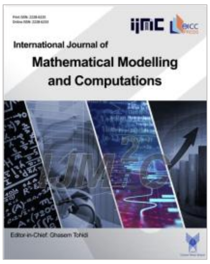

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A Two-Stage Approach for Modelling Undesirable Outputs in DEA with Production Trade-Offs: A Case on Chain Stores

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Abstract

Information about production trade-offs between inputs and outputs can be included in data envelopment analysis (DEA) models. In production processes, undesirable outputs are produced simultaneously with desirable outputs. We propose the production possibility set (PPS) with production trade-offs in the presence of undesirable outputs. This paper presents a two-stage process for measuring the efficiency of production units in the presence of undesirable outputs based on DEA with production trade-offs. In the first stage, the radial targets of the decision-making unit (DMU) under evaluation is calculated. In the second stage, we calculate the maximum amount of inefficiency slack corresponding to the components of inputs, desired outputs, and undesirable outputs. We prove that the targets obtained from the two-step process corresponding to inefficient DMUs are efficient units on the efficiency frontier of PPS. These targets have a minimum level of undesirable outputs. Also, by choosing the right direction in the presented models based on the directional distance function (DDF), we can obtain different efficient targets corresponding to each of the DMUs. We show that by changing the weight restrictions on the inputs and outputs, the efficiency score and the corresponding targets of the DMUs change. These weight restrictions are determined by the decision-maker (DM) in order to consider the importance of inputs and outputs. An application of the presented approach is presented to evaluate a set of chain stores, and at the end we present the results of the paper.

Keywords: Data envelopment analysis; Weight restrictions; Production trade-offs; Undesirable output; Weak disposability; Directional distance function.

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1. Intoduction

Traditional DEA models obtain the efficiency score of the DMU under evaluation based on the projection of this DMU on the efficiency frontier from the PPS[1][2]. This projection may not be a strongly efficient DMU and may be only weakly. A strongly efficient target corresponds to each DMU provided using two-stage approaches in literature DEA. These approaches, in the first stage, obtain the amount of radial improvement corresponding to the inefficient DMUs, and in the second stage, it obtains the maximum components slacks of inputs and outputs. This optimization can also be done in one step. This optimization is done in the multiplier DEA model by applying a lower bound on all input and output weights, which is equivalent to considering an additional term that multiplies $\epsilon > 0$ by the sum of input and output [3]. In addition to keep the assessment of radial efficiency as main goal and mix improvement as a secondary objective. Ali and Seiford [4] showed that the use of one-step approaches may lead to computational problems, and the value $\epsilon > 0$ should be chosen as a sufficiently small number. But using a sufficiently small number $\epsilon>0$ also has its own problems in calculating efficiency. Due to these drawbacks, two-step approaches are proposed to calculate radial efficiency (as radial improvement) and determine the maximum amount of the sum of component slacks (as non-radial improvement).

In production processes, in addition to producing desired outputs, other outputs are also produced simultaneously. These outputs are like sewage water and toxic gases. These outputs are dangerous for the environment and human health, and their production should reach the minimum possible level. However, in many cases, their production cannot be completely limited. We call these outputs undesirable. Several methods have been presented for measuring the efficiency score of DMUs in presence of undesirable outputs and calculating the optimal level of these outputs in the framework of DEA [5][6][7][8][9][10].

Now we examine some of the approaches provided in DEA to deal with undesirable outputs. Scheel [6] divided the methods of measuring efficiency score in the presence of undesirable outputs into two categories. Direct methods and indirect methods. Direct methods work on the original data, while indirect methods work on data transformations. Direct methods are closely related to the concept of weak disposability [11]. According to this assumption, undesirable outputs can only be reduced in proportion to desirable outputs. Song et al [7] divided the methods into data transformation, input reverse, and disposability-related methods. These methods apply to evaluating environmental efficiency in the presence of undesirable outputs Dakpo et al. [8] provided a review of papers on evaluating environmental efficiency in the presence of undesirable outputs and divided methods into five approaches as data transformation, free disposability of the inputs, weak disposability of the undesirable outputs, materials balance principles, and natural and managerial disposability, according to Sueyoshi and Goto [12].

Another method for dealing with undesirable outcomes in the framework of DEA is the concept of weak disposability. Considering the assumption of strong disposability, undesirable outputs can be reduced freely, but according to weak disposability, undesirable outputs can only be reduced in the same ratio as desirable outputs. Because undesirable outputs are produced simultaneously with desirable outputs. Färe et al. [13] developed an efficiency evaluation model based on weak disposability with common ratios for all DMUs. Färe et al. [14][15] applied the idea of Färe et al. [13] to calculate the maximum optimal outputs for all production units that can be produced if the undesirable outputs are optimally allocated among the production units. Ball et al. [16] built four models based on an idea to obtain shadow elasticities for undesirable outputs.

Kuosmanen [17] presented an individual-proportion weak disposability model to calculate efficiency. Kuosmanen and Podinovski [18] showed that the use of common ratios for all production units is not enough to correctly represent a convex technology that shows the weak disposability of desirable outputs and undesirable outputs, and if separate ratios are used to reduce desirable outputs and undesirable outputs, this technology is convex. Shen et al. [19] used the idea of individual-proportion weak disposability to calculate the efficiency of DMUs with undesirable outputs. Murty et al. [20] presented a by-product approach. They showed that production technology can be divided into two independent technologies: intended production and residual generation. In the first technology, inputs become intended outputs in production, and in the second, it reflects nature's residual generation mechanism, which is a relationship between undesirable outputs and inputs. The overall efficiency was defined as the average efficiency of producing desirable outputs and the efficiency of producing undesirable outputs. The main difference between the by-production approach and the weak disposability approach is that the former uses two different and independent technologies to describe the production of desirable and undesirable outputs, while the latter combines the production of desirable and undesirable outputs into one technology. Kao and Hwang [9] calculated the minimum amount of undesirable outputs that are produced in the production of desirable outputs under variable returns to scale (VRS) technology. This amount of undesirable output cannot be prevented in the production process and is produced simultaneously with desirable outputs. They measured the impact of producing an additional amount of undesirable output from each of the production units. Weak disposability assumes that the undesirable output is produced in equal proportion to the desirable output. Theoretically, the reduction ratio of undesirable outputs can be considered with the same ratio for all DMUs or different ratios for them in measuring the efficiency of a production unit. Kao and Hwang's [10] approach had two basic problems. First, they assumed that the undesirable outputs produced in the production process are only dependent on the produced desirable outputs and are independent of the amount of inputs, and this is inappropriate. Because the production of undesirable outputs is not independent of inputs, the role of inputs in calculating the minimum level of undesirable outputs cannot be ignored. They also used the common ratio of simultaneous reduction of desirable and undesirable outputs based on weak disposability and obtained the efficiency of the DMUs, which is relatively limited. To solve the above problems, they developed their previous approach. Then, Kao and Hwang [10] developed a new approach to calculate the minimum level of undesirable outputs that cannot be avoided in the production process of a production unit (based on the given input level). Their model was presented under two assumptions of weak disposability with common and separate ratios. They presented the efficiency of the production unit based on the minimum level of undesirable outputs while producing desirable outputs based on the inputs. They presented a new frontier of the PPS, which is defined based on the hypothetical units corresponding to each of the observed DMUs. These hypothetical units have a same level of inputs and desirable outputs equal to their corresponding original DMUs. However, they have a lower level of undesirable outputs than the original DMUs. The efficiency score of all DMUs is determined based on the new frontier. They also showed the effect of generating the undesirable outputs from the inefficiency of producing the desirable outputs in the production process. They stated that the difference between the efficiency (inefficiency) of the production unit and the corresponding hypothetical unit that produces the minimum level of undesirable outputs is a measure of the effect of producing an excessive amount of undesirable outputs on efficiency (inefficiency). They consider two forms of weak disposability: the former requires all production units to have the same proportion, and the latter allows each production unit to have its own proportion in reducing desirable and undesirable outputs. They show that the individual-proportion model is able to find lower levels of undesirable outputs in the production process.

To apply value judgement in DEA models, additional restrictions on components on input or output weights in multiplier models of DEA can be used. Applying weight restrictions in DEA models gives results an improved ability to differentiate unit efficiencies [21][22][23][24]. Podinovski [25][26] showed that applying weight restrictions to multiplier DEA models creates an additional term in envelopment DEA models. They considered this additional term as a production trade-off. They investigated the relationship between primal and dual models by applying weight restrictions and production trade-offs in the envelopment and multiplier DEA models, respectively. They showed that the application of weight restrictions in multiplier DEA models is equivalent to the presence of production trade-offs in envelopment DEA models. Podinovski [27] developed a three stages procedure for identifying efficient targets in DEA models with production trade-offs and weight restrictions. Podinovski and Bouzdine-Chameeva [28] showed that the application of weight restrictions in multiplier DEA models may lead to the infeasibility of models. They investigated these effects in detail. They showed that there are several fundamental problems in the presence of weight constraints in multiplier models and production trade-offs in envelopment DEA models. They showed that the use of weight restrictions may lead to zero or negative efficiency scores for some DMUs, and removing these problematic DMUs from the data set cannot solve the problems. They proved that in the presence of production trade-offs, the existence of free or unlimited production of outputs leads to problems. The multiplier model become infeasible and the envelopment model having an unbounded optimal solution. They provided the necessary and sufficient conditions to solve the above drawbacks. They developed analytical criteria and computational methods to identify problematic situations and test free and unlimited production. Podinovski and Bouzdine-Chameeva [29] proposed consistent weight restrictions in DEA. They investigated the necessary and sufficient conditions to have consistent weight restrictions. Podinovski [30] proved that the optimal solutions of DEA models in the presence of production trade-offs are optimal among all DMUs in the PPS and are not optimal only among the observed DMUs. Podinovski [31] proposed a single-stage DEA model with weight restrictions and obtained efficient targets for inefficient DMUs only by solving one model. Podinovski et al. [32] investigated DEA models with production trade-offs in the presence of ratio data. They applied their approach to measuring the efficiency of schools in England.

Efficiency evaluation using the DDF model was developed by Chambers, Chung, and Fare [33], and this model provides simultaneous expansion of outputs and reduction of inputs in an arbitrary direction. Chung et al. [34] developed this approach to include undesirable outputs in performance evaluation and showed that the model can be applied in a linear form. This model is used to measure efficiency in the framework of DEA in different subjects. For example, performance evaluation based on the slack-based measure (SBM) [13], the performance evaluation for network DEA structure [35], the super-efficiency analysis model [36], cost, revenue, and profit efficiency analysis [37] and performance analysis with undesirable factors [38].

