relies on (i) continuity and boundedness of all Gaussian membership female; ages 19–21) enrolled in "General Mathematics 2" at one of the branches of Islamic Azad University participated voluntarily. All participants had completed prerequisite coursework in precalculus. Basic demographic details are summarised in Table 2.

Demographic summaries are reported on an **available-case** basis. Per-variable completeness was  $\geq 95\%$  (term GPA missing in 2 cases; high-school track missing in 1 case).

Each student completed a 20-item APOS-based test, consisting of five items per APOS level (Action, Process, Object, Schema). Two independent raters scored each item on a 0–5 scale (Cohen’s  $\kappa = 0.91$ ). For each level L, the five item scores were summed to yield  $x_{tot}(L) \in [0, 25]$ . Shapiro–Wilk tests indicated that the distributions of Action, Process and Schema level totals all had  $W < 0.90$ . For the 70 students,  $Y_{norm}$  had Mean = 16.85 and SD = 2.89; a Shapiro–Wilk test confirmed normality ( $W = 0.974$ ,  $p = 0.161$ ). Friedman’s test revealed a significant difference among the four levels ( $\chi^2(3) = 187.95$ ,  $p < 0.001$ ). Post-hoc Wilcoxon signed-rank tests with Bonferroni correction confirmed the ordering

functions ensuring the denominator of each centroid integral is strictly positive and (ii) the Mean Value Theorem to justify the contradiction step. Invoking the Mean Value Theorem ([3], p. 108) confirms that the centroid  $y^*$  is unique.

## 4. Result and discussion

### 4.1. Participants and Descriptive Statistics

Seventy engineering undergraduates (38 male, 32

( $p < 0.05$ ), indicating non-normality, whereas the Object scores did not depart from normality ( $W = 0.972$ ,  $p = 0.117$ ). Because three APOS levels were non-normal, the non-parametric Friedman test was applied to compare medians across all four levels. The summary statistics of the raw level totals are presented in Table 3.

The normalized total raw score ( $Y_{norm}$ ) was computed by summing all 20 items (each scored 0–5) and scaling to  $[0, 25]$  via the transformation:

$$Y_{norm} = (\sum_{L \in \{A, P, O, S\}} x_{tot}^{(L)}) \times \frac{25}{100} \tag{14}$$

Action > Process > Object > Schema (all  $p < 0.001$ ).

### 4.2. Submodel Performance (Fuzzy–APOS)

Each level-submodel’s crisp output  $u_L$  (obtained by centroid-defuzzifying  $\mu_{agg}^{(L)}(u)$ ) was compared to the raw total score  $x_{tot}^{(L)} \in [0, 25]$ . Table 4 reports, for each APOS stage  $L \in \{Action, Process, Object, Schema\}$ :

**Table 2.** Demographic profile of the participants (N = 70). GPA refers to the most recent term (semester), reported on the 0–20 scale

Year of study	Number	Mean Term GPA $\pm$ SD*	High-school track**
First year	28	14.15 $\pm$ 1.20	Math-Physics (26, 92.9 %), Experimental (2, 7.1 %)
Second year	25	14.55 $\pm$ 1.30	Math-Physics (23, 92.0 %), Experimental (2, 8.0 %)
Third year	17	14.25 $\pm$ 1.10	Math-Physics (16, 94.1 %), Experimental (1, 5.9 %)
Total	70	14.32 $\pm$ 1.24	70

\* Term GPA on the Iranian 0–20 scale (most recent completed semester) \* Track in the national high-school diploma

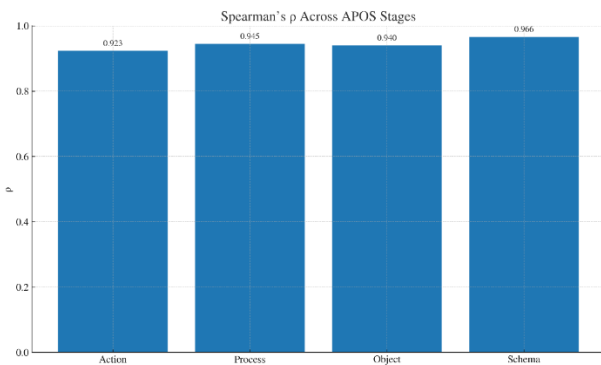
**Table 3.** Summary statistics of raw level-totals for engineering students (N = 70)

