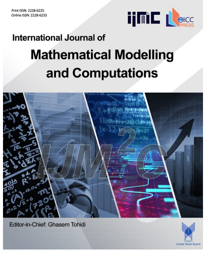

Int. J. Math. Model. Comput. 15(4) (2025) 261-272

International Journal of Mathematical Modelling & Computations (IJM^{2}C)

https://doi.org/10.57647/IJM2C.2025.150425

Closed-Form Formulas for Classical and Reverse Degree-Based Topological Indices in Two-Dimensional Grid Graphs

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Original Research

Received:
07 July 2025

Revised:
03 Agust 2025

Accepted:
12 Agust 2025

Published online:
31 Agust 2025

Published in Issue:
31 Decembre 2025

Abstract:

Topological indices are fundamental descriptors in mathematical chemistry, providing quantitative measures of molecular structure that correlate with physicochemical properties. This investigation presents a systematic computational approach to determine exact closed-form expressions for various degree-based topological indices applied to two-dimensional grid networks $P_m \times P_n$. We establish explicit formulas for classical Zagreb indices, Randić connectivity indices, atom-bond connectivity descriptors, geometric-arithmetic indices, harmonic indices, and the recently introduced reverse versions of these indices. Our methodology employs vertex degree distribution analysis combined with edge-based summation techniques to derive mathematically rigorous expressions. The obtained results demonstrate that grid networks exhibit predictable scaling behaviors for all examined indices, with computational complexity remaining polynomial in network dimensions. Moreover, for the first time, we provide a detailed graphical and comparative analysis of the specific degree-based topological indices addressed in this work and their reverse counterparts on grid networks, filling a gap in the literature by extending such comparisons beyond traditional molecular systems.

Keywords: Topological indices; Grid graphs; Degree-Based Indices; Reverse indices.

Cite this article: S. Shokrollahi Yancheshmeh,(2025) Closed-Form Formulas for Classical and Reverse Degree-Based Topological Indices in Two-Dimensional Grid Graphs. *Int. J. Math. Model. Comput* 15(4), 261-272.https://doi.org/10.57647/IJM2C.2025.150425

1. Introduction

In graph theory, a graph invariant is a numeric property that reflects fundamental structural characteristics of a graph and remains unchanged even under isomorphisms or relabeling. Graph invariants are important in analyzing and measuring graph structures and find extensive application in many areas of science, including network analysis, theoretical chemistry, and computational biology [1–5].

One particularly important category of these invariants is topological indices. These mathematical tools are used to characterize the connectivity, branching, and distance relationships within graphs, which are usually simple and connected. Their ability to represent molecular structures as abstract graphs makes them especially valuable in theoretical chemistry, where they assist in predicting the physical, chemical, and biological properties of compoundsbiology [6–10].

Degree-based indices are the notable subsets of topological descriptorss. Relying solely on the degrees of vertices, these indices are valued for their computational simplicity, intuitive interpretation, and strong predictive capabilities. They play a fundamental role in Quantitative Structure–Activity Relationship (QSAR) and Quantitative Structure–Property Relationship (QSPR) models, providing quantitative links between molecular graphs and observable properties [11–14].

Within graph theory, lattice graphs have attracted considerable interest due to their regular and periodic structures. These graphs effectively model crystalline materials, polymer frameworks, and solid-state systems, making them excellent candidates for exploring topological indices in materials science.

Grid networks, in particular, stand out in molecular graph theory due to their frequent appearance in crystalline structures, two-dimensional lattice arrangements, and polymer frameworks [15, 16]. The Cartesian product graph $P_m \times P_n$, formed by joining two path graphs P_m and P_n, serves as a fundamental building block in materials science. Such arrangements naturally arise in studies involving graphene-like materials, metal-organic frameworks (MOFs), and periodic crystal lattices [17, 18].

Reverse degree-based indices, such as the reverse Zagreb, reverse Randić, reverse ABC, and reverse GA indices, represent a newer class of topological descriptors that invert traditional degree-based formulations to offer alternative insights into molecular and network structures [19–23]. These indices have shown promise in characterizing various chemical and nanostructured graphs.

To the best of our knowledge, while such reverse indices have been studied extensively for specific graph families—such as trees, comet-like networks, and certain chemical nanostructures—their application to two-dimensional lattice graphs $P_m \times P_n$ remains unexplored. In this work, we fill this gap by deriving closed-form expressions for both classical topological indices (first and second Zagreb, Randić, ABC, GA, harmonic) and their reverse counterparts, comparing their behaviors numerically and graphically across varying lattice sizes.