In models based on the DDF, although the DM has the opportunity to choose an appropriate direction, this choice may increase the computational aspect. Therefore, knowing how to choose the right direction is important. Different directions also change the calculated efficiency scores. Wang et al.[39] studied how to choose the right direction in the DDF.

The main contribution of this paper is that we develop the PPS with production trade-offs in the presence of undesirable outputs. We present a two-stage approach to evaluate the efficiency score of production units that produce undesirable outputs at the same time as desirable outputs, based on the DDF model in DEA. This process obtains strongly efficient target for inefficient units on the efficiency frontier of production technology in the presence of production trade-offs. These targets also have a minimum level of undesirable outputs in the production process. The models in the two-stage process apply a general weight restriction to input, desirable output, and undesirable output components simultaneously. The weight restrictions are applied in a homogeneous form and as a link between the input and output components. In the two-stage process, we use radial models. Because in the presented model, the inefficiency slack values corresponding to undesirable outputs may be zero, and the approach provided by Kao and Hwang [9][10] in the form of SBM measures can no longer be used.

The continuation of the structure of this paper is as follows: In the second section, we introduce the production technology in the presence of undesirable outputs and production trade-offs. In the third section, we present a two-stage approach based on the DDF model in the presence of undesirable outputs and production trade-offs to evaluate the performance of the DMUs and obtain strongly efficient targets on the frontier of PPS. In the fourth section, we illustrate the model presented in this paper with a numerical example. In the fifth section, we use the proposed approach in this paper to evaluate the performance of a set of chain stores. In the sixth section, we present the results of the paper.

2. Production technology in the presence of production trade-offs

Consider a set of n DMUs in a production process. Each of these DMus has m inputs, s desirable outputs and l undesirable outputs. These DMU are as $DMU_j = (X_j, Y_j, U_j) \in R_+^{m+s+l}$, $j = 1, \dots, n$. Let $X_j = (x_{1j}, \dots, x_{mj}) \in R_+^m$, $Y_j = (y_{1j}, \dots, y_{sj}) \in R_+^s$, $U_j = (u_{1j}, \dots, u_{lj}) \in R_+^l$, $j = 1, \dots, n$ are inputs, desirable output, undesirable outputs vectors of DMU_j, $j = 1, \dots, n$, respectively. Undesirable outputs are by-products of the production of desirable outputs, and they cannot be freely disposed. They can be reduced only by the reduction ratio of desirable outputs under the assumption of weak disposability. There are two forms of weak disposability in the DEA literature, namely common-proportion and individual-proportion weak disposability. We can use these ratios to determine the minimum level of undesirable outputs that cannot be avoided in the process of producing outputs. Kuosmanen [17] stated that different mitigation factors can be considered to reduce undesirable outputs for different observed DMUs. Assume, we have k judgements specifying production trade-offs in the following form: (P_t, Q_t, φ_t), $t = 1, \dots, k$. The vectors P_t, Q_t, φ_t modify the inputs, desirable outputs and undesirable outputs of production unit respectively. Let $V \in R^m, U \in R^s, W \in R^l$ show the weight vectors correspond to the components of input, desirable output, and undesirable output respectively. The corresponding weight restrictions of these production trade-offs can be expressed as follows.

$$U^T Q_t - V^T P_t - W^T \varphi_t \le 0, t = 1, \dots, k.$$

In this section, by introducing the production axioms (postulates) of the PPS, we present the PPS in the presence of undesirable outputs and production trade-offs from inputs and outputs as follows:

(P1) Inclusion of observations:

This axiom states that all observed DMUs belong to the production technology. $DMU_j = \left(X_j, Y_j, U_j\right) \in T, \quad j = 1, \dots, n.$

(P2) Convexity:

This axiom states that the convex composition of all feasible DMU belongs to the production technology.

Let $(X_1, Y_1, U_1) \in T, \quad (X_2, Y_2, U_2) \in T,$ then $\lambda(X_1, Y_1, U_1) + (1-\lambda)(X_2, Y_2, U_2) \in T,$ $0 \le \lambda \le 1.$

(P3) Disposability.

(P3-1) Disposability of inputs:

This axiom states that if an input vector can be produced, then input vectors greater than or equal to this vector can also be produced.

$$\forall\, (X, Y, U), \forall \bar{X},$$
$$[(X, Y, U) \in T, \bar{X} \ge X \implies (\bar{X}, Y, U) \in T].$$

(P3-2): Disposability of desirable outputs:

This axiom states that if a desirable output vector can be produced, then desirable output vectors smaller or equal to this vector can also be produced.

$$\forall\, (X, Y, U), \forall \bar{Y},$$
$$[(X, Y, U) \in T, \bar{Y} \le Y \implies (X, \bar{Y}, U) \in T].$$

(P3-3): Weak disposability of undesirable outputs:

This axiom states that undesirable outputs can only be reduced by the same ratio of desirable outputs in production technology.

$$\forall (X, Y, U),$$
$$\forall \sigma, [(X, Y, U) \in T, 0 \le \sigma \le 1 \implies (X, \sigma Y, \sigma\, U) \in T]$$

(P3-4): Disposability of undesirable outputs (Modify):

This axiom states that if an undesirable output vector can be produced, then undesirable output vectors greater than or equal to this vector can also be produced.

$$\forall (X, Y, U), \forall \bar{U},$$
$$[(X, Y, U) \in T, U \le \bar{U} \implies (X, Y, \bar{U}) \in T].$$

(P4) Feasibility of trade-offs.

$$\forall\, (X, Y, U), \forall \pi_t \ge 0, \forall (P_t, Q_t, \varphi_t),$$
$$[(X, Y, U) \in T \implies (X + P_t, Y + Q_t, U + \varphi_t) \in T].$$

Provided $X + P_t \ge 0, Y + Q_t \ge 0, U + \varphi_t \ge 0.$

(P5) Minimum extrapolation principle:

This principle states that T is the smallest set that applies to the above axioms. In other words, T is the subscription of all sets that apply to axiom P1-P4.

By accepting production axioms **(P1), (P2), (P3-1), (P3-2), (P3-3), (P4), (P5)**, we can present the PPS in the presence of production trade-offs as follows.

$$T^{UC}_{VRS-To} = \tag{1}$$
$$\left\{ (X, Y, U) \;\middle|\; \begin{array}{c} \sum_{j=1}^{n} \lambda_j X_j + \sum_{t=1}^{k} \pi_t P_t \le X, \sum_{j=1}^{n} \lambda_j \sigma Y_j \\ + \sum_{t=1}^{k} \pi_t Q_t \ge Y, \\ \sum_{j=1}^{n} \lambda_j \sigma U_j + \sum_{t=1}^{k} \pi_t \varphi_t = U, \sum_{j=1}^{n} \lambda_j = 1, \\ 0 \le \sigma \le 1, \pi_t \ge 0 \end{array} \right\}$$

Undesirable outputs are by-products of the production of desirable outputs, and they cannot be produced freely. They can be reduced only by proportionally reducing the desired outputs under the assumption of weak disposability. To determine the minimum level of undesirable outputs, there are two forms of common-proportion and individual-proportion weak disposability, which cannot be avoided in producing desirable outputs. In the technology introduced above, we considered the common reduction ratio, but we can consider the reduction ratio differently for different observed DMUs. In the above discussion, the reduction factor σ is assumed to be the same for all DMUs. Kuosmanen[17] relaxed this assumption to allow each DMU to have its own corresponding reduction ratio, i.e., σ_j. Therefore, the PPS, under the assumption of individual-proportion weak disposability, is presented as follows:

$$T^{UV}_{VRS-To} =$$
$$\left\{ (X, Y, U) \;\middle|\; \begin{array}{c} \sum_{j=1}^{n} \lambda_j X_j + \sum_{t=1}^{k} \pi_t P_t \le X, \\ \sum_{j=1}^{n} \lambda_j \sigma_j Y_j + \sum_{t=1}^{k} \pi_t Q_t \ge Y, \\ \sum_{j=1}^{n} \lambda_j \sigma_j U_j + \sum_{t=1}^{k} \pi_t \varphi_t = U, \\ \sum_{j=1}^{n} \lambda_j = 1, 0 \le \sigma_j \le 1, \pi_t \ge 0 \end{array} \right\} \tag{2}$$

Now we introduce a two-stage process to obtain efficient targets in T_{VRS-To}^{UV} technology. We show that these targets are on the efficient frontier of the set T_{VRS-To}^{UV}. In the first stage, we obtain the radial efficiency of the DMU under evaluation, i.e. DMU_o, by solving model (3) based on the DDF model as follows.

$$\max \ \beta_o^{TO}$$

$$s.t. \ \sum_{j=1}^{n} \lambda_j x_{ij} + \sum_{t=1}^{k} \pi_t P_{it} \leq x_{io} - \beta_o^{TO} g_{x_i}, \quad i = 1, \dots, m,$$

$$\sum_{j=1}^{n} \lambda_j \sigma_j u_{fj} + \sum_{t=1}^{k} \pi_t \varphi_{ft} = u_{fo} - \beta_o^{TO} g_{u_f}, \quad f = 1, \dots, l,$$

$$\sum_{j=1}^{n} \lambda_j \sigma_j y_{rj} + \sum_{t=1}^{k} \pi_t Q_{rt} \geq y_{ro} + \beta_o^{TO} g_{y_r}, \quad r = 1, \dots, s,$$

$$\sum_{j=1}^{n} \lambda_j = 1, 0 \leq \sigma_j \leq 1, \qquad (3)$$

$$\sum_{j=1}^{n} \lambda_j x_{ij} + \sum_{t=1}^{k} \pi_t P_{it} \geq 0, \ i = 1, \dots, m,$$

$$\sum_{j=1}^{n} \lambda_j \sigma_j u_{fj} + \sum_{t=1}^{k} \pi_t \varphi_{ft} \geq 0, \ f = 1, \dots, l,$$

$$\lambda_j \geq 0, \ j = 1, .., n, \pi_t \geq o, t = 1, \dots, k.$$

Kuosmanen [17] used the following transformations to transform the nonlinear equations in the set T_{VRS-To}^{UV}.
$\mu_j = \lambda_j \sigma_j, (1 - \sigma_j)\lambda_j = \delta_j,$ then $\lambda_j = \mu_j + \delta_j$.
Model (3) becomes model (4).

