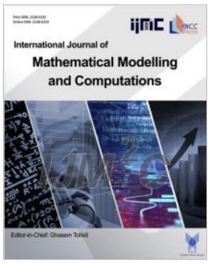

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A New Numerical Method for Solving the Delay Black-Scholes Model

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Abstract

Stochastic Delay Differential Equations (SDDEs) provide a powerful framework for modeling systems with memory effects. The objective of this study is to provide a numerical solution for stochastic delay differential equations, with a particular focus on the delayed Black-Scholes model, using the spectral collocation technique that employs radial basis functions. In this method, M-panels and r-point Newton-Cotes integration are used to approximate the Itô integral. The main advantage of the proposed method is that it is easy to apply and results in an algebraic equations system that is directly solved by numerical methods. Additionally, we analyze the stability and accuracy of the scheme through error estimation and comparisons with benchmark methods. To validate the approach, several numerical examples, including both linear and nonlinear SDDEs, are provided, demonstrating the method's fast convergence and computational robustness. The results highlight the effectiveness of the spectral collocation approach in handling stochastic delays, offering a reliable framework for financial and engineering applications where randomness and delay play a critical role.

Keywords: Delay Black-Scholes model; Spectral collocation technique; Radial basis functions, Newton-Cotes integration; Itô integral

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1. Introduction

1.1. The role of stochastic differential equations in financial analysis

The role of stochastic differential equations in financial analysis has grown significantly, making them indispensable for assessing asset valuation. Several mathematical models are used for pricing, including the Black-Scholes model [1] . Black-Scholes and Merton pioneered the use of SDEs to model stock price dynamics [2]. Although this model is used today, it has been severely criticized, primarily due to the assumptions on which it is based. One of the most significant criticisms concerns the assumption that volatility is constant [2] , [3] . Since empirical evidence shows that volatility is actually time-dependent so that it is not predictable. Sometimes researchers refer to incorrect predictions made by Black-Scholes model as a major limitation in its practical application. Recent developments in numerical methods have expanded our toolkit for financial modeling. While approaches like the homotopy perturbation method [4] and Adomian decomposition [5] have shown effectiveness for certain classes of nonlinear equations, they remain limited in their application to delay stochastic systems. This highlights the need for more specialized techniques in financial mathematics.

1.2. Challenges in modeling real-world phenomena

Accurately capturing real-world financial behavior is complicated by the influence of historical factors on systems. The fundamental principle of cause and effect, which ties the future of the system solely to its current conditions rather than its past, does not apply in this context. on the other hand, the presence of environmental noise that causes disturbances in the system leads to the emergence of stochastic delay differential equations (SDDEs).

1.3. Incorporating historical market memory

The reason for using SDDEs is institutional and expert traders rely on historical data (e.g., past stock prices) to forecast market trends and direct their investments. However, traditional and standard stochastic ordinary differential models do not account for feedback from this behavior. By incorporating delay parameter (τ) into the standard model, we integrated historical data and its feedback effects into our analysis. The SDDE provides a real formula for asset price estimation in an inefficient financial market than geometric Brownian motion based model. Research on these topics are available at [6,7,8, 9] and [10-13]. So it is necessary to develop a model that takes into account the impact of the past events on the current and future states of the system. Stochastic delay differential equations are considered as a powerful model and simulation tool for these systems and processes. Therefore, the evolution model of stock dynamics is as follows in SDDE form [14]:

$$\begin{cases} dS(t)=f(S(t),S(t\text{-}\tau),t)dt \\ \qquad +g(S(t),S(t\text{-}\tau),t)dB(t), \quad t\in[0.T], \\ S(t)=\phi(t), \qquad\qquad t\in[\text{-}\tau,0], \end{cases} \tag{1}$$

Where, the delay $\tau>0$ is fixed. $R_{+}=[0,\infty)$, $g: R\times R\times R_{+}\to R$ and $f: R\times R\times R_{+}\to R$. B is the Wiener process.
If f and g are drift and diffusion respectively, $\phi(t)$ is called the primary stochastic process. If the diffusion coefficient is a function of the stochastic process, we have a multiplicative noise. But if the diffusion coefficient is independent of the Stochastic process, it is called an additive noise. And finally, if the functions f and g are independent of time, we have an autonomous equation:

$$\begin{aligned} dS(t)&=f(S(t),S(t\text{-}\tau),t)dt \\ &\quad +g(S(t),S(t\text{-}\tau),t)dB(t),\ t\in[0,T], \\ S(t)&=\phi(t), \qquad t\in[\text{-}\tau,0]. \end{aligned} \tag{2}$$

We can also have equation Eq. (3) in the following form,

$$\begin{aligned} dS(t)&=S(0)+\int_0^t f\left(S(u),S(u\text{-}\tau)\right)du \\ &\quad +\int_0^t g\,(S(u),S(u\text{-}\tau))dB(u), \end{aligned} \tag{3}$$

1.4. Stability analysis of stochastic delay differential equations

Currently, there exists no efficient analytical solution or comprehensive numerical method for studying the stability of stochastic delay differential equations (SDDEs).Recently,some numerical approaches including Euler-Maruyama method [14] , [15]and Theta method [16] have been employed to generate approximate solution of for these equations.

1.5. Numerical approach using RBFs

This paper aims to develop a numerical approach for solving (SDDEs) employing the spectral collocation technique that utilizes Radial Basis Functions (RBFs). Previous studies have demonstrated the effectiveness of RBFs in related problems. Notably, [17] proposed a stable collocation approach for neutral delay stochastic differential equations of fractional order, while [18] investigated the collocation method specifically for stochastic delay differential equations. Ahmadi et al. (2017) applied RBFs-based collocation method to solve stochastic fractional differential equations (SFDEs) [19]. Kosec and Sarler (2008) investigated RBFs-based collocation model for Draw flow [20] . Recent advances in spectral methods have shown particular promise, with [21] developing a spectral collocation approach for SPDEs with fractional Brownian motion. In financial applications, U. Pettrsson et al. [22] developed a numerical method to price option based on radial basis functions, while [23] introduced an innovative RBF-LOD method for solving stochastic diffusion equations. Reference [24] employed radial basis functions method to solve fractional Schrödinger Black-Scholes equation in option pricing of financial problems. Researchers [25] created a technique utilizing radial basis functions to solve the Black-Scholes equation for both European and American options, evaluating European option pricing for multiple assets. This method evaluates European

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purchase pricing based on several assets. In the above researches, radial basis functions were applied to solve various differential equations.

1.6. Prior work and research objectives

Building upon these foundations, radial basis functions have emerged as an effective tool for solving stochastic delay differential equations. The literature documents several numerical and spectral methods previously applied to such equations: Akhtari et al (2014) developed a weak continuous adaptive Euler-Maruyama method for (SDDEs) [15]. Akhtari (2019) subsequently conducted comprehensive analyses of convergence, stability, and numerical solutions for delay-dependent SDEs [26] . Yin and Gan (2015) considered Chebyshev spectral collocation method for solving stochastic delay differential equations [27]. Maleknejad et al. [28] proposed a composite function approach for multi-delay dynamic systems. A thesis [14] applied the Euler-Maruyama method to SDDEs-based asset pricing models.