In the following sections, we first present the preliminaries and necessary definitions in Section 2. The main results are then provided in Section 3, followed by the numerical results and graphical comparisons in Section 4. Finally, the analysis of growth trends is presented in Section 5.

2. Preliminaries

Degree-based topological invariants find their genesis in the Zagreb descriptors, established by Gutman and Trinajstić [24] in 1972 as mathematical encodings of vertex connectivity:

$$M_1(G) = \sum_{v\in V(G)} d(v)^2 = \sum_{uv\in E(G)} (d(u)+d(v)),$$

$$M_2(G) = \sum_{uv\in E(G)} d(u)d(v),$$

where $d(v)$ denotes the degree of the vertex v in the graph G [25, 26].

This mathematical foundation enabled the construction of enhanced connectivity indices.

The Randić branching index, conceived by Randić [27] in 1975, encodes structural information via reciprocal geometric means of vertex connectivities:

$$R(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d(u)d(v)}},$$

Advanced topological measures subsequently emerged, including the Atom-Bond Connectivity (ABC) invariant and the Geometric-Arithmetic (GA) index, which encode molecular topology through degree-weighted summations.

Atom-Bond Connectivity (ABC) Index: Introduced by Estrada et al. [28], this index has demonstrated superior performance in QSAR applications:

$$ABC(G) = \sum_{uv\in E} \sqrt{\frac{d(u)+d(v)-2}{d(u)d(v)}}$$

Calculating the ABC index for such molecular graphs helps predict important physicochemical properties such as boiling point, melting point, molecular stability, and reactivity. For instance, in carbon-based network molecules like graphene or two-dimensional crystal structures, such structures can be modeled as grid graphs, and topological indices can provide insights into their physical and chemical properties. Geometric-Arithmetic (GA) Index: Developed by Vukičevi'c and Furtula [30], this index provides enhanced structure-property correlations:

$$GA(G) = \sum_{uv\in E} \frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)}$$

Harmonic Index: This connectivity-based descriptor is defined as:

$$H(G) = \sum_{uv\in E} \frac{2}{d(u)+d(v)}$$

Building upon these classical foundations, recent developments have introduced novel families of topological indices that provide alternative perspectives on molecular graph characterization.

The Reverse topological indices constitute a contemporary class of descriptors that employ a fundamental transformation of vertex degrees. For a graph G with maximum degree $\Delta(G)$, the reverse vertex degree of a vertex v is defined as $c_v = \Delta(G) - d_G(v) + 1$, where $d_G(v)$ represents the degree of vertex v.

This transformation emphasizes vertices with lower degrees by assigning them higher reverse degree values, thereby providing a complementary perspective to traditional degree-based measures.

Based on this transformation, reverse versions of classical topological indices have been defined as follows. In all definitions below, the summation is taken over all edges $uv \in E(G)$, and c_u, c_v represent the reverse degrees of vertices u and v, respectively.

- The first and second reverse Zagreb indices:

$$RM_1(G) = \sum_{uv\in E(G)} (c_u + c_v),$$

$$RM_2(G) = \sum_{uv\in E(G)} c_u \cdot c_v.$$

item Reverse Atom-Bond Connectivity (RABC) Index:

$$RABC(G) = \sum_{uv\in E(G)} \sqrt{\frac{c_u + c_v - 2}{c_u \cdot c_v}}.$$

https://doi.org/10.57647/IJM2C.2025.150425

- Reverse Geometric-Arithmetic (RGA) Index:

$$RGA(G) = \sum_{uv\in E(G)} \frac{2\sqrt{c_u c_v}}{c_u + c_v}.$$

- Reverse Harmonic (RH) Index:

$$RH(G) = \sum_{uv\in E(G)} \frac{2}{c_u + c_v}.$$

These reverse indices offer a complementary perspective to their classical counterparts and have been shown to enhance the discriminative power in distinguishing structurally similar graphs by highlighting the role of lower-degree vertices.

Definition 2.1 (Grid Graph) *The two-dimensional grid graph $P_m \times P_n$ is the Cartesian product of path graphs P_m and P_n, where P_k denotes the path graph on k vertices [29].*

The grid graph $P_m \times P_n$ possesses several important structural characteristics that facilitate topological index computation.

Lemma 2.2 *For the grid graph $P_m \times P_n$ with $m, n \geq 2$:*

1. *The total number of vertices is $|V| = mn$.*
2. *The total number of edges is $|E| = (m-1)n + m(n-1) = mn + m + n - 2$.*

Proof. By the definition of the grid graph as the Cartesian product of two path graphs P_m and P_n, the total number of vertices is simply the product of the numbers of vertices in each path, which gives mn. Similarly, the total number of edges corresponds to the sum of edges contributed by each dimension, resulting in $(m-1)n+m(n-1)$. Therefore, the statements of the lemma follow directly from the definition, and the proof is straightforward.