$$\beta_o^{TO^*} = \max \ \beta_o^{TO}$$

$$s.t. \ \sum_{j=1}^{n} (\mu_j + \delta_j) x_{ij} + \sum_{t=1}^{k} \pi_t P_{it} \leq x_{io} - \beta_o^{TO} g_{x_i}, \ i = 1, \dots, m,$$

$$\sum_{j=1}^{n} \mu_j u_{fj} + \sum_{t=1}^{k} \pi_t \varphi_{ft} = u_{fo} - \beta_o^{TO} g_{u_f}, \ f = 1, \dots, l,$$

$$\sum_{j=1}^{n} \mu_j y_{rj} + \sum_{t=1}^{k} \pi_t Q_{rt} \geq y_{ro} + \beta_o^{TO} g_{y_r}, \ r = 1, \dots, s,$$

$$\sum_{j=1}^{n} (\mu_j + \delta_j) = 1, \qquad (4)$$

$$\sum_{j=1}^{n} (\mu_j + \delta_j) x_{ij} + \sum_{t=1}^{k} \pi_t P_{it} \geq 0, \ i = 1, \dots, m,$$

$$\sum_{j=1}^{n} \mu_j u_{fj} + \sum_{t=1}^{k} \pi_t \varphi_{ft} \geq 0, f = 1, \dots, l, \ t = 1, \dots, k,$$

$$\mu_j \geq 0, \ \delta_j \geq o, j = 1, \dots, n, \pi_t \geq o, \ t = 1, \dots, k.$$

In model (4), x_{ij} is the i-th input component, y_{rj} is the r-th desired output component, and u_{fj} is the f-th undesirable output component of DMU_j.
$g = \left(-g_{x_i}, g_{y_r}, -g_{u_f}: \ i = 1, \dots, m, f = 1, \dots, l, r = 1, \dots, s\right)$, show changes in components of inputs, desirable outputs and undesirable outputs. Negative values related to inputs and undesirable outputs that decrease. Positive values correspond to desirable outputs that increase. The factor β_o^{TO} shows the spread of inefficiency in the DMUs. This factor corresponds to the maximum expansion of desirable outputs and the reduction (contraction) of inputs and undesirable outputs that are obtained simultaneously. In model (4), inputs and desirable outputs are assumed to be strongly disposable and undesirable outputs are assumed to be weakly disposable, as shown in the corresponding constraints. We assume that the undesirable outputs of by-products are desirable outputs and cannot be reduced without cost; this concludes that abatement of undesirable outputs is possible. If it is accompanied by a decrease in desirable outputs or an increase in an input, in model (4), the reduction ratio of undesirable outputs is different for all DMUs according to the principle of weak disposability.

Theorem 1. The model (4) is always feasible.

Proof. Put $\mu_{oo} = 1$ and $\mu_{jo} = 0, j = 1, \dots, n, \ j \neq o$, $\delta_j = 0, j = 1, \dots, n, \ g_{x_i} = 0, i = 1, \dots, m, \ g_{u_f} = 0, f = 1, \dots, l, g_{y_r} = 0, r = 1, \dots, s,$
$\pi_t = 0, t = 1, \dots, L$, in this case, we obtain a feasible solution for model (4) and the proof is completed. ■

To solve model (4), we choose the direction vector model (4) as follows.
$g = (-g_x, g_y, -g_u) = (-X_o, Y_o, -U_o)$.
In this case, suppose $\left(\mu^*, \delta^*, \pi^*, \beta_o^{TO^*}\right)$ is an optimal solution of model (4). Suppose that the radial projection corresponding to $DMU_o = (X_o, Y_o, U_o)$ resulting from model (4) is shown below.

$$(X_o^*, Y_o^*, U_o^*) = \left(\left((1 - \beta_o^{TO^*})X_o, (1 + \beta_o^{TO^*})Y_o, (1 - \beta_o^{TO^*})U_o\right)\right).$$

We define strong efficiency with undesirable outputs as follows.
Strong efficiency in the presence of undesirable outputs refers to a situation in which a DMU lies on the efficient frontier in such a way that no further improvement is possible in any input or output without causing a deterioration in at least one other dimension. More precisely:
1. No input can be reduced unless at least one other input increases, or desirable outputs decrease, or undesirable outputs increase.
2. No desirable output can be increased unless it requires more inputs, an increase in undesirable outputs, or a reduction in other desirable outputs.
3. No undesirable output can be reduced unless it causes an increase in inputs or a reduction in desirable outputs.
In other words, under strong efficiency, no slacks exist in inputs, desirable outputs, or undesirable outputs, and the DMU is located on the strictly efficient frontier of the PPS.
In order to obtain strongly efficient targets, in the second stage of the presented process, we solve model (5) to calculate the maximum of the sum of slacks in input, desirable output, and undesirable output components for accounting possible mix inefficiency.

$$\max \ \left(\sum_{i=1}^{m} d_i + \sum_{r=1}^{s} e_r + \sum_{f=1}^{l} s_f\right)$$

$$s.t. \ \sum_{j=1}^{n} (\mu_j + \delta_j) x_{ij} + \sum_{t=1}^{k} \pi_t P_{it} + d_i = x_{io}^*, \ i = 1, \dots, m,$$

$$\sum_{j=1}^{n} \mu_j \, u_{fj} + \sum_{t=1}^{k} \pi_t \, \varphi_{ft} + s_f = u_{fo}^*,$$

$f = 1, \dots, l,$

$$\sum_{j=1}^{n} \mu_j \, y_{rj} + \sum_{t=1}^{k} \pi_t \, Q_{rt} - e_r = y_{ro}^* \,,$$

$r = 1, \dots, s,$

$$\sum_{j=1}^{n} (\mu_j + \delta_j) = 1, \qquad (5)$$

$$\sum_{j=1}^{n} (\mu_j + \delta_j) \, x_{ij} + \sum_{t=1}^{k} \pi_t \, P_{it} \geq 0,$$

$i = 1, \dots, m,$

$$\sum_{j=1}^{n} \mu_j \, u_{fj} + \sum_{t=1}^{k} \pi_t \, \varphi_{ft} \geq 0, \; f = 1, \dots, l,$$

$$x_{io}^* - d_i \geq 0 \,, \; i = 1, \dots, m,$$

$$u_{fo}^* - s_f \geq 0 \,, \; f = 1, \dots, l,$$

$$\mu_j \geq 0, \; \delta_j \geq \text{o} \,, j = 1, \dots, n, \pi_t \geq \text{o},$$

$t = 1, \dots, k,$

$$d_i \geq 0, i = 1, \dots, m, \; s_f \geq 0, f = 1, \dots, l,$$

$e_r \geq 0, r = 1, \dots, s.$

Theorem 2. The model (5) is always feasible.

Proof. Put $\mu_{oo} = 1$ and $\mu_{jo} = 0, j = 1, \dots, n, \; j \neq o,$ $\delta_j = 0, \; j = 1, \dots, n, \; \pi_t = 0, \; t = 1, \dots, L, \; d_i = x_{io}^* - x_{io},$ $i = 1, \dots, m, \; s_f = u_{fo}^* - u_{fo}, \; f = 1, \dots, l, \; e_r = y_{ro} - y_{ro}^*,$ $r = 1, \dots, s,$ n this case, we obtain a feasible solution for model (5) and the proof is completed. ■

Let $(\mu^*, \delta^*, \pi^*, d^*, e^*, s^*)$ be an optimal solution of model (4). We show the strongly efficient target on the boundary of T_{VRS-To}^{UV} set corresponding to $DMU_o = (X_o, Y_o, U_o)$ resulting from models (4) and (5) as follows.
$(\bar{X}_o, \bar{Y}_o, \bar{U}_o) = (X_o^* - d^*, Y_o^* + e^*, U_o^* - s^*).$

Theorem 3. The target unit determined as $(\bar{X}_o, \bar{Y}_o, \bar{U}_o)$ of two-stage process is a strongly efficient DMU on the

frontier of the set T_{VRS-To}^{UV}.

Proof. It is clear that $(\bar{X}_o, \bar{Y}_o, \bar{U}_o)$ belongs to the set T_{VRS-To}^{UV}. Suppose this DMU is dominated by some units such as (X', Y', U') which belongs to the set T_{VRS-To}^{UV}. Therefore, there are vectors $d_i', e_r', s_f', \; i = 1, \dots, m, f = 1, \dots, l, r = 1, \dots, s$, such that

$$\sum_{j=1}^{n} (\mu_j' + \delta_j') \, x_{ij} + \sum_{t=1}^{k} \pi_t' \, P_{it} + d_i' = x_i',$$

$$\sum_{j=1}^{n} \mu_j' \, y_{rj} + \sum_{t=1}^{k} \pi_t' \, Q_t - e_r' = y_r',$$

$$\sum_{j=1}^{n} \mu_j' \, u_{fj} + \sum_{t=1}^{k} \pi_t' \, \varphi_{ft} + s_f' = u_f', \geq 0,$$

$$\sum_{j=1}^{n} (\mu_j' + \delta_j') = 1,$$

$$\delta_j', \mu_j', \pi_t', d_i', e_r', s_f' \geq 0,$$

$$i = 1, \dots, \text{m}, \; f = 1, \dots, l, r = 1, \dots, s, t = 1, \dots, k.$$

Replace this solution in model (5). Considering that $(\bar{X}_o, \bar{Y}_o, \bar{U}_o)$ is dominated by (X', Y', U'), so the sum of slacks related to the unit $(\bar{X}_o, \bar{Y}_o, \bar{U}_o)$ in the optimal solution from model (5) i.e. $d_i^*, e_r^*, s_f^*, i = 1, \dots, m, \; f = 1, \dots, l, r = 1, \dots, s$, is strictly greater than or equal than the sum of similar slack values related to the unit (X', Y', U'), i.e. $d_i', e_r', s_f', i = 1, \dots, m, \; f = 1, \dots, l, \; r = 1, \dots, s$, which has a contradiction with the optimality of the solution $(\mu^*, \delta^*, \pi^*, d^*, e^*, s^*)$ in model (5), so the contradiction assumption is invalid and the target unit determined as $(\bar{X}_o, \bar{Y}_o, \bar{U}_o)$ of two-stage process is a strongly efficient DMU on the frontier of the set T_{VRS-To}^{UV}. The proof is complete. ■

3. Numerical example

In this section, we explain the two-stage approach presented in the paper with a simple numerical example. Consider five DMUs according to Table 1. Each DMU consumes one input to produce one desirable output and one undesirable output. To find the strongly efficient targets corresponding to each of the DMUs, we first solve the model (4). If we do not want to consider the weight restrictions on the input and output components, we can choose the matrices corresponding to the production trade-offs as follows.

Production trade-offs 1:

$P_1 = (0), P_2 = (0),$

$Q_1 = (0), Q_2 = (0), \varphi_1 = (0), \varphi_2 = (0).$

Then $i = 1, f = 1, r = 1, t = 2.$

The results of model (4) are shown in Table 1. The efficiency score corresponding to each of the DMUs is given in the second column of Table 2. As can be seen in Table 2, DMUs 1, 3 and 4 are efficient DMUs and DMUs 2 and 5 are inefficient DMUs. In the following, we solve model (5) according to the two-stage process to obtain strongly efficient targets. The results of model (5) are shown in Table 3. Therefore, without considering the production trade-offs on input, desirable output, undesirable output components, DMUs 2, 3, 4 choose DMU 1 as their target in the two-stage process. DMU 5 has a different target.