1.7. Research contributions

The purpose of the present research is to extend spectral collocation method for solving stochastic delay differential equation especially delay Black-Scholes model based on radial basis functions. The organization of this paper is outlined as follows: Section 2 provides a set of key definitions relevant to solving stochastic delay differential equations. In Section 3, we detail significant operations, including the numerical solution and convergence analysis. Section 4 validates the approach through: numerical experiments to demonstrate the method's effectiveness, and Section 5 concludes the paper.

2. Definitions and requirements

In this section, we present an overview of the essential definitions and prerequisites needed for a comprehensive understanding of the following content.

2.1. Stochastic integrals

Stochastic integral is defined as follows:

$$I=\int_0^t f(s)dB(s),$$

where $\{B(t), t \geq 0\}$ is Brownian motion and $\{f(s)\}$ is a one-dimensional process. Since the trajectory of Brownian motion is not differentiable at any point, the standard definition of a stochastic integral does not apply. However, by leveraging the properties of Brownian motion, we can establish an integral. Such integrals were first defined in 1949 by Itô and since then they are called Itô integral [29].

Definition 2.1. Suppose $0 \leq a<b<\infty$. Denote by $M^2([a,b];R)$ the space of all real-valued measurable functions $\{F_t\}_{t\geq 0}$-adapted processes $f=\{f(t)\}_{a\leq t\leq b}$ [29], such that

$$\| f \|_{a,b}^2 =E\left(\int_a^b |f(t)|^2 dt\right)<\infty. \tag{4}$$

Definition 2.2. A real-valued stochastic process $g=\{g(t)\}_{a\leq t\leq b}$ is called a simple process when there exists a partition $P=\{a=t_0<t_1<\ldots<t_k=b\}$ of $[a,b]$ interval, and bounded random variables ξ_i, $0 \leq i \leq k$-1 [29] ., such that

$$g(t)=\sum_{i=0}^{k-1} \xi_i I_{(t_i,t_{i+1}]}(t). \tag{5}$$

We represent the space of such functions as $M_0([a,b];R)$

Definition 2.3. For simple processes of g that are in the form of (5) in $M_0([a,b];R)$ [29] , define

$$\int_a^b g(t)dB(t)=\sum_{i=0}^{k-1} g(t_i)\left(B_{t_{i+1}}-B_{t_i}\right). \tag{6}$$

The above integral is defined as stochastic or Itô integral according to Brownian motion.

Definition 2.4. Suppose $f \in M^2([a,b];R)$. The Itô integral of f corresponding to $\{B(t)\}$ [29] is defined as below

$$\int_a^b f(t)dB_t= \lim_{n\to\infty} \int_a^b g_n(t)dB_t \quad \text{in} \quad l^2(\Omega;R), \tag{7}$$

in which a sequence of simple processes like $\{g_n\}$ is presented so that

$$\lim_{n\to\infty} E\int_a^b |f(t)-g_n(t)|^2 dt=0. \tag{8}$$

Lemma 2.5. If $g \in M_0([a,b];R)$, then:

$$E\int_a^b g(t)dB(t)=0, \tag{9}$$

$$E\left|\int_a^b g(t)dB(t)\right|^2=E\int_a^b |g(t)|^2 dt. \tag{10}$$

The proof can be found in ([29] ,p.19).

Property 1. Suppose $g(s,w)=g(s)$ only depends on

s and g is continuous and bounded on the interval $[0, t]$ [30] .Then:

$$\int_0^t g(s)dB_s = g(t)B_t - \int_0^t B_s\, dg_s. \quad (11)$$

2.2. Radial basis functions

2.2.1. *Definition of RBFs*

In this section, radial basis function (RBF) is introduced. Suppose we want to approximate a function like $s(x)$ in $\Omega \subset R^d$ range, so we require the scattered points of $x_1, \dots, x_N \in \Omega$. the function $s(x)$ is approximated using the radial basis function, with the values $s(x_1), \dots, s(x_N)$ in nodal points,

$$s(x) = \sum_{i=1}^{N} \lambda_i\, \phi(\| x - x_i \|), x \in \Omega, \quad (12)$$

where λ_i is unknown cofficients. In mathematics, RBF has real values ϕ and its value only depends on the distance between the input and some fixed point. So, $\phi(\| x-x_i \|)$ is usually the Euclidean distance. Interpolation is performed to calculate the coefficients λ_i.

$$s(x_i) = f_i, \qquad i = 1, \dots, N, \quad (13)$$

After interpolation, a linear combination is obtained as follows:

$$A\lambda = f, \quad (14)$$

where, A_{ij}, λ, f are defined as follows,

$$A_{ij} = \phi(\| x_i - x_j \|),$$

$$\lambda = [\lambda_1, \dots, \lambda_N]^T,$$

$$f = [f(x_1), \dots, f(x_N)]^T.$$

Examples of radial basis functions (RBFs) employed in this study are listed in Table1 [31] .

Table 1. Common types of radial basis functions.

RBF	$\phi(r)$
Gaussian	$\phi(r) = e^{-(\epsilon r)^2}$
Multiquadric	$\phi(r) = \sqrt{1 + (\epsilon r)^2}$
Inverse Multiqudric	$1/\sqrt{r^2 + c^2}$
Linear	r
Cubic	r^3
Thin Plate Spline	$r^2 ln(r)$

In this paper, the use of radial basis functions (RBFs) is scientifically justified based on several key criteria, including their high flexibility and exceptional performance in handling scattered data that a significant advantage for scientific problems. These functions are well studied theoretically and guarantee properties such as the existence of a unique solution. They exhibit excellent compatibility with other numerical methods, reduce problem solving time, and deliver more stable and reliable results.

2.3. Numerical integration: Newton-Cotes quadrature rules

2.3.1. *Integration methods*

Integration encompasses a wide range of topics. The term quadrature rule refers to any numerical technique used to compute an approximation of an integral If of a function $f(s)$:

$$If := \int_a^b W(s) f(s) ds, \quad (15)$$

here $w(s)$ is the weight function that has a well-defined analytical expression. In principle, a technique can utilize any accessible information regarding the function $f(s)$, including the values of its derivatives at one or several specified points. However, when limited to nodal points $\{x_i, i = 1, \dots, r\}$, the approximation Q has the form:

$$Q_r f := \sum_{i=1}^{r} w_i\, f(x_i) = If - Ef, \quad (16)$$

where Ef is the error.