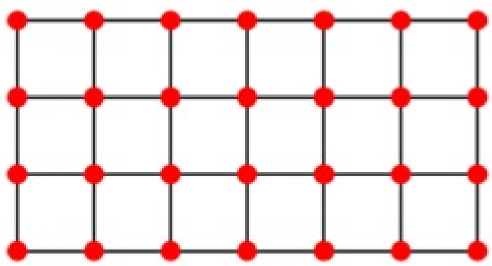

Figure 1. Grid graph $P_m \times P_n$

3. Main Results

Here we present a general formula for classical topological indices and, in particular, reverse indices that have not yet been computed for graphs in grid networks $P_m \times P_n$. These formulations leverage the structured nature of such graphs and systematically analyze vertex degree distributions. Furthermore, through comparative analysis and graphical representations, we provide deeper insights into the structural behavior of these indices to enhance the understanding of their geometric implications.

3.1 Classical Topological Indices

Theorem 3.1 *The Zagreb indices of the Grid graph $P_m \times P_n$ with $m, n \geq 3$, are:*

$$(i)\ M_1(P_m \times P_n) = 16mn - 14(m+n) + 8.$$
$$(ii)\ M_2(P_m \times P_n) = 32mn - 38(m+n) + 36.$$

Proof. We partition the vertex set $V(P_m \times P_n)$ into three subsets:

(a) $V_1 = \{v \in V \mid d(v) = 2\}$, (corner vertices),

(b) $V_2 = \{v \in V \mid d(v) = 3\}$, (boundary vertices excluding corners),

(c) $V_3 = \{v \in V \mid d(v) = 4\}$, (interior vertices).

(i) Based on the above partitioning of the vertex set $V(P_m \times P_n)$, we compute the first Zagreb index $M_1(P_m \times P_n)$ as follows:

$$\begin{aligned} M_1(P_m \times P_n) &= \sum_{v\in V(P_m\times P_n)} d(v)^2 \\ &= \sum_{v\in V_1} d(v)^2 + \sum_{v\in V_2} d(v)^2 + \sum_{v\in V_3} d(v)^2. \end{aligned}$$

Given that the degrees in each subset are constant, the above expression simplifies to:

$$\begin{aligned} M_1(P_m \times P_n) &= |V_1|\cdot 2^2 + |V_2|\cdot 3^2 + |V_3|\cdot 4^2 \quad (1) \\ &= 4|V_1| + 9|V_2| + 16|V_3|. \end{aligned}$$

The grid graph $P_m \times P_n$ has vertices labeled (i, j) for $1 \leq i \leq m$ and $1 \leq j \leq n$. We determine the sizes of each subset of the vertex set $V(P_m \times P_n)$:

- $V_1 = \{(1,1), (1,n), (m,1), (m,n)\}, |V_1| = 4.$
- The set V_2 (number of vertices of degree 3), corresponds to boundary vertices excluding corners, $V_2 = \{(i,1), (i,n), (1,j), (m,j) | 2 \leq j \leq n-1, 2 \leq i \leq m-1\}, |V_2| = 2m + 2n - 8$.
- Since the total number of vertices in $P_m \times P_n$ is $m \cdot n$, the number of vertices of degree 4, i.e., the size of the set V_3, can be determined by subtracting the sizes of V_1 and V_2 from the total number of vertices:

$$\begin{aligned} |V_3| &= m\cdot n - |V_1| - |V_2| \\ &= m\cdot n - 4 - (2m+2n-8) \\ &= mn - 2m - 2n + 4. \end{aligned}$$

Substituting these into the Zagreb index expression (1), we obtain:

$$\begin{aligned} M_1(P_m \times P_n) &= 4|V_1| + 9|V_2| + 16|V_3| \\ &= 4\cdot 4 + 9(2m+2n-8) \\ &\quad +16(mn-2m-2n+4) \\ &= 16 + 18m + 18n - 72 + \\ &\quad 16mn - 32m - 32n + 64 \\ &= 16mn - 14m - 14n + 8. \end{aligned}$$

https://doi.org/10.57647/IJM2C.2025.150425

(ii) To compute the second Zagreb index, the edges are partitioned as follows:

$$E(P_m \times P_n) = E_{2,3} \cup E_{3,3} \cup E_{3,4} \cup E_{4,4}$$

where each $E_{i,j}$ is defined as:

$$\begin{aligned} E_{2,3} &= \{uv \in E(P_m \times P_n) \mid d(u) = 2, d(v) = 3\}, \\ E_{3,3} &= \{uv \in E(P_m \times P_n) \mid d(u) = 3, d(v) = 3\}, \\ E_{3,4} &= \{uv \in E(P_m \times P_n) \mid d(u) = 3, d(v) = 4\}, \\ E_{4,4} &= \{uv \in E(P_m \times P_n) \mid d(u) = 4, d(v) = 4\}. \end{aligned} \tag{2}$$