Table 1. The set of data in the numerical example.

DMUs	x_1	y_1	u_1
1	4	27	7
2	5	25	8
3	6	22	10
4	4.5	12	8.5
5	7	18	11

Table 2. The results of model (4) with production trade-offs 1.

DMUs	Production trade-offs: $P_1 = (0), P_2 = (0),$ $Q_1 = (0), Q_2 = (0), \varphi_1 = (0), \varphi_2 = (0)$			
	$\beta_o^{TO^*}$	x_1^*	y_1^*	u_1^*
1	1	4	27	7
2	0.9722	4.8571	25.7143	7.7714
3	1	6	22	10
4	1	4.5	12	8.5
5	0.7121	4.1702	25.2766	6.5532

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Table 3. The results of model (5) with production trade-offs 1.

Production trade-offs: $P_1 = (0), P_2 = (0),$ $Q_1 = (0), Q_2 = (0), \varphi_1 = (0), \varphi_2 = (0).$						
DMUs	d_1^*	e_1^*	s_1^*	$\bar{x}_1$	$\bar{y}_1$	$\bar{u}_1$
1	0	0	0	4	27	7
2	0.8571	1.2857	0.7714	4	27	7
3	2	5	3	4	27	7
4	0.5	15	1.5	4	27	7
5	0.1702	0	0	4	25.2766	6.5532

Also, the inefficiency slack values obtained from model (5) are also shown in Table 3. As it can be seen, DMU 1 is a strongly efficient DMU, because the slack values corresponding to it in the input, desired and undesirable output components are equal to zero. DMUs 2, 3, and 4 have inefficiency slack values in all components of input, desirable output, and undesirable output. For example, DMU 4 has an inefficiency slack value of 1.5 units in its undesirable output, that is, in order to make this DMU efficient, we must reduce the undesirable output of this DMU by 1.5 units. In other words, after the radial improvement according to model (4), this DMU becomes a weakly efficient DMU, and model (5) achieves the strongly efficient target corresponding to this DMU by reducing the input component and desirable output component of the radial projection to the amount of 0.5 and 1.5 units, respectively. Also expanding the desired output component by the amount of 15 units. These values are the amount of optimal slack variables corresponding to this DMU in the input component, desirable output component, and undesirable output component in the optimal solution of model (5).

Now we consider two different weight restrictions to solve models (4) and (5). To perform production trade-offs, we select P_t, Q_t, φ_t matrixes as follows.

Production trade-offs 2:

$P_1 = (2), P_2 = (3),$

$Q_1 = (1), Q_2 = (2), \varphi_1 = (0), \varphi_2 = (1).$

Then $i = 1, f = 1, r = 1, t = 2.$

The weight restrictions corresponding to these matrixes on the components of input, desirable output, and undesirable output are as follows.

$$\begin{cases} u_1 - 2v_1 - 0w_1 \leq 0 \\ 2u_1 - 3v_1 - 1w_1 \leq 0 \end{cases}$$

where u_1, v_1, and w_1 are weights corresponding to input, desired output, and undesirable output components, respectively. In the first weight restriction, the importance corresponding to the desired output is less than or equal to twice the input component. Also, in the second weight restriction, we add 2 units to the desirable output component and subtract three units from the input component, and one unit is also subtracted from the undesirable output. To obtain strongly efficient targets, we first solve model (4). The radial efficiency score corresponding to each of the DMUs is given in the second column of Table 4. As can be seen, DMUs 1, 3, and 4 are efficient and DMUs 2 and 5 are inefficient. In the presence of these weight restrictions, similar to the solution of model (4) in the absence of weight restrictions, efficient DMUs do not change and remain efficient. Table 4 shows the results of model (4). In the following, we solve the model (5) based on the introduced two-stage process. The results are shown in sixth to eleventh columns of Table 4. Efficient targets are listed in Table 5. DMUs 2, 3 and 4 choose DMU 1 as their target. DMU 5 has a distinct target on the efficient frontier. The values of inefficiency slacks in the components of input, desirable output and undesirable output are given in Table 4. For example, DMU 3 has an undesirable output equal to 10. According to model (4), there is no improvement based on model (4) radially for this DMU in the undesirable output. However, the amount of slack corresponding to the undesirable output corresponding to this DMU 3 is equal to 3, and to achieve the strongly efficient targets, we must reduce the undesirable output corresponding to this DMU by three units in the second stage. The target of DMU 3 is DMU 1. As can be seen, the reduction ratio of undesirable outputs is not the same for all DMUs, and this corresponds to the axiom of weak disposability for undesirable outputs with individual-proportion in the approach presented in this paper.

Table 4. The results of model (4) with production trade-offs 2.

DMU	Production trade-offs: $P_1 = (2), P_2 = (3),$ $Q_1 = (1), Q_2 = (2), \varphi_1 = (0), \varphi_2 = (1).$			
	β_o^{TO*}	x_1^*	y_1^*	u_1^*
1	1	4	27	7
2	0.9286	4.6154	26.9231	7.3846
3	1	6	22	10
4	1	4.5	12	8.5
5	0.7114	4.1609	25.3006	6.5385

Tables 6 and 7 show the results of the two-stage approach by selecting the production trade-offs matrixes as follows.

Production trade-offs 3:

$P_1 = (1), P_2 = (0),$

$Q_1 = (2), Q_2 = (0), \varphi_1 = (2), \varphi_2 = (0).$

Then $i = 1, f = 1, r = 1, t = 2.$

Table 5. The results of model (5) with production trade-offs 2.

DMUs	Production trade-offs: $P_1 = (2),$ $P_2 = (3), Q_1 = (1), Q_2 = (2),$ $\varphi_1 = (0), \varphi_2 = (1).$					
	d_1^*	e_1^*	s_1^*	$\bar{x}_1$	$\bar{y}_1$	$\bar{u}_1$
1	0	0	0	4	27	7
2	0.6154	0.0769	0.3846	4	27	7
3	2	5	3	4	27	7
4	0.5	15	1.5	4	27	7
5	0.0034	0	0	4.1575	25.3006	6.539

In this case, we will have only one weight restrictions corresponding to these matrixes.

$$2u_1 - v_1 - 2w_1 \leq 0.$$

By applying these weight restrictions, DMUs 1 and 4 are the only efficient DMUs. DMU 3 becomes inefficient by applying this weight restriction. Only DMU 4 chooses DMU 1 as its target and the corresponding target is different from other DMUs. By checking the inefficiency slack values related to the undesirable output, the value of these slacks for all DMUs is equal to zero. Only DMU4 has an undesirable output slack value of 1.5.

Table 6. The results of model (4) with production trade-offs 3.

DMUs	Production trade-offs: $P_1 = (1), P_2 = (0), Q_1 = (2), Q_2 = (0), \varphi_1 = (2), \varphi_2 = (0)$.			
	β_o^{TO*}	x_1^*	y_1^*	u_1^*
1	1	4	27	7
2	0.9167	4.5455	27.2727	7.2727
3	0.8	4.5	27.5	7.5
4	1	4.5	12	8.5
5	0.7121	4.1702	25.2766	6.5532

Table 7. The results of model (5) with production trade-offs 3.

DMUs	Production trade-offs: $P_1 = (1),\ P_2 = (0), Q_1 = (2), Q_2 = (0), \varphi_1 = (2), \varphi_2 = (0)$.					
	d_1^*	e_1^*	s_1^*	$\bar{x}_1$	$\bar{y}_1$	$\bar{u}_1$
1	0	0	0	4	27	7
2	0.4091	0	0	4.1364	27.2727	7.2727
3	0.25	0	0	4.25	27.5	7.5
4	0.5	15	1.5	4	27	7
5	0.1702	0	0	4	25.2766	6.5532

Tables 8 and 9 show the results of the two-stage approach by selecting the production trade-offs matrices as follows.

Production trade-offs 4:

$$P_1 = (2), P_2 = (-1),$$

$$Q_1 = (3), Q_2 = (2), \varphi_1 = (-3), \varphi_2 = (-3).$$

Then $i = 1, f = 1, r = 1, t = 2$.

The weight restrictions corresponding to these matrices on the components of input, output, and undesirable output are as follows.

$$\begin{cases} 3u_1 - 2v_1 + 3w_1 \leq 0 \\ 2u_1 + v_1 + 3w_1 \leq 0 \end{cases}$$

By applying these weight restrictions, DMUs 3, 1 and 4 are efficient DMUs. DMUs 2 and 5 become inefficient. All the DMUs have a common target on the efficient frontier as follows.

$(1.6667, 31.6667, 0)$.

However, the remarkable thing about choosing this weight restriction is that the inefficiency slack in all input components, desirable output and undesirable output are non-zero. For DMUs 1, 3 and 4, the value of this slack is equal to this undesirable output, that is, these DMUs can reduce their undesirable output level to zero to be efficient. This shows the distinction between the models presented in this paper in the presence and absence of production trade-offs.

Table 8. The results of model (4) with production trade-offs 4.

DMUs	Production trade-offs: $P_1 = (2), P_2 = (-1), Q_1 = (3), Q_2 = (2), \varphi_1 = (-3),\ \varphi_2 = (-3)$.			
	β_o^{TO*}	x_1^*	y_1^*	u_1^*
1	1	4	27	7
2	0.9722	4.8571	25.7143	7.7714
3	1	6	22	10
4	1	4.5	12	8.5
5	0.6667	3.5	27	5.5

Table 9. The results of model (5) with production trade-offs 4.

DMUs	Production trade-offs: $P_1 = (2), P_2 = (-1), Q_1 = (3), Q_2 = (2), \varphi_1 = (-3),\ \varphi_2 = (-3)$.					
	d_1^*	e_1^*	s_1^*	$\bar{x}_1$	$\bar{y}_1$	$\bar{u}_1$
1	2.3333	4.6667	7	1.6667	31.6667	0
2	3.1904	5.9524	7.7714	1.6667	31.6667	0
3	4.3333	9.6667	10	1.6667	31.6667	0
4	2.8333	19.6667	8.5	1.6667	31.6667	0
5	1.8333	4.6667	5.5	1.6667	31.6667	0

4. Application of the presented approach to evaluate performance of chain stores

In this section, we propose an application of the two-stage process presented in this paper. We use it to evaluate the performance of a set of 25 chain stores in Iran. To evaluate these stores, we use the superior information of the DM and apply this information in the form of weight restrictions on inputs, desirable outputs, and undesirable outputs. These stores have a common structure, and all their activities are carried out under common management. Also, they have common suppliers. Each has a different number of human resources, physical space, and number of customers. The management of these stores uses the DEA technique to evaluate the chain stores under its management.