2.3.2. *Newton-Cotes rules*

The Newton-Cotes rules (Q_r) of degree r-1 use equally spaced points in $[a, b]$ to generate approximations $Q_r f$ to I_f. However, these rules often exhibit unsatisfactory convergence properties: the error $|If - Q_r f|$ may not converge smoothly to zero, even for functions that appear to be well-behaved f. For the common case where $w(s) = 1$, the rule is translation invariant and scales linearly. Comparing two rules:

$$Q_r(0,1) : \{x_i = \frac{(i-1)}{(r-1)}; w_i\}$$

And

$$Q_r(a. a + (r-1)h) : \{x_i = a + (i-1)h; w_i'\},$$

then

$$w_i' = h w_i.$$

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The M -panel r-point integration rule is obtained estimates $\int_a^b f(s)ds$ with $h=\frac{b\text{-}a}{M}$:

$$If=\int_a^b f(s)ds=\sum_{j=1}^{M}\int_{a+(j-1)h}^{a+jh} f(s)ds$$
$$=\sum_{j=1}^{M} Q_r\,(a+(j\text{-}1)h.\,a+jh)f+E_{r.M}. \qquad (17)$$

when r is fixed and M is increased and a sequence of approximations to f is obtained, [32] .

Property 2. For Riemann integrable functions [32].and fixed r as in (17) :

$$\lim_{M\to\infty} E_{r,M}(f)=0. \qquad (18)$$

3. Solution procedure

This section deals with the approximation of the solution of equation (3) using the collocation spectral method based on $RBFs$.

$$ds(t)=s_0+\int_0^t f\left(s(u),s(u\text{-}\tau)\right)du$$
$$+\int_0^t g\,(s(u),s(u\text{-}\tau))dB(u). \qquad (19)$$

so that, the approximation s is a linear combination of a set of basis functions $\{\phi_i(t)\}_{i=0}^{N}$ with respect to the coefficientλ_i, where λ_i is an unknown coefficient. To find λ_i we use equation (20). So, we have:

$$s(t)\simeq s_N(t)=\sum_{i=0}^{N}\lambda_i\,\phi_i(t). \qquad (20)$$

$$s_N(t) = s_0+\int_0^t f\left(s_N(u),s_N(u\text{-}\tau)\right)du$$
$$+\int_0^t g\left(s_N(u),s_N(u\text{-}\tau)\right)dB(u) \quad t\in[0,1] \qquad (21)$$
$$+Res_N(t),$$

here, $Res_N(t)$ is the residual error when $t\in[0,1]$. This error is considered when we replace the equation Eq. (20) in Eq. (19) . Using Property 1, results driven:

$$s_N(t) = s_0+t\int_0^1 f\left(s_N(tv),s_N(tv\text{-}\tau)\right)dv \qquad (22)$$
$$+g\left(s_N(t),s_N(t-\tau)\right)B(t)$$
$$-t\int_0^1 g'\left(s_N(tv).\,s_N(tv-\tau)\right)$$
$$B(tv)dv$$
$$+Res_N(t),$$

To solve the Itô integral in Eq. (21), the Newton-Cotes rule is used as follows. Therefore, the integration interval is transferred to the interval of $[0,t]$ into interval $[0,1]$.

$$u=tv,\qquad v\in[0,1],\qquad u\in[0,t].$$

$$s_N(t) = s_0+t\int_0^1 f\left(s_N(tv),s_N(tv\text{-}\tau)\right)dv$$
$$+(g(s_N(t),s_N(t\text{-}\tau))B(t)$$
$$\text{-}t\int_0^1 g'\,(s_N(tv),s_N(tv\text{-}\tau))B(tv))dv \qquad (23)$$
$$+Res_N(t).$$

So, Eq. (23) is rewritten as follows,

$$s_N(t)=s_0+g\left(s_N(t),s_N(t\text{-}\tau)\right)B(t)$$
$$+t\sum_{j=1}^{M}\sum_{i=1}^{r} w_i^{(j)}\,[f(s_N(tv_i^{(j)}),s_N(tv_i^{(j)}\text{-}\tau))$$
$$\text{-}g'(s_N(tv_i^{(j)}),s_N(tv_i^{(j)}\text{-}\tau))B(tv_i^{(j)})] \qquad (24)$$
$$+E_{r,M}+Res_N(t).$$

Where, $\{v_i^j\}_{i=1}^r,\{w_i^j\}_{i=1}^r, j=1,\ldots,M$ are explained in [33] . $E_{r,M}$ is the error between the the Newton-Cotes rule and the exact integral. Finally, by replacing the collocated points t_l in Eq. (24), we have:

$$s_N(t_l)=s_0+g\left(s_N(t_l),s_N(t_l\text{-}\tau)\right)B(t_l)$$
$$+t_l\sum_{j=1}^{M}\sum_{i=1}^{r} w_i^{(j)}\,[f(s_N(t_lv_i^{(j)}),s_N(t_lv_i^{(j)}\text{-}\tau)) \qquad (25)$$
$$\text{-}g'(s_N(t_lv_i^{(j)}),s_N(t_lv_i^{(j)}\text{-}\tau))B(t_lv_i^{(j)})].$$

Now, a set of arbitrary points $\{t_0,\ldots,t_l\}$ from the interval $[a,b]$ is utilized, which are called collocated points. To find the unknown coefficients we cosider,

$$t_l=a+lh,\quad l=0,1,\ldots,N.$$

By applying the above conditions, a system of linear equations is obtained. To solve the system of linear equations, the unknown coefficients are found with one of the numerical methods such as Newton method. As a result, the approximate solution of the equation is obtained.

3.1. Convergence of numerical approximation

This section analyzes the convergence behavior of the proposed method in solving the SDDE (3) , particularly the delay Black-Scholes model, is examined. The following definitions are provided.

Definition 3.1. Functions f and g satisfy the local Lipschitz condition [14] . in Eq. (3), if for every integer of $n \geq 1$, there is a positive constant K_n, so,

$$|f(x_1,y_1,t)-f(x_2,y_2,t)| \vee |g(x_1,y_1,t)-g(x_2,y_2,t)| \leq K_n(|(x_1-x_2|+|y_1-y_2|),$$

For every $x_1,y_1,x_2,y_2 \in \mathbb{R}$ with $|x_1| \vee |y_1| \vee |x_2| \vee |y_2| \leq n$ and for every $t \in \mathbb{R}_+$, where $|x| \vee |y|$=max $(|x| \vee |y|)$.

Defenition 3.2. Functions f and g satisfy the linear growth condition [14] . in Eq. (3). If there is a positive constant K, so,

$$|f(x,y,t)| \vee |g(x,y,t)| \leq K(1+|x|+|y|).$$

For every $(x,y,t) \in R \times R \times R_+$.