The second Zagreb index is computed as:

$$\begin{aligned} M_2(P_m \times P_n) &= \sum_{uv \in E(P_m \times P_n)} d(u)d(v) \quad (3) \\ &= \sum_{uv \in E_{2,3}} d(u)d(v) + \sum_{uv \in E_{3,3}} d(u)d(v) \\ &\quad + \sum_{uv \in E_{3,4}} d(u)d(v) + \sum_{uv \in E_{4,4}} d(u)d(v) \\ &= |E_{2,3}| \cdot (2 \cdot 3) + |E_{3,3}| \cdot (3 \cdot 3) \\ &\quad + |E_{3,4}| \cdot (3 \cdot 4) + |E_{4,4}| \cdot (4 \cdot 4) \\ &= 6|E_{2,3}| + 9|E_{3,3}| + 12|E_{3,4}| + 16|E_{4,4}|. \end{aligned}$$

The sizes of the edge sets are determined as follows:

$$\begin{aligned} &|E_{2,3}| = 8, \quad |E_{3,3}| = 2m + 2n - 12, \\ &|E_{3,4}| = 2m + 2n - 8, \quad |E_{4,4}| = 2mn - 5n - 5m + 12. \end{aligned} \tag{4}$$

- $|E_{2,3}| = 8$, Edges connect corner vertices (degree 2) to boundary vertices (degree 3).
- $E_{3,3}$: Edges that connect degree-3 boundary vertices:

$$\begin{aligned} |E_{3,3}| &= |\{(i,1) - (i+1,1), (i,n) - (i+1,n) : \\ &\qquad 2 \le i \le m-2\}| \\ &\quad + |\{(1,j) - (1,j+1), (m,j) - (m,j+1) : \\ &\qquad 2 \le j \le n-2\}| \\ &= 2(m-3) + 2(n-3) \\ &= 2m + 2n - 12. \end{aligned}$$

- $E_{3,4}$: Edges connect boundary vertices (degree 3) to interior vertices (degree 4):

$$\begin{aligned} |E_{3,4}| &= |\{(i,1) - (i,2), (i,n-1) - (i,n) : \\ &\qquad 2 \le i \le m-1\}| \\ &\quad + |\{(1,j) - (2,j), (m-1,j) - (m,j) : \\ &\qquad 2 \le j \le n-1\}| \\ &= 2(m-2) + 2(n-2) \\ &= 2m + 2n - 8. \end{aligned}$$

- $E_{4,4}$: Edges connect interior vertices:

$$\begin{aligned} |E_{4,4}| &= |\{(i,j) - (i+1,j), (i,n-1) - (i,n) : \\ &\qquad 2 \le i \le m-2, 2 \le j \le n-1\}| \\ &\quad + |\{(i,j) - (i,j+1), (m-1,j) - (m,j) : \\ &\qquad 2 \le i \le m-1, 2 \le j \le n-2\}| \\ &= (m-3)(n-2) + (n-3)(m-2) \\ &= 2mn - 5n - 5m + 12. \end{aligned}$$

According to (3), (4):

$$\begin{aligned} M_2(P_m \times P_n) &= 6|E_{2,3}| + 9|E_{3,3}| + 12|E_{3,4}| + 16|E_{4,4}| \\ &= 6 \cdot 8 + 9(2m + 2n - 12) \\ &\quad + 12(2m + 2n - 8) \\ &\quad + 16(2mn - 5m - 5n + 12) \\ &= 32mn - 38m - 38n + 36. \end{aligned}$$

Proposition 3.2 *The Randić index of the Grid graph $P_m \times P_n$ with $m, n \ge 3$:*

$$\begin{aligned} R(P_m \times P_n) &= \frac{4\sqrt{6}}{3} + \frac{2(m+n-6)}{3} \\ &\quad + \frac{(m+n-4)\sqrt{3}}{3} + \frac{2mn - 5(m+n) + 12}{4}. \end{aligned}$$