Considering that the importance of each of the input and output components for store management is different, management tries to check the performance of stores by applying different restrictions. This restriction is applied in such a way that his opinion can be included in the evaluation. The data sets related to these stores are shown in Table 10. Corresponding to each of the stores, two inputs are considered. The first input is current costs, which include employee and personnel salaries, store rent, and energy consumption costs. The second input also includes commodity costs. These costs include the cost of transporting goods to the store by suppliers, storage costs, maintenance costs, the cost of distributing goods to customers, and the cost of software services. The unit of measurement for these two inputs is based on million Tomans. For the convenience of calculations in the Table 10, in the second and third columns, six zeros have been removed from the numbers.

In this evaluation, we considered two desirable outputs for the store. The first desirable output includes the net income of the store during the evaluation period. These revenues are from the sale of goods after deducting the cost of their purchase by the company. In other words, these revenues are net profit. The unit of measurement for this output is also based on a million Tomans. For the convenience of calculations in Table 6 in the fourth column, 5 zeros have been removed from the numbers.
Another desirable output is the number of products that have been delivered to customers by the store during the evaluation period online and through the application and have not been returned. In other words, it is the number of products that have been successfully delivered to customers during the evaluation period. Its unit is the number of sent goods. Also, in this evaluation, we consider an undesirable output. This undesirable output is the number of products ordered by customers. These goods have been returned for various reasons, including the lack of required quality, the lack of an expiration date, the lack of customer satisfaction, and the fact that the customer's request is not the same as the one sent by the store. The measurement of this output is the number of goods sent. The data is given in Tables 10 and 11.

Table 10. The inputs data set of twenty-five chain stores in Iran.

Chains stores	Input 1	Input 2
1	139	286
2	225	772
3	378	1314
4	542	637
5	801	2552
6	876	3427
7	855	1943
8	674	1184
9	543	1422
10	968	2638
11	4359	14276
12	763	3549
13	3379	3704
14	10494	4537
15	572	12943
16	475	11543
17	364	1793
18	2548	13943
19	9576	8963
20	1427	487
21	11654	9648
22	657	10754
23	692	9937
24	2738	8311
25	807	957

We now use the two-stage process presented in this paper to evaluate these stores. To apply the opinion of the store management in the evaluation process, we use the weight restrictions method in DEA. Based on the two-stage process, at first, in order to obtain the amount of radial improvement in the input and output components, we solve model (4). Next, we solve model (5) in the second stage and get the strongly efficient targets corresponding to each of the stores. These targets have the best level of inputs, desirable outputs, and undesirable outputs.

Table 11. The outputs data set of twenty-five chain stores in Iran.

Chains stores	Output Desirable 1	Output Desirable 2	Output Undesirable 1
1	5376	7632	74
2	224765	5497	284
3	55328	6539	876
4	658248	10432	1394
5	1917321	3592	2036
6	1647532	4428	1947
7	2575375	9659	2032
8	1123692	8437	2215
9	1743257	6826	2491
10	1445829	5632	3593
11	7854285	8592	3288
12	1875355	7392	3255
13	2354934	5439	6362
14	4234829	6472	5235
15	9654	5384	3714
16	76834	6683	4572
17	556923	5834	847
18	1789432	9372	1845
19	4963471	10426	2847
20	1954296	9253	2659
21	1017894	7937	3531
22	6874493	7482	4857
23	9739	12593	839
24	674392	9372	1293
25	2341838	8428	2197

Table 12. The results of model (4), including the efficiency scores and radial improvement of inputs and outputs with production trade-offs 5.

Chains stores	β_o^*	x_1^*	x_2^*
1	1	139	286
2	0.854	186.547	640.0633
3	0.7767	269.3025	936.1466
4	1	542	637
5	0.8617	672.4517	2142.443
6	0.7993	656.014	2566.393
7	1	855	1943
8	1	674	1184
9	1	543	1422
10	0.7255	601.7807	1639.977
11	1	4359	14276
12	1	763	3549
13	1	3379	3704
14	1	10494	4537
15	0.7085	336.7026	7618.778
16	1	475	11543
17	0.7612	249.8041	1230.491
18	0.8317	2032.577	11122.54
19	1	9576	8963
20	1	1427	487
21	0.7566	7905.461	6544.696
22	1	657	10754
23	1	692	9937
24	0.8454	2237.256	6791.029
25	1	807	957

If we do not want to consider the weight restrictions on the input and output components, we can choose the matrices corresponding to the production trade-offs namely P_t, Q_t, φ_t as follows:

Table 13. The results of model (4), including the radial improvement of outputs with production trade-offs 5.

Chains stores	y_1^*	y_2^*	u_1^*
1	5376	7632	74
2	263177.9	6436.451	235.4637
3	71238.1	8419.352	624.0978
4	658248	10432	1394
5	2225022	4168.461	1709.253
6	2061269	5539.984	1458.059
7	2575375	9659	2032
8	1123692	8437	2215
9	1743257	6826	2491
10	1992823	7762.73	2233.676
11	7854285	8592	3288
12	1875355	7392	3255
13	2354934	5439	6362
14	4234829	6472	5235
15	13625.26	7598.758	2186.212
16	76834	6683	4572
17	731643.6	7664.271	581.275
18	2151408	11267.82	1471.784
19	4963471	10426	2847
20	1954296	9253	2659
21	1345302	10489.96	2395.245
22	6874493	7482	4857
23	9739	12593	839
24	797729.3	11086.01	1056.528
25	2341838	8428	2197

Table 14. The results of model (5), including the efficient targets of inputs with production trade-offs 5.

Chains stores	$\bar{x}_1$	$\bar{x}_2$
1	139	286
2	186.547	640.0633
3	269.3025	936.1466
4	542	637
5	672.4517	2142.443
6	656.014	2566.393
7	855	1943
8	674	1184
9	543	1422
10	601.7807	1639.977
11	4359	14276
12	763	3549
13	3379	3704
14	10494	4537
15	336.7026	4274.126
16	475	7076.054
17	249.8041	1230.491
18	1667.293	10668.93
19	9576	8963
20	1427	487
21	6666.751	6544.696
22	657	10754
23	692	9937
24	1831.05	6791.029
25	807	957

Table 15. The results of model (5), including the efficient targets of outputs with production trade-offs 5.

Chains stores	$\bar{y}_1$	$\bar{y}_2$	$\bar{u}_1$
1	5376	7632	74
2	263178.2	6436.451	235.464
3	422759.6	8419.352	595.6375
4	658248	10432	1394
5	2225022	7078.044	1709.253
6	2061270	5539.984	1458.059
7	2575375	9659	2032
8	1998457	8437	1892.749
9	1789666	8057.094	1588.214
10	2076063	8114.88	1825.142
11	7854285	8592	3288
12	3528159	8172.241	2888.413
13	3676267	7730.647	3497.217
14	4234829	6472	5235
15	2592722	7598.758	1879.154
16	4461019	7534.703	3176.487
17	731648.5	7664.271	581.28
18	2151408	11267.82	1471.784
19	4963471	10426	2847
20	1954296	9253	2659
21	3556382	10489.96	2362.465
22	6874493	7482	4857
23	9739	12593	839
24	797733.9	11086.01	1056.53
25	2341838	8428	2197

Table 16. The slacks of input components of model (5) with production trade-offs 5.

Chains stores	d_1^*	d_2^*
1	0	0
2	0	0
3	0	0
4	0	0
5	0	0
6	0	0
7	0	0
8	0	0
9	0	0
10	0	0
11	0	0
12	0	0
13	0	0
14	0	0
15	0	3344.652
16	0	4466.946
17	0	0
18	365.2844	453.6027
19	0	0
20	0	0
21	1238.71	0
22	0	0
23	0	0
24	406.2059	0
25	0	0

Production trade-offs 5:

$$P_1 = \begin{pmatrix}0\\0\end{pmatrix}, P_2 = \begin{pmatrix}0\\0\end{pmatrix}, Q_1 = \begin{pmatrix}0\\0\end{pmatrix}, Q_2 = \begin{pmatrix}0\\0\end{pmatrix}, \varphi_1 = (0),$$

In the following, we solve the model (5) to obtain the strongly efficient targets. The results of model (5) in the evaluation of these stores with considering production trade-offs 5 are shown in Tables 14, 15, 16 and 17. $\varphi_2 = (0)$. Then $i = 2, f = 1, r = 2, t = 2$.

The results of model (4) are shown in Tables 12 and 13. The efficiency scores and input and output components corresponding to stores after radial improvement based on model (4) are shown in Table 8. As can be seen in this evaluation, stores 1, 4, 7, 8, 9, 11, 12, 13, 14, 16, 19, 20, 22, 23, and 25 are efficient, and other stores are inefficient.

Table 17. The slacks of desirable outputs and undesirable outputs components of model (5) with production trade-offs 5.

Chains stores	e_1^*	e_2^*	s_1^*
1	0	0	0
2	0.2871	0	0
3	351521.5	0	28.4603
4	0	0	0
5	0.0069	2909.583	0
6	0.4658	0	0
7	0	0	0
8	874764.6	0	322.251
9	46408.77	1231.094	902.7863
10	83239.3	352.1498	408.5336
11	0	0	0
12	1652804	780.2407	366.5869
13	1321333	2291.647	2864.783
14	0	0	0
15	2579097	0	307.0583
16	4384185	851.7027	1395.514
17	4.8567	0	0
18	0.6606	0	0
19	0	0	0
20	0	0	0
21	2211080	0	32.7802
22	0	0	0
23	0	0	0
24	4.5595	0	0
25	0	0	0

As can be seen, if the slack components of inputs and outputs are non-zero, that means these stores will have the possibility of improving their inputs and outputs after radial improvement. For example, store number 12 can bring its input and output components to the following levels after radial improvement, according to model (4). $(763,3549,1875355,7392,3255)$.

This store can increase its first and second desirable output by 1652804, 780.2407 respectively and decrease its undesirable output by 366.5869. In this way, the target corresponding to store 12 is determined as

$$(\bar{X}_o, \bar{Y}_o, \bar{U}_o) =$$

$$(763,3549,3528159,8172.241,2888.413\ ).$$

Now we consider two different weight constraints to solve models (4) and (5). To consider production trade-offs, we choose the matrices P_t, Q_t, φ_t as follows.