Theorem 3.3. (Convergence of numerical approximation)

Let the following assumptions hold:

- The exact solution $s(t)$ of the SDDE (3) satisfies:

$$|s|_{L^2}^2 := E[|s(t)|^2] < \infty, \forall t \in [0,T].$$

- The delay $\tau>0$ is fixed and satisfies $\tau<T$,
- $t \in [0,T], \qquad T \geq \tau$.
- For any stochastic process $s(t)$ define the time interval $[0.T]$, we define the L^2 norm as:

$$|s|_{L^2(\Omega)} := \left(E[|s|^2]\right)^{\frac{1}{2}}.$$

- The constants K_1, K_2 from definition (5) satisfy:

$$K_1^2+K_2^2>0.$$

Then, the numerical solution $s_N(t)$ converges to $s(t)$ in the L^2-sense:

$$\lim_{N\to\infty} |s-s_N|_{L^2}=0.$$

Proof. Let $\varepsilon_N=s(t)-s_N(t)$ be the absolute error between the exact solution $s(t)$ and the numerical solution $s_N(t)$ obtained using the spectral collocation technique that utilizes (RBFs), the exact solution satifies the (SDDE):

$$s(t)=s_0+\int_0^t f\left(s(\upsilon),s(\upsilon-\tau)\right)d\upsilon + \int_0^t g\left(s(\upsilon),s(\upsilon-\tau)\right)dB(\upsilon), \tag{26}$$

and the numerical solution $s_N(t)$ satisfies:

$$s_N(t)=s_0+\int_0^t f\left(s_N(\upsilon),s_N(\upsilon-\tau)\right)d\upsilon + \int_0^t g\left(s_N(\upsilon),s_N(\upsilon-\tau)\right)dB(\upsilon) + \mathrm{Res}_N(t). \tag{27}$$

Where $\mathrm{Res}_N(t)$ is the residual term, the absolute error can be expressed as:

$$\varepsilon_N(t) = \int_0^t f(s(\upsilon),s(\upsilon-\tau))d\upsilon + \int_0^t g(s(\upsilon),s(\upsilon-\tau))dB(\upsilon) - \int_0^t f\left(s_N(\upsilon),s_N(\upsilon-\tau)\right)d\upsilon - \int_0^t g\left(s_N(\upsilon),s_N(\upsilon-\tau)\right)dB(\upsilon) + E_{r,M} + Res_N(t) \tag{28}$$

In this method, $E_{r,M}$ measures the error between the Newton-Cotes quadrature (with parameters M and r, where M is the number of subintervals and r is the number of points) and the exact value of the integral.

$$\varepsilon_N(t) = \int_0^t f(s(\upsilon),s(\upsilon-\tau))-f(s_N(\upsilon),s_N(\upsilon-\tau))d\upsilon + \int_0^t g\left(s(\upsilon),s(\upsilon-\tau)\right) - g\left(s_N(\upsilon),s_N(\upsilon-\tau)\right)dB(\upsilon) + E_{r,M} + Res_N(t) \tag{29}$$

it is concluded from Eq. (29) that

$$\varepsilon_N(t) = \mu_1(t)+\mu_2(t)+\mu_3(t). \tag{30}$$

Where,

$$\mu_1(t)=\int_0^t f(s(\upsilon),s(\upsilon-\tau))-f(s_N(\upsilon),s_N(\upsilon-\tau))d\upsilon, \tag{31}$$

$$\mu_2(t)=\int_0^t g(s(\upsilon),s(\upsilon-\tau))-g(s_N(\upsilon), s_N(\upsilon-\tau))dB(\upsilon), \tag{32}$$

and

$$\mu_3(t) = E_{r,M} + Res_N(t). \tag{33}$$

We find upper bound for Eqs. (31), (32) and (33)

$$|\mu_1(t)|\leq \int_0^t |f(s(\upsilon),s(\upsilon-\tau)) -f(s_N(\upsilon),s_N(\upsilon-\tau))|d\upsilon. \tag{34}$$

According to Lipschitz property in the definition 3.1, we have

$$|\mu_1(t)|\leq \int_0^t k_1 (|s(\upsilon)-s_N(\upsilon)| +|s(\upsilon-\tau)-s_N(\upsilon-\tau)|)d\upsilon, \tag{35}$$

$$|\mu_1(t)|\leq k_1(\int_0^t |s(\upsilon)-s_N(\upsilon)|d\upsilon + \int_0^t |s(\upsilon-\tau)-s_N(\upsilon-\tau)|d\upsilon), \tag{36}$$

consequently,

$$|\mu_1(t)|\leq k_1(\int_0^t |\varepsilon_N(\upsilon)|d\upsilon + \int_0^t |\varepsilon_N(\upsilon-\tau)|d\upsilon). \tag{37}$$

Using the change of variable $u = \upsilon - \tau$, the second integral of Eq. (37) becomes:

$$\int_0^t |\varepsilon_N(\upsilon-\tau)\Big|d\upsilon = \int_{-\tau}^{t-\tau} |\epsilon_N(u)\Big|du,$$

for $t \geq \tau$,

$$\int_{-\tau}^{t-\tau} |\epsilon_N(u)|du= \int_{-\tau}^{0} |\epsilon_N(u)|du+ \int_0^{t-\tau} |\epsilon_N(u)|du,$$

we combine the integrals:

$$|\mu_1(t)|\leq k_1(\int_0^t |\varepsilon_N(\upsilon)|d\upsilon + \int_0^{t-\tau} |\varepsilon_N(u)|du), \tag{38}$$

and we can further bound:

$$|\mu_1(t)| \leq 2k_1 \int_0^t |\varepsilon_N(u)|du, \tag{39}$$

squaring both sides and according to the Cauchy-Schwarz inequality, it is stated that:

$$|\mu_1(t)|^2 \leq 4k_1^2(\int_0^t |\varepsilon_N(u)|du)^2, \tag{40}$$

$$|\mu_1(t)|^2 \leq 4k_1^2 t \int_0^t |\varepsilon_N(u)|^2 du, \tag{41}$$

so, we have:

$$\| \mu_1 \| \leq 2k_1\sqrt{t} \| \varepsilon_N \|_{L^2}. \tag{42}$$

We obtain an upper bound for equation Eq. (32)

$$|\mu_2(t)|\leq \int_0^t |g(s(\upsilon),s(\upsilon-\tau))-g(s_N(\upsilon) ,s_N(\upsilon-\tau))|dB(\upsilon), \tag{43}$$

according to the Lipschitz property in definition 3.1, the following holds:

$$|\mu_2(t)|\leq \int_0^t k_2 (|s(\upsilon)-s_N(\upsilon)| +|s(\upsilon-\tau)-s_N(\upsilon-\tau)|)dB(\upsilon), \tag{44}$$

$$|\mu_2(t)|\leq k_2[\int_0^t |\varepsilon_N(\upsilon)|dB(\upsilon) + \int_0^t |\varepsilon_N(\upsilon-\tau)|dB(\upsilon)], \tag{45}$$

using Lemma1(itô isometry) and squaring, it is concluded that:

$$E[|\mu_2(t)|^2]\leq k_2^2 E[\int_0^t (|\varepsilon_N(\upsilon)| +|\epsilon_N(\upsilon-\tau)|)^2], \tag{46}$$

like μ_1 using the change of variable and interval, we have:

$$\mathbb{E}[|\mu_2(t)|^2] \leq 4k_2^2 \int_0^t \mathbb{E}\,[|\varepsilon_N(u)|^2]du, \tag{47}$$

so, we have

$$\| \mu_2 \| \leq 2K_2 \| \varepsilon_N \|_{L^2}. \tag{48}$$

For μ_3, it is clear

$$\| \mu_3 \| \leq \| Res_N \| + \| E_{r,M} \|. \tag{49}$$

According to Eq. (30) and the following inequality:

$$(a+b+c)^2 \leq 3(a^2+b^2+c^2), \tag{50}$$

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By combining the results of equations (42) (deterministic part), (48) (stochastic part), (49) (method error), and the fundamental inequality (50), we arrive at the following key:

$$\| \varepsilon_N \|^2 \leq 3(\| \mu_1 \|^2 + \| \mu_2 \|^2 + \| \mu_3 \|^2), \quad (51)$$

$$\|\varepsilon_N\|^2 \leq 3(4K_1^2(t)\|\varepsilon_N\|^2 + 4K_2^2\|\varepsilon_N\|^2 + \|Res_N\|^2 + \|E_{r,M}\|^2), \quad (52)$$

consequently,

$$\|\varepsilon_N\|^2 \leq 3(4(K_1^2(t)+K_2^2))\|\varepsilon_N\|^2 + \|Res_N\|^2 + \|E_{r,M}\|^2), \quad (53)$$

and

$$\|\varepsilon_N\|^2 \leq 12(K_1^2(t)+K_2^2)\|\varepsilon_N\|^2 + 3\|Res_N\|^2 + 3\|E_{r,M}\|^2), \quad (54)$$

from Eq. (54), we obtain the following equation:

$$\| \varepsilon_N \|^2 \left(1 - 12(K_1^2(t) + K_2^2)\right) \leq 3(\| Res_N \|^2 + \| E_{r,M} \|^2), \quad (55)$$

finally, according to $12(K_1^2(t) + K_2^2) < 1$,we have

$$\| \varepsilon_N \|^2 \leq \frac{3\left(\| Res_N \|^2 + \| E_{r,M} \|^2\right)}{\left(1 - 12(K_1^2(t) + K_2^2)\right)}. \quad (56)$$

because the spectral method converges as N increases and the Newton-Cotes method becomes more accurate as M increases.

- $\|RES_N\| \to 0$
- $\|E_{r.M}\| \to 0$

Consequently, $\| \varepsilon_N \| \to 0$, and convergence is achieved. □

4. Numerical results

In this section, we present various numerical examples of stochastic delay differential equations, highlighting the accuracy and effectiveness of our proposed method. All calculations were performed using Maple 2019 software on a laptop with the following specifications: Intel ®Core i4 processor, 2 · 10. GHz, 8 GB RAM. In this paper, to evaluate the accuracy of the proposed method, the absolute error is calculated as follows:
$\varepsilon_N = |s(t) - s_N(t)|$,

This error has been computed for various examples and different radial basis functions such as multiquadric and Gaussian functions. The results demonstrate the rapid convergence and high accuracy of the proposed method. Here, $s(t)$ represents the solution of equation (3) for $\mathcal{T} = 500$. Since equation (3) does not have an exact solution, we consider this numerical solution as the reference exact solution for the problem. Using the spectral collocation method with a sufficiently small step size $\Delta\zeta = \frac{1}{\mathcal{T}}$ we obtain the approximate solution $s_N(t)$.

Example 4.1. The linear SDDE is considered [15] . :

$$\begin{cases} dS(t) = (aS(t)+bS(t-1))dt + (\beta_1+\beta_2 S(t) + \beta_3 S(t-1))dW(t) & t\in[0,2] \\ S(t) = 1+t & t\in[-1,0] \end{cases} \quad (57)$$

where, a and b are real values, and $\{\beta_i = 0, \beta_j = 0, \beta_k = 1\}$ in which $i, j, k \in \{1,2,3\}$ with $i \neq j \neq k$. We take $a = -2$ and $b = 0.1$. $W(t)$ is a m-dimensional standard wiener process($m = 1$). For the case of SDDE (57) with additive noise (i.e., $\beta_2 = \beta_3 = 0$) the exact solution on $[0,1]$ is given by

$$S(t)=e^{at}\left(1+\frac{b}{a^2}\right)-\left(\frac{b}{a}\right)t-\frac{b}{a^2}+\beta_1 e^{at}\int_0^t e^{-as}\, dW(s),$$

The numerical solution of the SDDE (57) was obtained by applying the method outlined in Section 3 . Keep in mind that, N represents the total number of collocation points, while r denotes the number of points and M signifies the number of panels used in the Newton-Cotes quadrature. So, we adjust $r = 8$, $M = 10$, delay $\tau = 1$, $a = -2$, $b = 0.1$. In this example, multiquadric and Gaussian radial basis functions have been employed. The rationale for this selection is as follows: The multiquadric function have High accuracy in approximating smooth functions, Flexibility through shape parameter tuning to control accuracy and stability. The Gaussian function provides: Adaptability to random components, Rapid convergence when approximating functions with abrupt changes, Lower sensitivity to shape parameters compared to multiquadric functions. Numerical errors with different values of N are presented in the following Tables 2 , 3 and 4 and CPU times show that the proposed method is fast and easy to implement. Additionally, Table 5 shows the l_2 error for different N values using radial basis function approximations.

Example 4.2. Consider the following nonlinear SDDE [15].

$$\begin{cases} dS(t) & = aS(t)\big(-S(t-1)\big)dt \\ & +bS^2(t)dW(t) & t \in [0,10] \\ S(t) & = 1+t & t \in [-1,0] \end{cases} \quad (58)$$

To address this example, we apply the proposed method with $\tau = 1, b = 0.01$ and $a = 1$. Table 6 shows the absolute errors, mean errors and CPU times of the proposed method. To show that the proposed method is better, the results of Table 6 are compared with the results of adaptive algorithm method [15] in Table 7.

Example 4.3. The SDDE is considered:
To solve this example, we implement the proposed method with two sets of parameters as follows: $a = -2, b = 0.1, c = 0, d = 0.5$ and $a = -2, b = 0.1, c = 0.5$ and $d = 0$ with $\tau = 1$. The following Tables 8 and 9 show the absolute errors, mean errors and CPU times of the proposed method.

Table2. Error of the proposed method in Example 4.1 various value of N with $\beta_1=1$, $\beta_2=0$, $\beta_3=0$.