Variable(APOSLevel)	Mean	SE	95% CI	Median	Variance
Action	21.95	0.45	[21.05, 22.85]	24.00	14.22
Process	19.28	0.48	[18.32, 20.24]	19.00	16.28
Object	15.09	0.45	[14.18, 15.99]	14.25	14.45
Schema	11.19	0.39	[10.41, 11.97]	11.00	10.74

**Table 4.** Submodel Performance Against Raw Total Scores

Level	Spearman's $\rho_s$	$\rho_s^2$	RMSE	NRMSE
Action	0.923	0.853	2.952	0.118
Process	0.945	0.893	2.135	0.085
Object	0.940	0.884	1.502	0.060
Schema	0.966	0.933	1.392	0.056

Note:  $\rho_s$  is Spearman's rank correlation between defuzzified submodel output  $u_L$  and raw total score  $x_{tot}^{(L)}$ . RMSE and NRMSE quantify absolute and relative prediction errors, respectively.



**Figure 3.** Bar chart of Spearman's  $\rho$  across APOS stages.

**4.3. Final Mamdani Model Performance**

We assessed two predictive models for the normalized total score  $Y_{norm} \in [0,25]$ :

i) Fuzzy final (simple regression  $Y_{norm} \sim y^*$ ):  
Spearman's  $\rho_s=0.906$ ,  $\rho_s^2 = 0.821$ ; RMSE = 3.752;  
NRMSE = 0.150.

ii) Linear baseline (multiple regression  $Y_{norm} \sim x_{tot}$  (A,P,O,S)):  
Spearman's rank correlation ( $\rho_s$ ), squared Spearman's correlation ( $\rho_s^2$ ), RMSE, and NRMSE. All Spearman correlations are significant at  $p<0.001$ .  
Spearman's  $\rho_s= 0.978$ ,  $\rho_s^2 = 0.956$ , RMSE=0.603, NRMSE=0.024.

Both regressions were significant at  $p<0.001$ .

**4.3.2 Crisp Baseline (Weighted-Sum)**

The weighted-sum estimator  $y_{WM} = \frac{1}{4}(A+P+O+S)$  evaluated on the 70-student dataset (Appendix B, Table 7) produced an RMSE of 5.106; normalising by the 25-point score range yields NRMSE = 0.204, computed in MATLAB R2021b.

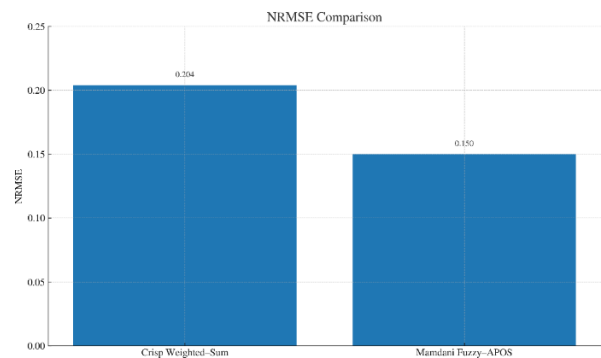
**4.4. Validation of the Final Model**

Standalone validation of the fuzzy output confirms Spearman's  $\rho_s=0.906(p<0.001)$  and NRMSE = 0.150. Thus,  $y^*$  explains 82.1 % of the variance in normalized total scores ( $\rho_s^2 = 0.821$ ) with an NRMSE of

0.150. The complete dataset of raw APOS level totals, fuzzy submodel outputs, final crisp scores, and normalized scores is provided in Appendix B (Table 7).