Production trade-offs 6:

$$P_1 = \binom{2}{-2}, P_2 = \binom{-2}{3}, Q_1 = \binom{-2}{-1},$$

$$Q_2 = \binom{3}{2}, \varphi_1 = (2), \varphi_2 = (-3).$$

Then $i = 2, f = 1, r = 2, t = 2$.

Table 18. The results of model (4), including the efficiency scores and radial improvement of inputs with production trade-offs 6.

Chains stores	β_o^*	x_1^*	x_2^*
1	1	139	286
2	0.7513	150.5293	516.4827
3	0.7003	216.2442	751.7059
4	1	542	637
5	0.7904	588.6025	1875.2978
6	0.7064	511.9822	2002.9256
7	1	855	1943
8	1	674	1184
9	1	543	1422
10	0.6777	507.5739	1383.2436
11	1	4359	14276
12	1	763	3549
13	0.7054	1967.7088	2156.9675
14	0.9261	9656.2459	4174.8035
15	0.5255	55.4433	1254.5495
16	0.7316	300.707	7307.4958
17	0.6846	196.2786	966.8336
18	0.7728	1798.9416	9844.0514
19	0.9322	8879.3025	8310.901
20	1	1427	487
21	0.6244	4643.4656	3844.1871
22	1	657	10754
23	1	692	9937
24	0.7642	1893.0991	5746.3647
25	1	807	957

The weight restrictions corresponding to these matrices on the components of input, desirable output and undesirable output are as follows.

$$\begin{cases}(-u_2 - 2u_1) - (-2v_2 + 2v_1) - (2w_1) \leq 0 \\ (2u_2 + 3u_1) - (3v_1 - 2v_1) - (-3w_1) \leq 0\end{cases}$$

Where $u_1, u_2,\ v_1$ ‘v_2 and w_1 are the weights corresponding to the components of input, desirable output, and undesirable output, respectively. The results of model (4) are shown in Tables 18 and 19. The efficiency score and input and output components corresponding to stores, after radial improvement based on model (4), is shown in Table 14. As can be seen, in this evaluation, stores 1, 4, 7, 8, 9, 11, 12, 20, 22, 23 and 25 are efficient. Other stores are inefficient.

In the following, we solve the model (5) to obtain strongly efficient targets. The results of model (5) in the evaluation of these stores in the presence of production trade-offs are shown in Tables 20, 21, 22 and 23.

As can be seen, if the slack components of inputs and outputs are non-zero, that means these stores will have the possibility of improving their inputs and outputs after radial improvement. For example, store number 9 can bring

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its input and output components to the following levels after radial improvement, according to model (4).

$(543,1422,1743257,6826,2491)$.

This store can increase its first and second desirable output by 631649.1, 1858.1 respectively and decrease its undesirable output by 669.2034. In this way, the target corresponding to store 9 is determined as

$$(\bar{X}_o,\bar{Y}_o,\bar{U}_o) =$$
$$(543, 1422,2374906.1,8684.0995,1821.7966).$$

Now, in order to sensitivity analysis of the results of the models to the different selection of production trade-offs matrices, we consider two other different weight restrictions to solve models (4) and (5). To consider production trade-offs, we choose the matrices P_t, Q_t, φ_t as follows.
The strongly efficient targets corresponding to the stores based on model (5) by choosing these weight restrictions and the inefficiency slack values in the evaluation of these stores are listed in Tables 26, 27, 28 and 29, respectively.

Production trade-offs 7:

$$P_1 = \binom{2}{1}, P_2 = \binom{-1}{1}, Q_1 = \binom{0}{0},$$
$$Q_2 = \binom{3}{-1}, \varphi_1 = (3), \varphi_2 = (-2).$$

Then $i = 2, f = 1, r = 2, t = 2$.

Table 19. The results of model (4), including the radial improvement of outputs with production trade-offs 6.

Chains stores	y_1^*	y_2^*	u_1^*
1	5376	7632	74
2	299157.93	7316.4021	190.1
3	79004.261	9337.2048	501.1373
4	658248	10432	1394
5	2425728.3	4544.4742	1496.1232
6	2332156.5	6268.0354	1137.94
7	2575375	9659	2032
8	1123692	8437	2215
9	1743257	6826	2491
10	2133533	8310.843	1884.0009
11	7854285	8592	3288
12	1875355	7392	3255
13	3338508.3	7710.6819	3704.82
14	4572902.7	6988.6709	4817.1
15	18372.249	10246.135	359.9936
16	105026.91	9135.2113	2894.383
17	813538.2	8522.1509	456.73
18	2315487.4	12127.171	1302.6088
19	5324586.1	11184.539	2639.868
20	1954296	9253	2659
21	1630214.3	12711.551	1406.9055
22	6874493	7482	4857
23	9739	12593	839
24	882498.07	12264.042	894
25	2341838	8428	2197

Table 20. The results of model (5), including the efficient targets of inputs with production trade-offs 6.

Chains stores	$\bar{x}_1$	$\bar{x}_2$
1	139	286
2	150.5293	516.4827
3	216.2442	751.7059
4	542	637
5	382.3679	1875.2978
6	104.6084	2002.9256
7	855	1943
8	674	1184
9	543	1422
10	507.5739	1383.2436
11	4359	14276
12	763	3549
13	1967.7088	2156.9675
14	4424.1215	4174.8035
15	55.4433	1254.5495
16	300.707	7307.4958
17	196.2786	966.8336
18	1798.9416	9844.0514
19	7769.6051	8310.901
20	1427	487
21	4643.4656	3844.1871
22	657	10754
23	692	9937
24	1893.0991	5746.3647
25	807	957

Table 21. The results of model (5), including the efficient targets of outputs with production trade-offs 6.

Chains stores	$\bar{y}_1$	$\bar{y}_2$	$\bar{u}_1$
1	5376	7632	74
2	299240.26	7316.4021	190.1
3	354494.56	9337.2048	501.1373
4	658248	10432	1394
5	2425728.3	8908.6998	1496.1232
6	2332162.2	9088.7352	1137.94
7	2575375	9659	2032
8	2355059	8557.8512	2005.7885
9	2374906.1	8684.0995	1821.7966
10	2310298.2	8707.3908	1732.5196
11	7854285	8592	3288
12	3537988.2	8182.7714	2892.3362
13	3338510.6	7769.809	3704.82
14	4572910	7405.3888	4817.1
15	496240.27	10246.135	359.9936
16	4791549	9135.2113	2663.5738
17	813542.13	8522.1509	456.73
18	4909809.7	12127.171	440.8807
19	5324586.1	11184.539	2639.868
20	1954296	9253	2659
21	2869969.5	12711.551	924.6441
22	6874493	7482	4857
23	2675570.1	12593	0
24	3255829.4	12264.042	63.8214
25	2341838	8428	2197

Table 22. The slacks of input components of model (5) with production trade-offs 6.

Chains stores	d_1^*	d_2^*
1	0	0
2	0	0
3	0	0
4	0	0
5	206.2346	0
6	407.3738	0
7	0	0
8	0	0
9	0	0
10	0	0
11	0	0
12	0	0
13	0	0
14	5232.124	0
15	0	0
16	0	0
17	0	0
18	0	0
19	1109.697	0
20	0	0
21	0	0
22	0	0
23	0	0
24	0	0
25	0	0

Table 23. The slacks of desirable outputs and undesirable outputs components of model (5) with production trade-offs 6.

Chains stores	e_1^*	e_2^*	s_1^*
1	0	0	0
2	82.3321	0	0
3	275490.3	0	0
4	0	0	0
5	0.0024	4364.226	0
6	5.7447	2820.7	0
7	0	0	0
8	1231367	120.8512	209.2115
9	631649.1	1858.1	669.2034
10	176765.2	396.5478	151.4813
11	0	0	0
12	1662633	790.7714	362.6638
13	2.2776	59.1271	0
14	7.2831	416.7179	0
15	477868	0	0
16	4686522	0	230.8092
17	3.9375	0	0
18	2594322	0	861.7281
19	0.0575	0	0
20	0	0	0
21	1239755	0	482.2614
22	0	0	0
23	2665831	0	839
24	2373331	0	830.1786
25	0	0	0

The weight restrictions corresponding to these matrices on the components of input, desirable output, and undesirable output are as follows:

$$\begin{cases} (-1v_2 - 2v_1) - (3w_1) \le 0 \\ (-u_2 + 3u_1) - (v_2 - v_1) - (-2w_1) \le 0 \end{cases}$$

The results of model (4) are shown in Tables 24 and 25. The efficiency scores and input and output components corresponding to stores after radial improvement based on model (4) are shown in Tables 24 and 25. As can be seen in this evaluation, stores 1, 4, 7, 8, 9, 11, 14, 12, 19, 20, 22, 23, and 25 are efficient. Other stores are inefficient. Like before, in order to conduct a conduct a sensitivity analysis of the results of the models to the different selections of production trade-off matrices, we consider two other different weight restrictions to solve models (4) and (5). To consider the production trade-offs, we choose the matrices as follows:

Production trade-offs 8:

$$P_1 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}, P_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, Q_1 = \begin{pmatrix} 2 \\ -2 \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \varphi_1 = (2), \varphi_2 = (0).$$

Then $i = 2, f = 1, r = 2, t = 2$.

We will have a weight restriction corresponding to these matrices on the components of input, desirable output and undesirable output as follows.

$$(-2u_2 + 2u_1) - (3v_2 - 2v_1) - (2w_1) \le 0.$$

Table 24. The results of model (4), including the efficiency scores and radial improvement of inputs and with production trade-offs 7.

Chains stores	β_o^*	x_1^*	x_2^*
1	1	139	286
2	0.7176	136.4409	468.144
3	0.7028	218.1475	758.3223
4	1	542	637
5	0.7638	553.26	1762.696
6	0.6757	455.4774	1781.873
7	1	855	1943
8	1	674	1184
9	1	543	1422
10	0.7077	568.2105	1548.491
11	1	4359	14276
12	1	763	3549
13	0.8203	2638.558	2892.34
14	1	10494	4537
15	0.5403	85.2361	1928.691
16	0.8675	402.4294	9779.459
17	0.6733	187.3494	922.8503
18	0.8199	1988.326	10880.39
19	1	9576	8963
20	1	1427	487
21	0.7127	6956.734	5759.273
22	1	657	10754
23	1	692	9937
24	0.8214	2142.825	6504.391
25	1	807	957

Table 25. The results of model (4), including the efficiency scores and radial improvement of outputs with production trade-offs 7.