	$\phi(r)=\sqrt{(r^2+c^2)}$, c = 0,1			$\phi(r)=e^{-(cr)^2}$, c = 1		
t	$N=2^2$	$N=2^4$	$N=2^5$	$N=2^2$	$N=2^4$	$N=2^5$
0.0	7.25e-04	5.35e-04	3.14e-04	3.00e-03	2.90e-03	2.00e-03
0.2	3.80e-03	2.14e-03	1.05e-03	1.16e-02	1.07e-02	1.00e-02
0.4	8.95e-03	3.35e-03	2.21e-03	5.43e-02	3.02e-02	1.22e-02
0.6	1.05e-02	1.79e-03	1.02e-03	7.20e-02	2.70e-02	1.04e-02
0.8	8.95e-03	1.97e-03	1.75e-03	5.75e-02	3.00e-02	2.25e-02
1.0	1.10e-02	1.09e-03	1.00e-03	2.99e-02	1.56e-02	1.45e-02
1.2	2.25e-02	1.56e-03	1.01e-03	2.70e-02	1.98e-02	1.79e-02
1.4	3.48e-02	4.36e-03	2.15e-03	5.22e-02	4.78e-02	2.59e-02
1.6	2.99e-02	1.89e-03	1.00e-03	7.39e-02	1.47e-02	1.37e-02
1.8	8.40e-03	2.45e-03	1.90e-03	5.70e-02	2.00e-02	2.96e-03
2.0	3.35e-04	2.48e-04	1.33e-04	2.68e-02	1.88e-02	1.39e-02
mean of error	4.32e-03	1.26e-03	1.10e-03	1.41e-02	1.26e-02	1.23e-02
CPU time(s)	16.14	17.37	26.60	14.62	14.91	18.96

Table 3. Error of the proposed method in Example 4.1 for various value of N with $\beta_1=0$, $\beta_2=1$, $\beta_3=0$.

	$\phi(r)=\sqrt{(r^2+c^2)}$, c = 0.1			$\phi(r)=e^{-(cr)^2}$, c = 0.123		
t	$N=2^4$	$N=2^5$	$N=2^6$	$N=2^4$	$N=2^5$	$N=2^6$
0.0	6.78e-04	5.94e-04	4.04e-04	5.46e-02	2.34e-02	1.90e-02
0.2	1.63e-02	1.01e-02	0.86e-02	1.95e-01	5.98e-02	3.40e-02
0.4	5.10e-03	4.49e-03	1.55e-03	1.16e-01	2.40e-02	2.07e-02
0.6	4.95e-03	2.98e-03	2.01e-03	1.10e-02	1.10e-02	1.00e-02
0.8	3.02e-03	1.07e-03	0.65e-03	4.25e-02	3.11e-02	2.95e-02
1.0	3.10e-03	2.56e-03	1.90e-03	4.87e-02	2.65e-02	1.18e-02
1.2	2.44e-03	2.29e-03	1.31e-03	5.60e-03	3.46e-03	2.94e-02
1.4	3.59e-03	3.39e-03	3.00e-03	1.46e-02	1.29e-02	1.19e-02
1.6	4.00e-03	3.99e-03	2.27e-03	7.67e-02	5.27e-02	5.21e-02
1.8	0.98e-03	0.76e-03	0.16e-03	5.59e-02	4.77e-02	4.00e-02
2.0	4.37e-04	3.78e-04	1.33e-04	2.61e-01	6.10e-02	1.73e-02
mean of error	1.81e-02	1.60e-02	1.30e-02	1.84e-02	1.76e-02	1.30e-02

$$\begin{cases} dS(t) & =[aS(t)+bS(t-1)]dt \\ & +[cS(t)+dS(t-1)]dW(t) & t\in[0,2] \\ S(t) & =1+t & t\in[-1,0] \end{cases} \quad (59)$$

Example 4.4 The linear SDDE is considered [14] :

$$\begin{cases} dS(t) & =[-3S(t)+2e^{-1}S(t-1)+3-2e^{-1}]dt \\ & +[cS(t)+dS(t-1)]dW(t) & t\in[0,2] \\ S(t) & =1+e^{-t} & t\in[-1,0] \end{cases} \quad (60)$$

In this example, we examine the effect of different values of the delay:
$\tau = 1, \tau = \frac{1}{2}, \tau = \frac{1}{16}$
and no-delay $\tau = 0$. So, we set two groups of parameters: $a = -3, b = 2e^{-1}$, $c = 0.1, d = 0.1$ and $a = -3, b = 2e^{-1}$, $c = 0.1, d = 0.01$, In Fig.1, four colorful curves start from the point of (0,2). Our goal is to show the effect of different delays on the diffusion part. According to Fig.1, there is a significant change before and after the point of (0,2).

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The red curve is lower than other curves. A notable change is observed both before and after this point. The green curve is very close to red curve when the delay length of $\frac{1}{16}$. Therefore, we can infer that a minor disturbance during the delay will not case in a substantial change; however, as the delay increases from $\frac{1}{16}$ to $\frac{1}{2}$ or 1, The curve shifts upward and diverges from both the green and red curve. The curve of $\tau = 1$ is above the rest of curves, showing that the higher the delay value, the higher its effect.

Table 4. Error of the proposed method in Example 4.1 for various N with $\beta_1=0$, $\beta_2=0$, $\beta_3=1$.

	$\phi(r) = \sqrt{(r^2 + c^2)}$, c = 0.01			$\phi(r) = e^{-(cr)^2}$, c = 0.2		
t	N=2^2	N=2^4	N=2^5	N=2^2	N=2^4	N=2^5
0.0	1.13e-03	1.05e-03	1.00e-03	3.10e-02	2.25e-03	1.12e-03
0.2	3.52e-03	2.31e-03	1.70e-03	1.80e-02	6.18e-03	3.21e-03
0.4	8.76e-03	6.12e-03	4.26e-03	3.00e-02	1.32e-02	1.04e-02
0.6	1.17e-02	1.33e-03	1.25e-03	1.92e-02	6.00e-03	3.94e-03
0.8	1.23e-02	3.62e-03	1.87e-03	2.03e-02	1.96e-02	1.44e-02
1.0	1.34e-02	6.94e-03	3.19e-03	2.60e-02	1.56e-02	1.34e-02
1.2	1.77e-02	4.27e-03	1.53e-03	3.84e-02	3.02e-03	1.23e-03
1.4	2.20e-02	5.29e-03	1.68e-03	4.10e-02	2.05e-02	1.41e-02
1.6	1.90e-02	4.70e-03	3.70e-03	4.50e-02	2.20e-02	1.06e-02
1.8	8.75e-03	2.45e-03	2.32e-05	2.40e-02	1.22e-02	4.73e-03
2.0	5.17e-04	2.11e-04	2.65e-05	1.25e-02	4.59e-03	6.48e-04
mean of error	4.07e-03	2.79e-03	1.27e-03	1.64e-02	1.76e-03	1.10e-03

Table 5. l_2 Error for different values of N with radial basis function with $\beta_1 = 0, \beta_2 = 1, \beta_3 = 0$ of Example 4.1.

N=2^3	N=2^4	N=2^5	N=2^6
1.92e-02	1.81e-02	1.60e-02	1.30e-02

Table 6. Error of proposed method with with $a = 1$ and $b = 0.01$ for example 4.2.