Proof. The Randić index sums the reciprocal geometric means of vertex degrees over all edges:

$$\begin{aligned} R(P_m \times P_n) &= \sum_{uv \in E(P_m \times P_n)} \frac{1}{\sqrt{d(u)d(v)}} \quad (5) \\ &= \sum_{uv \in E_{2,3}} \frac{1}{\sqrt{6}} + \sum_{uv \in E_{3,3}} \frac{1}{\sqrt{9}} \\ &\quad + \sum_{uv \in E_{3,4}} \frac{1}{\sqrt{12}} + \sum_{uv \in E_{4,4}} \frac{1}{\sqrt{16}} \\ &= |E_{2,3}| \frac{1}{\sqrt{6}} + |E_{3,3}| \frac{1}{3} \\ &\quad + |E_{3,4}| \frac{1}{\sqrt{12}} + |E_{4,4}| \frac{1}{4}. \end{aligned}$$

Now considering the sizes of the edge sets mentioned in (4), (5):

$$\begin{aligned} R(P_m \times P_n) &= 8\frac{1}{\sqrt{6}} + (2m + 2n - 12)\frac{1}{3} \\ &\quad + (2m + 2n - 8)\frac{1}{\sqrt{12}} \\ &\quad + (2mn - 5n - 5m + 12)\frac{1}{4} \\ &= \frac{4\sqrt{6}}{3} + \frac{2(m+n-6)}{3} + \frac{(m+n-4)\sqrt{3}}{3} \\ &\quad + \frac{2mn - 5(m+5n) + 12}{4}. \end{aligned}$$

Proposition 3.3 *The Atom-Bond Connectivity and Geometric-Arithmetic indices of the Grid graph $P_m \times P_n$ with $m, n \ge 3$ are:*

$$\begin{aligned} (i) \quad ABC(P_m \times P_n) &= 4\sqrt{2} + \frac{4(m+n-6)}{3} \\ &\quad + \frac{(m+n-4)\sqrt{15}}{3} \\ &\quad + \frac{(2mn - 5m - 5n + 12)\sqrt{6}}{4}, \\ (ii) \quad GA(P_m \times P_n) &= \frac{16\sqrt{6}}{5} + (2mn - 3m - 3n) \\ &\quad + \frac{8\sqrt{3}}{7}(m + n - 4). \end{aligned}$$

https://doi.org/10.57647/IJM2C.2025.150425

Proof. (i) The ABC index is:

$$\begin{aligned} ABC(P_m \times P_n) &= \sum_{uv\in E(P_m\times P_n)} \sqrt{\frac{d(u)+d(v)-2}{d(u)\cdot d(v)}} \qquad (6)\\ &= \sum_{uv\in E_{2,3}} \sqrt{\frac{2+3-2}{2\cdot 3}} + \sum_{uv\in E_{3,3}} \sqrt{\frac{3+3-2}{3\cdot 3}}\\ &+ \sum_{uv\in E_{3,4}} \sqrt{\frac{3+4-2}{3\cdot 4}} + \sum_{uv\in E_{4,4}} \sqrt{\frac{4+4-2}{4\cdot 4}}\\ &= |E_{2,3}|\sqrt{\frac{1}{2}} + |E_{3,3}|\sqrt{\frac{4}{9}}\\ &+|E_{3,4}|\sqrt{\frac{5}{12}} + |E_{4,4}|\sqrt{\frac{3}{8}}. \end{aligned}$$

Now considering the sizes of the edge sets mentioned in (4), (6):

$$\begin{aligned} ABC(P_m \times P_n) &= 8\sqrt{\frac{1}{2}} + (2m+2n-12)\sqrt{\frac{4}{9}}\\ &+(2m+2n-8)\frac{\sqrt{15}}{6}\\ &+(2mn-5m-5n+12)\sqrt{\frac{3}{8}}\\ &= 8\cdot\frac{1}{\sqrt{2}} + (2m+2n-12)\cdot\frac{2}{3}\\ &+(2m+2n-8)\cdot\frac{\sqrt{5}}{\sqrt{12}}\\ &+(2mn-5m-5n+12)\cdot\frac{\sqrt{6}}{4}\\ &= \frac{8}{\sqrt{2}} + \frac{2(2m+2n-12)}{3}\\ &+\frac{(2m+2n-8)\sqrt{5}}{\sqrt{12}}\\ &+\frac{(2mn-5m-5n+12)\sqrt{6}}{4}\\ &= 4\sqrt{2} + \frac{4(m+n-6)}{3}\\ &+\frac{(m+n-4)\sqrt{15}}{3}\\ &+\frac{(2mn-5m-5n+12)\sqrt{6}}{4}. \end{aligned}$$