Chains stores	y_1^*	y_2^*	u_1^*
1	5376	7632	74
2	313231.6	7660.596	172.2188
3	78725.67	9304.279	505.5482
4	658248	10432	1394
5	2510326	4702.964	1406.289
6	2438427	6553.655	1012.345
7	2575375	9659	2032
8	1123692	8437	2215
9	1743257	6826	2491
10	2042965	7958.048	2109.071
11	7854285	8592	3288
12	1875355	7392	3255
13	2870972	6630.851	4967.891
14	4234829	6472	5235
15	17869.42	9965.708	553.4388
16	88572.71	7704.03	3873.489
17	827199.9	8665.262	435.9477
18	2182485	11430.58	1439.741
19	4963471	10426	2847
20	1954296	9253	2659
21	1428167	11136.09	2107.794
22	6874493	7482	4857
23	9739	12593	839
24	820988.4	11409.24	1011.933
25	2341838	8428	2197

Table 26. The results of model (5) including the efficient targets of inputs with production trade-offs 7.

Chains stores	$\bar{x}_1$	$\bar{x}_2$
1	139	286
2	136.4409	468.144
3	179.8574	758.3223
4	542	637
5	357.0651	1762.696
6	183.7438	1781.873
7	855	1943
8	674	1184
9	543	1422
10	568.2105	1548.491
11	4359	14276
12	763	3549
13	2638.558	2892.34
14	10494	4537
15	85.2361	1928.691
16	402.4294	9779.459
17	187.3494	922.8503
18	1988.326	10880.39
19	9576	8963
20	1427	487
21	2854.04	5759.273
22	657	10754
23	692	9937
24	1526.619	6504.391
25	807	957

Table 27. The results of model (5) including the efficient targets of outputs with production trade-offs 7.

Chains stores	$\bar{y}_1$	$\bar{y}_2$	$\bar{u}_1$
1	5376	7632	74
2	313232.4	7660.596	172.22
3	450427.3	9304.279	505.5482
4	658248	10432	1394
5	2510326	7948.71	1406.289
6	2438428	7788.108	1012.35
7	2575375	9659	2032
8	2345280	8437	1902.34
9	2437061	8147.415	1730.673
10	2508255	8160.108	1827.641
11	7854285	8592	3288
12	3539032	8173.758	2890.806
13	3282455	7933.47	3117.178
14	4234829	6472	5235
15	867895.8	9965.708	425.6706
16	6005065	7704.03	3737.677
17	827199.9	8665.262	435.9477
18	2182485	11430.58	1439.741
19	4963471	10426	2847
20	1954296	9253	2659
21	1524982	11136.09	1576.769
22	6874493	7482	4857
23	9739	12593	839
24	820988.4	11409.24	1011.933
25	2341838	8428	2197

Table 28. The slacks of input components of model (5) with production trade-offs 7.

Chains stores	d_1^*	d_2^*
1	0	0
2	0	0
3	38.2901	0
4	0	0
5	196.1949	0
6	271.7336	0
7	0	0
8	0	0
9	0	0
10	0	0
11	0	0
12	0	0
13	0	0
14	0	0
15	0	0
16	0	0
17	0	0
18	0	0
19	0	0
20	0	0
21	4102.694	0
22	0	0
23	0	0
24	616.2066	0
25	0	0

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Table 29. The slacks of desirable outputs and undesirable outputs components of model (5) with production trade-offs 7.

Chains stores	e_1^*	e_2^*	s_1^*
1	0	0	0
2	0.7929	0	0
3	371701.6	0	0
4	0	0	0
5	0.0002	3245.747	0
6	0.9292	1234.453	0
7	0	0	0
8	1221588	0	312.66
9	693803.7	1321.415	760.3269
10	465290.6	202.0601	281.4298
11	0	0	0
12	1663677	781.7576	364.1943
13	411483.1	1302.619	1850.713
14	0	0	0
15	850026.4	0	127.7682
16	5916492	0	135.8128
17	0.073	0	0
18	0.0285	0	0
19	0	0	0
20	0	0	0
21	96814.97	0	531.0243
22	0	0	0
23	0	0	0
24	0.0417	0	0
25	0	0	0

Table 30. The results of model (4) including the efficiency scores and radial improvement of inputs with production trade-offs 8.

Chains stores	β_o^*	x_1^*	x_2^*
1	1	139	286
2	0.854	186.547	640.0633
3	0.7499	251.9006	875.6546
4	1	542	637
5	0.8617	672.4517	2142.443
6	0.7993	656.014	2566.393
7	1	855	1943
8	0.8697	573.0507	1006.665
9	0.8801	469.03	1228.289
10	0.6943	541.8059	1476.533
11	1	4359	14276
12	0.7841	552.9315	2571.892
13	1	3379	3704
14	1	10494	4537
15	0.5701	140.5836	3181.073
16	0.6378	205.2866	4988.681
17	0.7612	249.8041	1230.491
18	0.8317	2032.577	11122.54
19	1	9576	8963
20	1	1427	487
21	0.7566	7905.461	6544.696
22	1	657	10754
23	1	692	9937
24	0.8454	2237.256	6791.029
25	1	807	957

The results of model (4) are shown in Tables 30 and 31. The efficiency scorer and input and output components corresponding to stores after radial improvement based on model (4) are shown in Tables 30 and 31. As can be seen in this evaluation, stores 1, 4, 7, 11, 14, 13, 19, 20, 22, 23, and 25 are efficient. Other stores are inefficient.

The strongly efficiency targets corresponding to the stores based on model (5) with the selection of this weight restriction and inefficiency slack values in the evaluation of these stores are listed in Tables 32, 33, 34 and 35 respectively.

Table 31. The results of model (4) including the efficiency scores and radial improvement of outputs with production trade-offs 8.

Chains stores	y_1^*	y_2^*	u_1^*
1	5376	7632	74
2	263177.9	6436.451	235.4637
3	73785.21	8720.386	583.7697
4	658248	10432	1394
5	2225022	4168.461	1709.253
6	2061269	5539.984	1458.059
7	2575375	9659	2032
8	1291994	9700.663	1883.245
9	1980732	7755.869	2151.664
10	2082403	8111.675	2011.063
11	7854285	8592	3288
12	2391676	9427.16	2358.836
13	2354934	5439	6362
14	4234829	6472	5235
15	16935.28	9444.745	912.8103
16	120461.7	10477.73	1975.938
17	731643.6	7664.271	581.275
18	2151408	11267.82	1471.784
19	4963471	10426	2847
20	1954296	9253	2659
21	1345302	10489.96	2395.245
22	6874493	7482	4857
23	9739	12593	839
24	797729.3	11086.01	1056.528
25	2341838	8428	2197

As it can be seen, by applying weight restrictions in the multiplier DEA models and equivalent production trade-offs in the envelopment DEA models, we can obtain the efficient targets corresponding to each of the stores. These stores must bring the level of their inputs and outputs to the level of these targets for they to be efficient. In this paper, to solve the models (4) and (5) from a suitable direction, which in most of the DEA literature is the distance model of the proportional direction, i.e. $g = (-g_x, g_y, -g_u) = (-X_o, Y_o, -U_o)$. To solve the model (4), we can use other directions such as the following directions.

$$g = (-g_x, g_y, -g_u) = (-X_o, 0,0),$$

$$g = (-g_x, g_y, -g_u) = (0, Y_o, 0),$$

$$g = (-g_x, g_y, -g_u) = (0,0, -U_o).$$

Also, the models presented in this paper were in envelope form. We can also develop models for multiplier forms. Figs 1, 2, 3 and 4 show the efficiency scores corresponding to chain stores with different choices of production trade-offs matrixes.

Table 32. The results of model (5) including the efficient targets of inputs with production trade-offs 8.

Chains stores	$\bar{x}_1$	$\bar{x}_2$
1	139	286
2	186.547	640.0633
3	251.9006	875.6546
4	542	637
5	672.4517	2142.443
6	656.014	2566.393
7	855	1943
8	573.0507	1006.665
9	469.03	1228.289
10	541.8059	1476.533
11	4359	14276
12	552.9315	2571.892
13	3379	3704
14	10494	4537
15	140.5836	3181.073
16	205.2866	4988.681
17	249.8041	1230.491
18	1667.293	10668.93
19	9576	8963
20	1427	487
21	6666.751	6544.696
22	657	10754
23	692	9937
24	1831.044	6791.029
25	807	957

Table 33. The results of model (5) including the efficient targets of outputs with production trade-offs 8.

Chains stores	$\bar{y}_1$	$\bar{y}_2$	$\bar{u}_1$
1	5376	7632	74
2	263178.2	6436.451	235.464
3	220352.6	8720.386	583.7697
4	658248	10432	1394
5	2225022	7078.044	1709.253
6	2061270	5539.984	1458.059
7	2575375	9659	2032
8	1359944	9700.663	1775.595
9	2012957	8071.999	2141.817
10	2082403	8111.675	2011.063
11	7854285	8592	3288
12	2517507	9427.16	2303.328
13	3676267	7730.647	3497.217
14	4234829	6472	5235
15	200613.1	9444.745	912.8103
16	996285	10477.73	1869.085
17	731648.5	7664.271	581.28
18	2151408	11267.82	1471.784
19	4963471	10426	2847
20	1954296	9253	2659
21	3556382	10489.96	2362.465
22	6874493	7482	4857
23	9739	12593	839
24	797730.2	11086.01	1056.528
25	2341838	8428	2197

Table 34. The slacks of input components of model (5) with production trade-offs 8.

Chains stores	d_1^*	d_2^*
1	0	0
2	0	0
3	0	0
4	0	0
5	0	0
6	0	0
7	0	0
8	0	0
9	0	0
10	0	0
11	0	0
12	0	0
13	0	0
14	0	0
15	0	0
16	0	0
17	0	0
18	365.2844	453.6027
19	0	0
20	0	0
21	1238.71	0
22	0	0
23	0	0
24	406.2122	0
25	0	0

Table 35. The slacks of desirable outputs and undesirable outputs components of model (5) with production trade-offs 8.

Chains stores	e_1^*	e_2^*	s_1^*
1	0	0	0
2	0.2871	0	0
3	146567.3	0	0
4	0	0	0
5	0.0069	2909.583	0
6	0.4658	0	0
7	0	0	0
8	67949.52	0	107.6506
9	32225.65	316.13	9.8474
10	0.1682	0	0
11	0	0	0
12	125830.4	0	55.5084
13	1321333	2291.647	2864.783
14	0	0	0
15	183677.8	0	0
16	875823.3	0	106.8525
17	4.8567	0	0
18	0.6606	0	0
19	0	0	0
20	0	0	0
21	2211080	0	32.7802
22	0	0	0
23	0	0	0
24	0.8657	0	0
25	0	0	0

In Fig 5, we compare the initial undesirable output level and target undesirable output level of chains stores for different production trade-offs. As can be seen, undesirable output level of target obtained of two-process for chains stores number 13, 10, 15, 16 and 21, compared to their initial level, they have decreased more than other stores.