	$\phi(r) = \sqrt{r^2 + c^2}$. c = 2.1		$\phi(r) = e^{-(cr)^2}$. c = 2.5
t	N=5	N=10	N=5
0.0	7.25e-06	6.45e-06	7.24e-06
0.2	1.56e-04	1.35e-04	6.17e-06
0.4	2.74e-04	1.43e-04	3.76e-06
0.6	3.48e-04	1.18e-04	1.05e-07
0.8	3.81e-04	2.48e-04	4.72e-08
1.0	3.74e-04	1.56e-04	4.72e-06
1.2	3.31e-04	1.78e-04	1.30e-05
1.4	2.53e-04	1.36e-04	3.97e-04
1.6	1.41e-04	1.23e-04	8.83e-04
1.8	6.34e-06	5.26e-04	1.43e-03
2.0	1.91e-04	1.58e-04	1.68e-03
mean of error	1.71e-04	1.47e-04	6.00e-06
CPU time(s)	19.50	49.27	19.56

Table 7. The results of adaptive algorithm [15] for example 4.2 with $a = 1, b = 0.01$

k	h	M	E_T	E_{S_1}	E_{TS}	E_{S_2}	ε_c	E
1	0.3000	10	0.0603	0.0156	0.0036	-	0.1385	0.0795
2	0.3000	116	0.0599	0.0045	0.0009	-	0.1213	0.0653
3	0.0520	116	0.0013	0.0041	0.000	-	0.0047	0.0054

1	0.0520	116	0.0013	-	0.000	0.0041	0.0047	0.0054

Table 8. Error of proposed method with a=-2, b=0.1, c=0, d=0.5 for example 4.3.

t	$\phi(r)=\sqrt{(r^2+c^2)}$, c=1 (N=10)	$\phi(r)=e^{-(cr)^2}$, c=1.5 (N=10)
0.0	6.68e-04	3.68e-05
0.2	1.72e-03	7.98e-04
0.4	3.96e-03	2.18e-03
0.6	5.82e-03	3.98e-03
0.8	7.12e-03	5.45e-03
1.0	7.12e-03	5.45e-03
1.2	7.66e-03	6.42e-03
1.4	7.02e-03	6.58e-03
1.6	6.06e-03	6.55e-03
1.8	5.03e-03	6.44e-03
2.0	4.19e-03	5.34e-03
mean of error	5.83e-03	5.34e-03
CPU time(s)	19.78	16.03

Table 9. Error of proposed method with a=-2, b=0.1, c=0.5, d=0 for example 4.3.

t	$\phi(r)=\sqrt{(r^2+c^2)}$, c=3.4 (N=10)	$\phi(r)=e^{-(cr)^2}$, c=1.4 (N=10)
0.0	3.62e-04	3.62e-04
0.2	3.38e-05	1.41e-04
0.4	3.65e-04	2.45e-04
0.6	5.70e-04	7.05e-04
0.8	6.22e-04	7.65e-04
1.0	5.20e-04	1.18e-04
1.2	3.09e-04	9.63e-04
1.4	7.37e-05	4.68e-04
1.6	7.53e-05	1.19e-04
1.8	3.72e-05	3.39e-04
2.0	2.41e-04	6.95e-04
mean of error	3.59e-04	7.05e-04
CPU time(s)	18.26	17.73

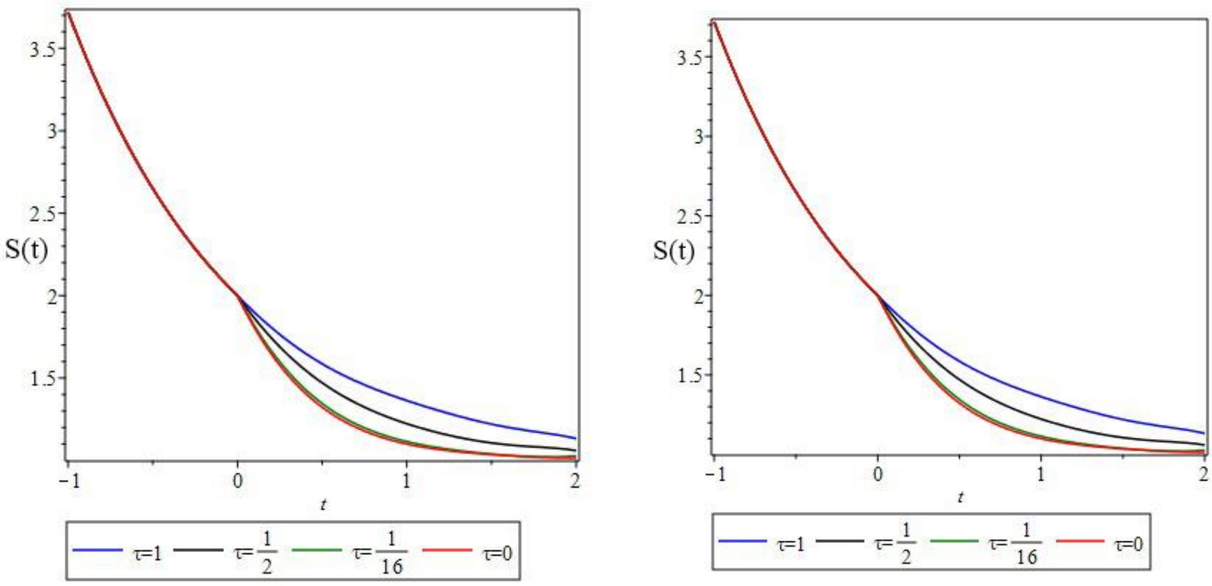

Figure 1. The graph of delay effect SDDE with N=20, $c = 0.1$ and $d = 0.01$ (left) and the graph of delay effect SDDE N=20, c=0.1 and d=0.1 (right)

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5. Conclusion

Finding an exact solution for many SDDEs is difficult. Therefor, we used a spectral collocation method based on radial basis functions (RBFs). The radial basis functions in our approach include Multiquadric and Gaussian functions, among others. In this method, the M-panel and r-point Newton-Cotes integration were used to estimate the $It\hat{o}$ integral. We performed a convergence assessment of the proposed approach and provided multiple examples including both linear and nonlinear cases, to demonstrate the effectiveness and accuracy of the method.

Authors Contribution

All the authors have participated sufficiently in the intellectual content, conception and design of this work or the analysis and interpretation of the data (when applicable), as well as the writing of the manuscript.

Availability of data and materials

The data that support the findings of this study are available from the corresponding author, upon reasonable request.

Conflict of interests

The author states that there is no conflict of interest.

Reference

[1] Merton, R.c. (1991). Continuous-Time Finance, Oxford University Press. 4(4), 793-803.

[2] Rubinstein, M. (1994). Implied Binomial Trees. The Journal of Finance, 49(3), 771–818.

[3] Scott, L.O. (1987). Option pricing when the variance changes randomly: Theory, estimation, and an application. Journal of Financial and Quantitative Analysis, 22(4), 419–438. doi: https://doi.org/10.2307/2330793.