(ii) The GA index is:

$$\begin{aligned} GA(P_m \times P_n) &= \sum_{uv\in E((P_m\times P_n))} \frac{2\sqrt{d_u d_v}}{d_u+d_v}\\ &= \sum_{uv\in E_{2,3}} \frac{2\sqrt{2\cdot 3}}{2+3} + \sum_{uv\in E_{3,3}} \frac{2\sqrt{3\cdot 3}}{3+3}\\ &+ \sum_{uv\in E_{3,4}} \frac{2\sqrt{3\cdot 4}}{3+4} + \sum_{uv\in E_{4,4}} \frac{2\sqrt{4\cdot 4}}{4+4}\\ &= |E_{2,3}|\frac{2\sqrt{6}}{5} + |E_{3,3}|\cdot 1\\ &+|E_{3,4}|\frac{2\sqrt{12}}{7} + |E_{4,4}|\cdot 1 \end{aligned}$$

Now, according to (4), (7):

$$\begin{aligned} GA(P_m \times P_n) &= 8\frac{2\sqrt{6}}{5} + (2m+2n-12)\cdot 1\\ &+(2m+2n-8)\frac{2\sqrt{12}}{7}\\ &+(2mn-5m-5n+12)\cdot 1\\ &= \frac{16\sqrt{6}}{5} + (2mn-3m-3n)\\ &+\frac{8\sqrt{3}}{7}(m+n-4). \end{aligned}$$

Proposition 3.4 *The Harmonic index of the Grid graph $P_m \times P_n$ with $m, n \geq 3$ is:*

$$\begin{aligned} H(P_m \times P_n) &= \frac{16}{5} + \frac{2(m+n-6)}{3}\\ &+\frac{4(m+n-4)}{7}\\ &+\frac{2mn-5m-5n+12}{4}. \end{aligned}$$

Proof. The harmonic index is:

$$\begin{aligned} H(P_m \times P_n) &= \sum_{uv\in E(P_m\times P_n)} \frac{2}{d(u)+d(v)} \qquad (7)\\ &= \sum_{uv\in E_{2,3}} \frac{2}{2+3} + \sum_{uv\in E_{3,3}} \frac{2}{3+3}\\ &+ \sum_{uv\in E_{3,4}} \frac{2}{3+4} + \sum_{uv\in E_{4,4}} \frac{2}{4+4}\\ &= |E_{2,3}|\frac{2}{5} + |E_{3,3}|\frac{1}{3} + |E_{3,4}|\frac{2}{7} + |E_{4,4}|\frac{1}{4}. \end{aligned}$$

Now considering (4), (7):

$$\begin{aligned} H(P_m \times P_n) &= \frac{16}{5} + (2m+2n-12)\cdot\frac{1}{3}\\ &+(2m+2n-8)\cdot\frac{2}{7}\\ &+(2mn-5m-5n+12)\cdot\frac{1}{4}\\ &= \frac{16}{5} + \frac{2(m+n-6)}{3}\\ &+\frac{4(m+n-4)}{7} + \frac{2mn-5m-5n+12}{4}. \end{aligned}$$

https://doi.org/10.57647/IJM2C.2025.150425

3.2 Reverse Topological Indices of the Grid graph $P_m \times P_n$

Based on the definition of the reverse degree of a vertex v, for graph $G = P_m \times P_n$ we have:
$\Delta(G) = 4$, $c_v = \Delta(G) - d_G(v) + 1 = 4 - d_G(v) + 1 = 5 - d_G(v)$. Thus:

- $d(v) = 2 \Rightarrow c_v = 5 - 2 = 3$
- $d(v) = 3 \Rightarrow c_v = 5 - 3 = 2$
- $d(v) = 4 \Rightarrow c_v = 5 - 4 = 1$

Therefore, according to 2 and 4, the reverse edges set can be we partitioned as following:

$$\begin{aligned} E_{2,3} &= \{uv \in E(P_m \times P_n) \mid c_u = 3, c_v = 2\}, \quad (8)\\ E_{3,3} &= \{uv \in E(P_m \times P_n) \mid c_u = 2, c_v = 2\},\\ E_{3,4} &= \{uv \in E(P_m \times P_n) \mid c_u = 2, c_v = 1\},\\ E_{4,4} &= \{uv \in E(P_m \times P_n) \mid c_u = 1, c_v = 1\},\\ & |E_{2,3}| = 8, \quad |E_{3,3}| = 2m + 2n - 12,\\ & |E_{3,4}| = 2m + 2n - 8,\\ & |E_{4,4}| = 2mn - 5n - 5m + 12. \end{aligned}$$