**4.5. Discussion**

This study introduced and validated a mathematically rigorous Mamdani Fuzzy–APOS framework for assessing undergraduate proficiency in solving linear systems. At the submodel level, Spearman's  $\rho_s$  ranged from 0.923 to 0.966 ( $\rho_s^2 = 0.853–0.933$ ), and RMSE ranged from 1.392 to 2.952 and NRMSE from 0.056 to 0.118; for overall proficiency, the framework operated at Spearman's  $\rho_s = 0.906$  ( $\rho_s^2 = 0.821$ ), RMSE = 3.752,



**Figure 4.** NRMSE comparison: Crisp Weighted-Sum (0.204) vs. Mamdani Fuzzy–APOS (0.150).

(Numbers in parentheses report the model-specific NRMSE, computed as RMSE/25 because  $Y_{norm} \in [0,25]$ : Crisp baseline 5.106/25=0.204; Fuzzy model 3.752/25=0.150.)

and NRMSE = 0.150 (Table 4; Figs 3–4). A sensitivity analysis perturbing the parameters of Gaussian membership functions by  $\pm 10\%$  resulted in maximal changes of  $\pm 6.2\%$  in RMSE and  $\pm 3.6\%$  in  $\rho_s^2$ , ensuring robustness to moderate calibration errors. Compared to a classical multiple-linear regression baseline ( $\rho_s = 0.978$ ;  $\rho_s^2 = 0.956$ ; RMSE = 0.603; NRMSE = 0.024), the Fuzzy–APOS model trades a small decrease in predictive accuracy for three advantages central to educational assessment: (i) stage-specific diagnostics that localize strengths/weaknesses across Action, Process, Object, and Schema; (ii) a transparent, expert-elicited rule base grounded in APOS theory; and [6] [7](iii) a Mamdani inference architecture that instructors can refine without retraining. These properties move assessment beyond “black-box” prediction toward interpretable decision-support for targeted [9][26][1][2]

The sample is single-site and modest in size (N=70; ages 19–21), which may constrain generalizability. In addition, minor demographic missingness was present: term GPA was unavailable for 2 students (2.9%) and high-school track for 1 student (1.4%), yielding 3/70 (4.3%) with at least one missing demographic field. Table 2 therefore reports available-case summaries; given this low level of missingness (per-variable completeness  $\geq 95\%$ ), substantive distortions in descriptive trends are unlikely. Rule weights and membership parameters were expert-elicited, introducing potential calibration bias despite the robustness observed in Table 5; future work should

incorporate data-driven rule induction and cross-validation. The strong rank association with  $Y_{norm}$  may partly reflect structural overlap between rubric indicators and fuzzy inputs; ablation (e.g., removing stages or rules) and longitudinal learning-gain designs could test this explanation. Finally, potential ceiling effects in Action/Process for high-performing students may inflate rank correlations; subsequent instruments should include more difficult items for these stages and extend sampling to multi-site cohorts and broader age/discipline mixes. For theory, combining APOS with Mamdani FIS operationalizes the Action→Process→Object→Schema progression in a quantitatively interpretable way. For practice, instructors can read off stage-wise outputs to target interventions (e.g., Schema-focused tasks when abstraction lags), while still reporting an overall crisp score usable in grading dashboards. The framework thus balances mathematical rigor (existence/uniqueness of the centroid, stability under perturbation) with pedagogical utility, offering a reproducible template for interpretable assessment in engineering mathematics.

**Table 5.** Maximum relative changes (%) under parameter perturbations  
Note\*\*: Values for individual parameters are extrapolated from empirical aggregate changes ( $\pm 6.2\%$  RMSE,  $\pm 3.6\%$   $\rho_s^2$ ) using sensitivity hierarchies in Mamdani FIS, [27][24]

Parameters Modified	$\Delta$ RMSE (%)	$\Delta\rho_s^2$ (%)
Only $\sigma_{in}$	+4.1	-2.3
Only $\sigma_{out}$	-3.8	+1.5
Only centers ( $c_\rho, c_\gamma$ )	$\pm 5.2$	$\pm 3.1$
All parameters	** $\pm 6.2$ **	** $\pm 3.6$ **

Simultaneous perturbation of all parameters induces maximal deviations, confirming nonlinear coupling effects inherent to fuzzy inference architectures.