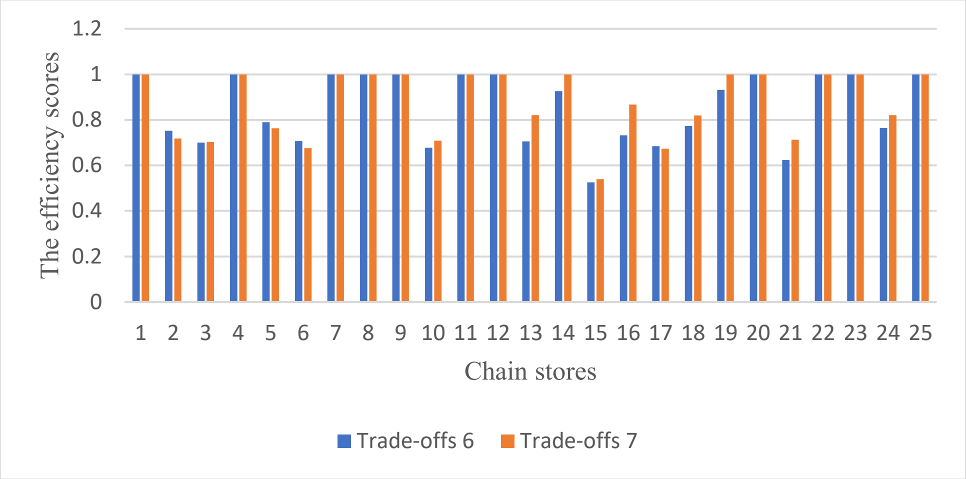

Figure 1. Comparison of efficiency scores of chains stores base on the production trade-offs 6 and 7.

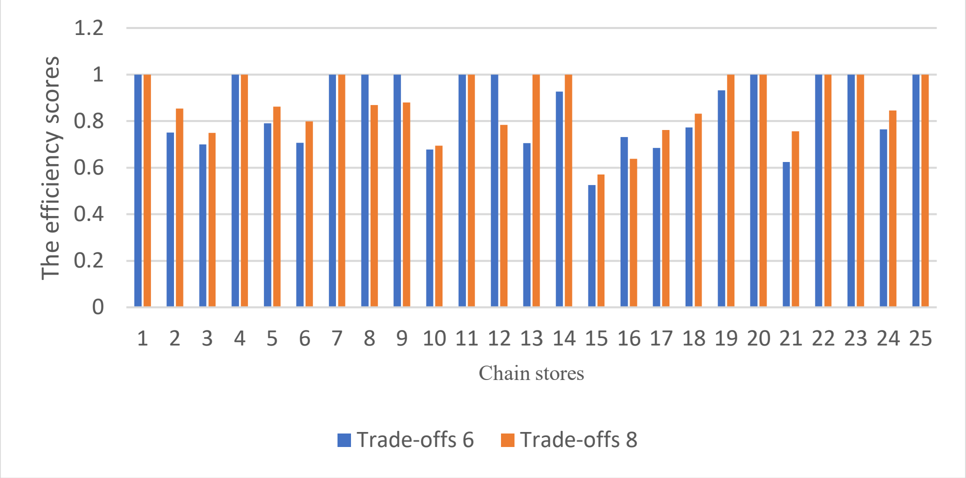

Figure 2. Comparison of efficiency scores of chains stores base on the production trade-offs 6 and 8.

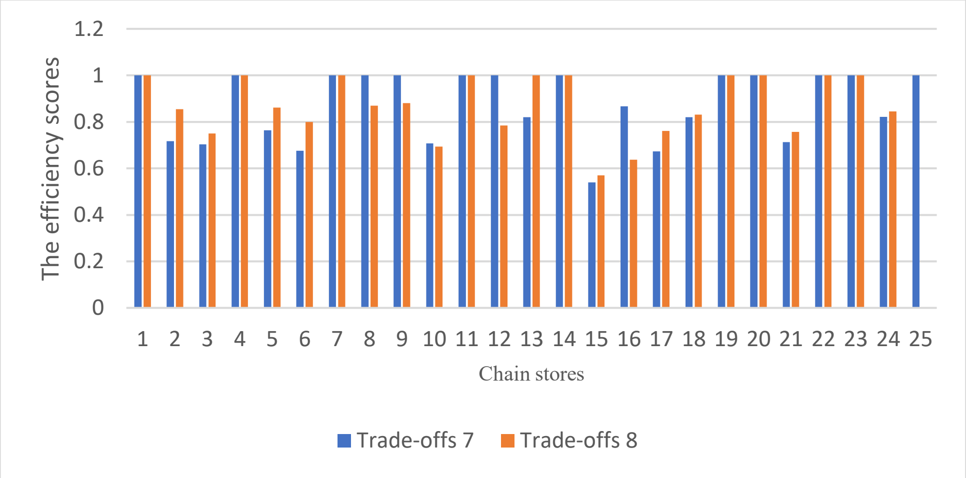

Figure 3. Comparison of efficiency scores of chains stores base on the production trade-offs 7 and 8.

10.57647/ijm2c.2025.152526

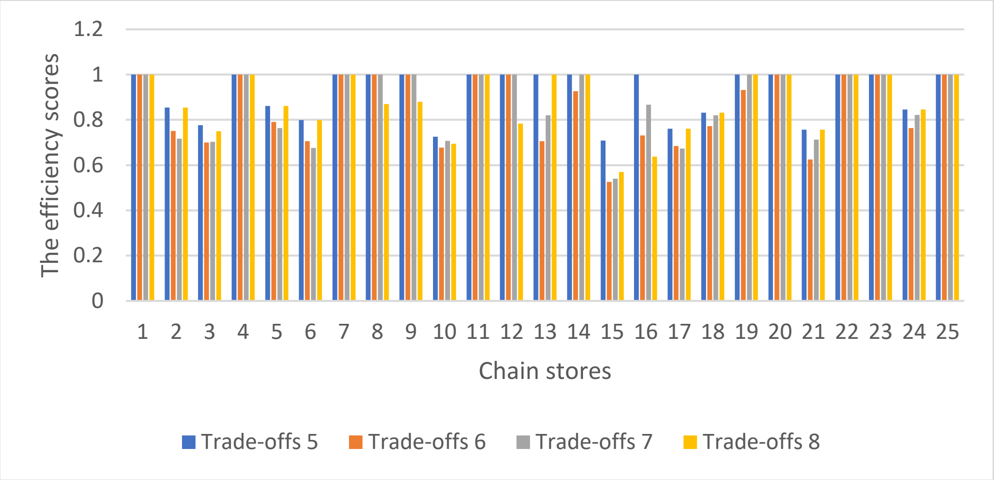

Figure 4. Comparison of the efficiency scores of chains stores base on the different production trade-offs.

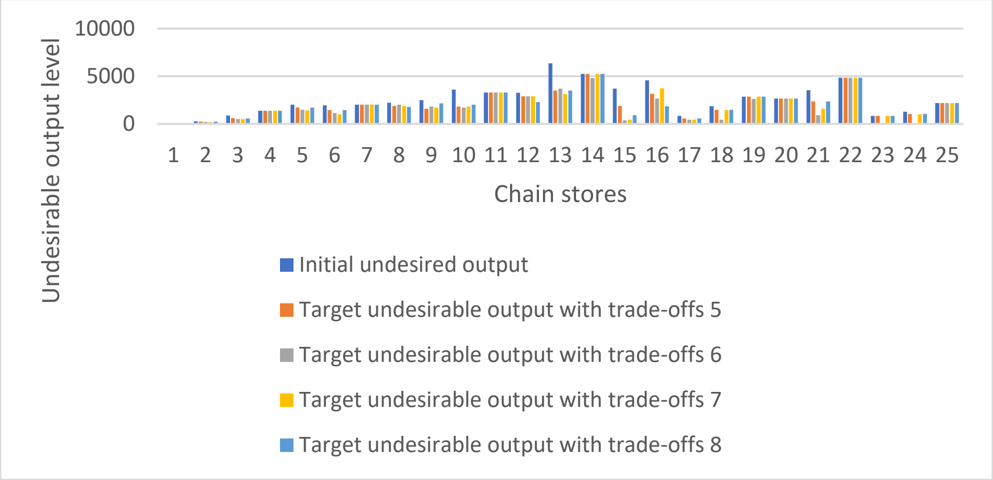

Figure 5. Comparison of initial undesirable output level and target undesirable output level of chains stores for different production trade-offs.

5. Conclusion

In this paper, we present a two-step process to obtain strongly efficient targets corresponding to DMUs that produce undesirable outputs at the same time as desirable outputs. In the first stage, we obtain radial improvement corresponding to components of inputs, desirable output and the undesirable output of the DMU under evaluation based on the DDF. In the second stage, we obtain the maximum amount of the sum of inefficiency slacks in the components of inputs, desirable output and the undesirable output of this DMU. The optimal values of slack variables indicate the level of inefficiency in these components. In other words, if a DMU has an amount of inefficiency slack, in the first stage, it is depicted on the weak efficiency frontier from the PPS introduced based on the production trade-offs from inputs and outputs. In the second stage, we obtained strongly efficient targets corresponding to these DMUs. We proved that the targets obtained based on the proposed process are strongly efficient DMUs on the efficiency frontier. These targets have a minimum level of inputs and undesirable outputs and a maximum level of desirable outputs among DMUs. These targets are feasible and can be produced. The two-stage approach is presented based on the DDF model in DEA. By choosing the right directions based on the opinion of the DM, we can achieve different strongly efficient targets for each of the DMUs. In this paper, we used individual-proportion weak disposability and disposability of undesirable outputs (modify) in the first and second stages, respectively, to reduce the undesirable outputs. Also, in the numerical and applied example sections, we showed that by choosing different production trade-off matrices according to the opinion of the DM, we can achieve different targets for the DMUs. In other words, the choice of targets is affected by the choice of directions to solve the model and the matrixes corresponding to production trade-offs. In future work, we can develop the approach presented in this paper for other structures of weight restriction in DEA. We can also develop and investigate the stability of models in the case that we have unlimited and free productions of desirable outputs. Also, the models presented in this paper were in envelopment form. We can also develop these models for multiplier forms. Another suggestion is to determine the optimal level of inputs and outputs based on the predetermined target efficiency score based on inverse DEA models in the presence of production trade-offs.

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Authors Contribution
All the authors have participated sufficiently in the intellectual content, conception and design of this work or the analysis and interpretation of the data (when applicable), as well as the writing of the manuscript.

Availability of data and materials
The data that support the findings of this study are available from the corresponding author, upon reasonable request.

Conflict of interests
The author states that there is no conflict of interest.

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