Theorem 3.5 *The reverse Zagreb indices of the Grid graph $P_m \times P_n$ with $m, n \geq 3$, are:*

$$(i) \quad RM_1(P_m \times P_n) = 4mn + 4(m + n) - 8.$$
$$(ii) \quad RM_2(P_m \times P_n) = 2mn + 7(m + n) - 4.$$

Proof. In both cases, we will have the following expression based on the corresponding relation (8):

$$\begin{aligned} (i)\ RM_1(P_m \times P_n) &= \sum_{uv \in E(P_m \times P_n)} (c_u + c_v)\\ &= \sum_{uv \in E_{2,3}} (3 + 2) + \sum_{uv \in E_{3,3}} (2 + 2)\\ &\quad + \sum_{uv \in E_{3,4}} (2 + 1) + \sum_{uv \in E_{4,4}} (1 + 1)\\ &= |E_{2,3}| \cdot 5 + |E_{3,3}| \cdot 4\\ &\quad +|E_{3,4}| \cdot 3 + |E_{4,4}| \cdot 2\\ &= 8 \cdot 5 + (2m + 2n - 12) \cdot 4\\ &\quad +(2m + 2n - 8) \cdot 3\\ &\quad +(2mn - 5m - 5n + 12) \cdot 2\\ &= 40 + (8m + 8n - 48)\\ &\quad +(6m + 6n - 24)\\ &\quad +(4mn - 10m - 10n + 24)\\ &= 4mn + 4(m + n) - 8 \end{aligned}$$

$$\begin{aligned} (ii)\ RM_2(P_m \times P_n) &= \sum_{uv \in E(P_m \times P_n)} c_u \cdot c_v\\ &= \sum_{uv \in E_{2,3}} (3 \cdot 2) + \sum_{uv \in E_{3,3}} (2 \cdot 2)\\ &\quad + \sum_{uv \in E_{3,4}} (2 \cdot 1) + \sum_{uv \in E_{4,4}} (1 \cdot 1)\\ &= |E_{2,3}| \cdot 6 + |E_{3,3}| \cdot 4\\ &\quad +|E_{3,4}| \cdot 2 + |E_{4,4}| \cdot 1\\ &= 8 \cdot 6 + (2m + 2n - 12) \cdot\\ &\quad 4 + (2m + 2n - 8) \cdot 2\\ &\quad +(2mn - 5m - 5n + 12) \cdot 1\\ &= 48 + (8m + 8n - 48)\\ &\quad +(4m + 4n - 16)\\ &\quad +(2mn - 5m - 5n + 12)\\ &= 2mn + 7(m + n) - 4. \end{aligned}$$

Proposition 3.6 *The reverse Randić index of the Grid graph $P_m \times P_n$ with $m, n \geq 3$:*

$$RR(P_m \times P_n) = 2mn + (-4 + \sqrt{2})(m + n) + \frac{4\sqrt{6}}{3} + 6 - 4\sqrt{2}.$$

Proof. Based on the (8):

$$\begin{aligned} RR(P_m \times P_n) &= \sum_{uv \in E(P_m \times P_n)} \frac{1}{\sqrt{c_u c_v}}\\ &= \sum_{uv \in E_{2,3}} \frac{1}{\sqrt{6}} + \sum_{uv \in E_{3,3}} \frac{1}{\sqrt{4}}\\ &\quad + \sum_{uv \in E_{3,4}} \frac{1}{\sqrt{2}} + \sum_{uv \in E_{4,4}} \frac{1}{\sqrt{1}}\\ &= |E_{2,3}| \frac{1}{\sqrt{6}} + |E_{3,3}| \frac{1}{4}\\ &\quad +|E_{3,4}| \frac{1}{\sqrt{2}} + |E_{4,4}|\\ &= 8 \frac{1}{\sqrt{6}} + (2m + 2n - 12) \frac{1}{2}\\ &\quad +(2m + 2n - 8) \frac{1}{\sqrt{2}}\\ &\quad +(2mn - 5n - 5m + 12)\\ &= 2mn + (-4 + \sqrt{2})(m + n) + \frac{4\sqrt{6}}{3}\\ &\quad +6 - 4\sqrt{2}. \end{aligned}$$

Proposition 3.7 *The reverse Atom-Bond Connectivity and the Geometric-Arithmetic indices of the Grid graph $P_m \times P_n$ with $m, n \geq 3$ are:*

$$\begin{aligned} (i) \quad RABC(P_m \times P_n) &= 2\sqrt{2}(m + n - 3),\\ (ii) \quad RGA(P_m \times P_n) &= \frac{16\sqrt{6}}{5} + (2mn - 3m - 3n)\\ &\quad + \frac{4\sqrt{2}}{3}(m + n - 4). \end{aligned}$$