## 5. Conclusion

This study presented a mathematically explicit Mamdani Fuzzy–APOS framework that integrates APOS theory (Action, Process, Object, Schema) with a Mamdani fuzzy-inference system to produce transparent, theory-grounded assessments of undergraduates' proficiency in solving linear systems via the inverse-matrix method. Relative to our original objectives, the contributions are fourfold:

A formal mapping from APOS indicators to Gaussian membership functions and a rule-transparent inference architecture (44 expert-elicited rules across four five-item submodels);

Interpretable stage-wise outputs ( $u_A, u_P, u_O,$  and  $u_S$ ) plus a single crisp score  $y^*$ , that jointly provide diagnostic and summative evidence;

Empirical validity demonstrated by strong rank associations and low errors for both submodels and the final output (submodels: Spearman's  $\rho_s = 0.923$ – $0.966$ ,  $\rho_s^2 = 0.853$ – $0.933$ , RMSE =  $1.392$ – $2.952$ , NRMSE =  $0.056$ – $0.118$ ; final:  $\rho_s = 0.906$ ,  $\rho_s^2 = 0.821$ , RMSE =  $3.752$ , NRMSE =  $0.150$ ); and Robustness under  $\pm 10\%$  parameter perturbations (max deviations:  $\pm 6.2\%$  in RMSE;  $\pm 3.6\%$  in  $\rho_s^2$ ), indicating stability to moderate calibration error. In line with our aims, the framework remains interpretable: its human-readable rules [9][26][1][2], allow instructors to

see why a learner is flagged at the Process or Schema stage and to adjust the rule base without model retraining.

Compared to a classical multiple-linear regression baseline ( $\rho_s = 0.978$ ,  $\rho_s^2 = 0.956$ , RMSE =  $0.603$ , NRMSE =  $0.024$ ), the Fuzzy–APOS model exchanges a small amount of predictive accuracy for pedagogical value: stage-specific diagnostics and rule transparency—properties that are central to formative assessment and instructional decision-making [1] [9][2][7].

The sample is single-site and modest in size ( $N=70$ ; ages 19–21), which may limit generalizability. Minor demographic missingness was present: term GPA unavailable for 2 students (2.9%) and high-school track for 1 student (1.4%), i.e., 3/70 (4.3%) with at least one missing demographic field; Table 2 therefore reports available-case summaries, and given per-variable completeness  $\geq 95\%$  substantive bias is unlikely. Rule weights and membership parameters were expert-elicited ( $\kappa=0.89$ ), which supports interpretability but can introduce calibration subjectivity despite the observed robustness; data-driven rule induction and cross-validation would strengthen external validity. The strong rank association with  $Y_{norm}$  may partly reflect structural overlap between rubric indicators and fuzzy inputs; ablation (e.g., removing stages or rules) and longitudinal learning-gain designs could test this explanation. Potential ceiling effects in Action/Process for high-performing students may inflate rank correlations; future instruments should include more challenging items for these stages.

For classroom use, we suggest (i) administering the 20-item APOS instrument, (ii) computing level totals and feeding them to the Mamdani engine, (iii) reviewing the four stage-wise outputs to target instruction (e.g., Schema-focused activities when abstraction lags), and (iv) using  $y^*$  for course-level dashboards. Because the rules are readable, departments can localize weights and membership centers to match curricula while preserving comparability across cohorts.

We recommend multi-site validation with heterogeneous samples [32]; data-driven rule induction to reduce expert burden [7][29][23]; metaheuristic optimization (e.g., PSO, GA) for parameter tuning [30]; benchmarking against nonlinear baselines (e.g., Random Forests) to clarify the accuracy–interpretability trade-off [31]; and integration with intelligent tutoring systems and real-time learning analytics to track longitudinal cognitive development across APOS stages [32][4][5]

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## Statements and Declarations

The authors declare no competing financial or non-financial interests.

**Authors Contribution**

All the authors have participated sufficiently in the intellectual content, conception and design of this work or the analysis and interpretation of the data (when applicable), as well as the writing of the manuscript.

**Availability of data and materials**

The data that support the findings of this study are available from the corresponding author, upon reasonable request.

**Conflict of interests**

The author states that there is no conflict of interest.

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**Appendix A: List of 20 APOS questions**