[4] Vahidi, A. R., Babolian, E., Azimzadeh, Z. (2018). An Improvement to the Homotopy Perturbation Method for Solving Nonlinear Duffing's Equations, Bulletin of the Malaysian Mathematical Sciences Society, 41(3), 1105–1117. doi: https://doi.org/10.1007/s40840-015-0191-4.

[5] Vahidi, A. R., Azimzadeh, Z., Mohammadifar, S. (2012). Restarted Adomian Decomposition Method for Solving Duffing-van der Pol Equation. Applied Mathematical Sciences, 6(11), 499-507. URL: https://www.researchgate.net/publication/265227996

[6] Anh, V., Inoue, A. (2005). Financial markets with memory I: Dynamic models. Stochastic Analysis and Applications, 23(2),275300. URL: https://www.researchgate.net/publication/27466029

[7] Arriojas, M., Hu, Y., Mohammed, S. E. A., Pap, G. (2007). A Delayed Black and Scholes Formula I. Stochastic Analysis and Applications, 25, 471–492. doi: https://doi.org/10.48550/arXiv.math/0604640

[8] Higham, D. J. (2001). An Algorithmic Introduction to Numerical Simulation of Stochastic Differential Equations. SIAM Review, 43(3), 525–546. doi: https://doi.org/10.1137/S0036144500378302.

[9] Hu, Y., Mohammed, S. E. A., Yan, F. (2004). Discrete-time approximations of stochastic delay equations: the Milstein scheme. The Annals of Probability, 32(1A), 265–314. doi: https://doi.org/10.1214/aop/1078415836.

[10] Stein, EM, Stein JC. 1991. Stock price distributions with stochastic volatility: an analytic approach. Rev Financ Stud. 4(4):727-52. URL: https://www.jstor.org/stable/2962156.

[11] Shreve, SE. (2004). Stochastic Calculus for Finance II: Continuous-Time Models. New York: Springer.

[12] Shinde, AS, Takale KC. (2012). Study of Black-Scholes Model and its Applications. Procedia Eng. 38:270-9. doi: https://doi.org/10.1016/j.proeng.2012.06.035

[13] Stoica, G. (2004). A stochastic delay financial model. Proc Am Math Soc. 133(6):1837-41. doi: https://doi.org/10.1090/S0002-9939-04-07765-2

[14] Zheng, Y. (2015). Asset Pricing Based on Stochastic Delay Differential Equations.

[15] Akhtari B, Babolian E, Foroush Bastani A. (2014). An adaptive weak continuous Euler-Maruyama method for stochastic delay differential equations. Numer Algorithms. 69(1):29-57. doi: https://doi.org/10.1007/s11075-014-9880-6

[16] Zong X, Wu F, Huang C. (2015). Exponential mean square stability of the theta approximations for neutral stochastic differential delay equations. J Comput Appl Math. 286:172-85. doi: https://doi.org/10.1016/j.cam.2015.03.016.

[17] Banihashemi, S., Jafari, H., Babaei, A. (2022). A stable collocation approach to solve a neutral delay stochastic differential equation of fractional order. Journal of Computational and Applied Mathematics, 403, 113845. doi: https://doi.org/10.1016/j.cam.2021.113845.

[18] Fodor, G., Sykora, H. T., & Bachrathy, D. (2023). Collocation method for stochastic delay differential equations. Probabilistic Engineering Mechanics, 74, 103515. doi: https://doi.org/10.1016/j.probengmech.2023.103515

[19] Ahmadi, N., Vahidi, A. R., & Allahviranloo, T. (2017). An efficient approach based on radial basis functions for solving stochastic fractional differential equations. Mathematical Sciences, 11(2), 113–118. doi: https://doi.org/10.1007/s40096-017-0211-7

[20] Kosec, G., Sarler, B. (2008). Local RBF Collocation Method for Darcy Flow. CMES: Computer Modeling in Engineering & Sciences, 25(3), 197–208. URL: https://www.researchgate.net/publication/258396427_Local_RBF_Collocation_Method_for_Darcy_flow

[21] Arezoomandan, M., Soheili, A. R. (2021). Spectral collocation method for stochastic partial differential equations with fractional Brownian motion. Journal of Computational and Applied Mathematics, 389, 113369. https://doi.org/10.1016/j.cam.2020.113369

[22] Pettersson U, Larsson E, Marcusson G, Persson J. (2005)Option pricing using radial basis functions. In: Proceedings of the ECCOMAS Thematic Conference on Meshless Methods; Nov 11-13; Lisbon, Portugal. URL: https://www.researchgate.net/publication/247078464

[23] Mokhtari, S., Mesforush, A., Mokhtari, R., Akbari, R. (2024). An RBF-LOD Method for Solving Stochastic Diffusion Equations. Journal of Mathematics, 2024, Article ID 9955109. https://doi.org/10.1155/2024/9955109.

[24] Anco, S. C., Nualsaard, N., Luadsong, A., Aschariyaphotha, N. (2020). The Numerical Solution of Fractional Black-Scholes-Schrödinger Equation Using the RBFs Method. Advances in Mathematical Physics, 2020, 8868940. doi: https://doi.org/10.1155/2020/1942762.

[25] Rad, J. A., Parand, K., Ballestra, L. V. (2015). Pricing European and American options by radial basis point

interpolation. Applied Mathematics and Computation, 251, 363–377. doi: https://doi.org/10.1016/j.amc.2014.11.016.

[26] Akhtari, B. (2019). Numerical solution of stochastic state-dependent delay differential equations: convergence and stability. Advances in Difference Equations, 2019(1), 396. doi: https://doi.org/10.1186/s13662-019-2323-x.

[27] Gan, S., Yin, Z. (2015). Chebyshev spectral collocation method for stochastic delay differential equations. Advances in Difference Equations, 2015(1), 228. doi: https://doi.org/10.1186/s13662-015-0447-1.

[28] Maleknejad, K., Ezzati, R., Damercheli, T. (2014). Solution of Multi-Delay Dynamic Systems by Using Hybrid Functions. Applied Mathematics, 5(13), 1969-1982. doi: http://dx.doi.org/10.4236/am.2014.513194.

[29] Mao X.(2007). Stochastic Differential Equations and Applications. 2nd ed. Chichester: Horwood Publishing.

[30] Oksendal, B. (2000). Stochastic Differential Equations: An Introduction with Applications (5th ed.). Springer-Verlag.

[31] Khattak, A. J., Tirmizi, S. I. A., Islam, S. U. (2009). Application of meshfree collocation method to a class of nonlinear partial differential equations. Engineering Analysis with Boundary Elements, 33(5), 661-667. https://doi.org/10.1016/j.enganabound.2008.10.001

[32] Delves, L. M., & Mohamed, J. L. (1988). Computational Methods for Integral Equations. Cambridge University Press.

[33] Phillips, G. M., & Taylor, P. J. (1996). Theory and Applications of Numerical Analysis (2nd ed.). Academic Press.