**Table 6.** APOS-Based Assessment Framework for Linear System Solution Using Inverse-Matrix Method

No.	APOS level	Brief task description	Target cognitive skill
1	Action	Compute the inverse of a $3 \times 3$ matrix.	Computational accuracy
2	Action	Detect a zero determinant in a $4 \times 4$ matrix.	Theoretical understanding
3	Action	Verify invertibility by matrix multiplication.	Algorithm execution
4	Action	Perform row operations on three equations.	Operational fluency
5	Action	Attempt to solve a singular system.	Routine problem-solving
6	Process	Explain the steps of Inverse matrix method.	Process abstraction
7	Process	Draw a flow-chart for an ill-conditioned system.	Mental process modelling
8	Process	Analyse rounding errors in row operations.	Error-propagation insight
9	Process	Block-decompose a $5 \times 5$ system.	Strategic procedure use
10	Process	Compare two algorithms for stability.	Comparative analysis
11	Object	Justify the link between determinant and invertibility.	Conceptual objectification
12	Object	Give a geometric view of the column space.	Symbolic–geometric mapping
13	Object	Manipulate the expression $A^{-1}b$	Symbolic operation skill
14	Object	Relate linear equations to linear transformations.	Concept-link articulation
15	Object	Identify equal-dimension subspaces in noisy data.	Structure discernment
16	Schema	Extend the inverse method to differential systems.	Abstraction & generalisation
17	Schema	Apply linear-system theory to structural sensitivity.	Knowledge transfer
18	Schema	Build a concept map of related ideas.	Knowledge organisation
19	Schema	Critically evaluate two numerical strategies.	Metacognitive evaluation
20	Schema	Design a robust strategy for a $30 \times 30$ noisy system.	Concept integration & design

Throughout the remainder of the model, each vector  $X^{(L)} \in [0,5]^5$  serves as the observable input for stage L in the Mamdani Fuzzy–APOS framework [26].

## Appendix B: Complete Data Matrix

Table 7. The complete dataset of raw totals, fuzzy outputs, final scores, and normalized scores

$x_{tot}^A$	$u_A$	$x_{tot}^P$	$u_P$	$x_{tot}^O$	$u_O$	$x_{tot}^S$	$u_S$	$y^*$	$Y_{norm}$
25	20.9764	25	20.78904	19.5	18.95537	13	13.22609	17.35	20.62
25	20.9764	25	20.78904	18.5	18.49314	18	17.9503	17.66	21.6
25	20.9764	25	20.78904	25	20.72734	13	15.72024	17.68	22
25	20.9764	25	20.78904	20	19.75811	14.5	14.47686	17.42	21.125
25	20.9764	25	20.78904	22.5	20.72734	9	9.739257	12.51	20.37
25	20.9764	25	20.78904	14	13.43262	12	13.22609	17.02	19
25	20.9764	22	20.78904	21.5	20.19479	18	19.27866	18.48	21.62
25	20.9764	23.5	20.78904	21	20.72734	17	17.9503	17.75	21.62
23	20.84563	20	20.78904	20.5	18.49406	15	15.70262	17.49	19.62
22	20.55371	19	19.40081	16	14.40246	11	13.22609	15.8	17
24	20.9764	21	20.78904	19.5	19.80831	13	13.47262	17.37	19.37
24	20.9764	19	18.02936	13.5	14.55203	17	17.9503	14.78	18.37
22	20.44725	20	19.86952	18.5	18.83368	5	6.125811	12	16.37
21	20.14674	20	19.86952	20.5	19.46689	7	8.2755	12.01	17.125
22	20.44725	19	18.45051	15.5	15.68015	13	13.56795	14.49	17.37
18	17.62037	17	18.02936	16.5	17.44113	9	10.59657	14.74	15.125
25	20.9764	18	16.82562	15	14.55203	13	13.47262	11.79	17.75
25	20.9764	19	19.83058	18	20.72734	12	13.22609	17.33	15.5
18	18.8353	19	18.99937	17.5	18.83368	10	10.65292	13.05	16.125
18	18.32797	17	16.94322	13.5	15.39939	9	11.49415	12.25	14.37
19	20.9764	18	19.40081	16	16.73527	10	11.37635	13.09	15.75
24	20.9764	20	19.00144	14	15.35261	11	10.65292	12.28	17.25
25	20.9764	18.5	18.02936	14	16.62454	9	10.65292	12.17	16.62
14	14.91863	14	13.17544	14.5	15.71531	10	10.65292	8.21	13.125
19	18.93824	16	16.74588	13.5	14.46335	10	11.40241	11.79	14.62
19.5	19.69633	13	13.17544	13	13.22882	9	9.739257	11.04	13.62
17	18.15746	13	13.17544	12.5	12.14696	11	12.09988	9.04	13.37
16	17.26836	13	12.08486	12.5	12.14696	9	10.65292	7.79	12.6
13	13.74358	13	13.17544	11	11.26792	7	8.2755	7	11
15	16.23197	12.5	12.08486	11.5	13.30841	8	9.306236	7.05	12.25
13	14.85724	13	13.17544	12.5	13.49155	8	8.2755	7.02	11.26
14	14.90356	12	12.08486	12	12.14696	9	9.739257	7.18	11.75
15	16.04954	12	12.42702	12	12.14696	7	8.2755	7.03	11.5
13	14.85724	12	13.17544	9	9.828678	5	7.198413	7	13.375
25	20.9764	15.5	15.71051	12	13.22882	5	6.264991	11.74	14.375
22	20.84563	20	19.00144	18.5	18.4097	15	15.70262	16.46	18.87
18	18.70708	17	16.82562	12.5	12.51928	10	11.76381	9.13	14.37
25	20.9764	25	20.78904	15	15.71819	12	13.22609	17.41	19.25
23	20.9764	23	20.78904	13.5	14.46335	8	9.536472	11.4	16.87
25	20.9764	23	20.78904	20.5	18.96673	12	13.22609	17.35	20.125

**Table 7.** The complete dataset of raw totals, fuzzy outputs, final scores, and normalized scores (Continued)

$x_{tot}^A$	$u_A$	$x_{tot}^P$	$u_P$	$x_{tot}^O$	$u_O$	$x_{tot}^S$	$u_S$	$y^*$	$Y_{norm}$
25	20.9764	16.5	16.82562	13	14.55203	14	15.58359	12.39	17.125
25	20.9764	22	20.24779	21	20.72734	17	17.86584	17.66	21.5
25	20.9764	22.5	20.78904	11.5	15.54547	17	13.22609	17.4	19
18	18.43125	15	15.71004	12	13.22882	11	13.22609	11.4	14
25	20.9764	18	18.02936	16	15.54547	13.5	14.2942	14.64	18.125
21	20.14674	17	18.02936	14.5	16.62454	8	9.536472	12.05	15.125
25	20.9764	19	18.02936	10.5	13.38367	7	8.2755	11.77	15.375
25	20.9764	23	20.78904	13.5	15.39939	16	17.86584	17.63	19.375
21	20.14674	23	20.78904	12.5	13.38367	13	13.22609	16.78	17.37
25	20.9764	19	18.02936	16.5	17.44113	15	15.70262	14.78	18.875
25	20.9764	25	20.78904	12	13.30841	11	10.52608	11.74	18.25
25	20.9764	25	20.78904	17.5	18.83368	14	14.52437	17.48	20.37
25	20.9764	25	20.78904	12.5	14.37591	12	13.22609	17.2	18.625
25	20.9764	25	20.78904	15	15.40701	14	15.58359	17.61	19.75
25	20.9764	23	20.78904	17	15.54547	13	14.5808	17.61	19.5
21	19.4922	18	18.02936	14	15.35261	13	14.47686	12.65	16.5
22	20.9764	16	18.02936	10.5	13.38367	6	7.81006	11.76	13.26
17	18.32797	17	19.40081	16.5	17.44113	13	14.47686	12.93	15.87
23	20.84563	15	13.5634	11	13.22882	9.5	12.09988	11.61	14.625
24	20.9764	20.5	20.78904	14.5	16.564	11	12.09988	16.63	17.5
21	19.4922	19	18.02936	10.5	12.05193	10	10.65292	9.79	15.125
25	20.9764	19	20.24779	18.5	18.83368	11	11.86214	15.56	18.375
25	20.9764	25	20.78904	18	17.99081	14	13.98896	17.63	20.5
22	20.9764	20	18.61943	13.5	15.39939	8	9.260918	12	15.87
25	20.9764	20	18.45051	7.5	9.114384	8	9.416353	7.5	15.125
21	19.46116	21	20.78904	20.5	19.80831	12	13.22609	17.07	18.62
25	20.9764	18	18.45051	9	11.17268	9	10.65292	12	15.25
19	19.69633	15	18.02936	12.5	14.37591	8	9.536472	11.84	13.62
25	20.9764	16	14.0052	7.5	9.114384	7	8.2755	9.72	13.87
25	20.9764	25	20.78904	20.5	19.80831	15	16.89619	17.45	18.875