Proof. In both cases, using the relation (8) we have:

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(i)

$$\begin{aligned} RABC(P_m \times P_n) &= \sum_{uv\in E(P_m\times P_n)} \sqrt{\frac{c_u+c_v-2}{c_u\cdot c_v}} \\ &= \sum_{uv\in E_{2,3}} \sqrt{\frac{3+2-2}{3\cdot 2}} \\ &+ \sum_{uv\in E_{3,3}} \sqrt{\frac{2+2-2}{2\cdot 2}} \\ &+ \sum_{uv\in E_{3,4}} \sqrt{\frac{2+1-2}{2\cdot 1}} \\ &+ \sum_{uv\in E_{4,4}} \sqrt{\frac{1+1-2}{1\cdot 1}} \\ &= |E_{2,3}|\sqrt{\frac{1}{2}} + |E_{3,3}|\sqrt{\frac{1}{2}} \\ &+|E_{3,4}|\sqrt{\frac{1}{2}} + |E_{4,4}|\cdot 0 \\ &= 8\sqrt{\frac{1}{2}} + (2m+2n-12)\sqrt{\frac{1}{2}} \\ &+(2m+2n-8)\sqrt{\frac{1}{2}} \\ &= 2\sqrt{2}(m+n-3). \end{aligned}$$

(ii)

$$\begin{aligned} RGA(P_m \times P_n) &= \sum_{uv\in E(P_m\times P_n)} \frac{2\sqrt{c_u c_v}}{c_u+c_v} \\ &= \sum_{uv\in E_{2,3}} \frac{2\sqrt{3\cdot 2}}{3+2} + \sum_{uv\in E_{3,3}} \frac{2\sqrt{2\cdot 2}}{2+2} \\ &+ \sum_{uv\in E_{3,4}} \frac{2\sqrt{2\cdot 1}}{2+1} + \sum_{uv\in E_{4,4}} \frac{2\sqrt{1\cdot 1}}{1+1} \\ &= |E_{2,3}|\frac{2\sqrt{6}}{5} + |E_{3,3}| \\ &+|E_{3,4}|\frac{2\sqrt{2}}{3} + |E_{4,4}| \\ &= 16\frac{\sqrt{6}}{5} + (2m+2n-12) \\ &+(2m+2n-8)\frac{2\sqrt{2}}{3} \\ &+(2mn-5m-5n+12) \\ &= \frac{16\sqrt{6}}{5} + (2mn-3m-3n) \\ &+\frac{4\sqrt{2}}{3}(m+n-4). \end{aligned}$$

Proposition 3.8 *The reverse Harmonic index of the Grid graph $P_m \times P_n$ with $m,n \geq 3$ is:*

$$RH(P_m \times P_n) = 2mn - \frac{8}{3}(m+n) + \frac{58}{15}.$$

Proof. According to (8):

$$\begin{aligned} RH(G) &= \sum_{uv\in E(P_m\times P_n)} \frac{2}{c_u+c_v} \\ &= \sum_{uv\in E_{2,3}} \frac{2}{3+2} + \sum_{uv\in E_{3,3}} \frac{2}{2+2} \\ &+ \sum_{uv\in E_{3,4}} \frac{2}{2+1} + \sum_{uv\in E_{4,4}} \frac{2}{1+1} \\ &= |E_{2,3}|\cdot\frac{2}{5} + |E_{3,3}|\cdot\frac{1}{2} + |E_{3,4}|\cdot\frac{2}{3} + |E_{4,4}|\cdot 1 \\ &= 8\cdot\frac{2}{5} + (2m+2n-12)\cdot\frac{1}{2} \\ &+(2m+2n-8)\cdot\frac{2}{3} \\ &+(2mn-5m-5n+12)\cdot 1 \\ &= \frac{16}{5} + (m+n-6) + (m+n-4)\cdot\frac{4}{3} \\ &+(2mn-5m-5n+12) \\ &= 2mn - \frac{8}{3}(m+n) + \frac{58}{15}. \end{aligned}$$

In the following, we present an example of calculating the Zagreb and Reverse Zagreb indices. In this example, we first calculate indices for graph $P_3 \times P_3$ based on the definitions and the partitioning of its vertices and edges. Then, it is demonstrated that substituting into the formulas established in Theorems 3.1, 3.5 yields identical results.

Example 3.9 *Topological Indices of Grid Graph $P_3 \times P_3$*:
Degree vertex distribution:
$|V_{(2)}| = 4$ (corners), $|V_{(3)}| = 4$ (boundaries), $|V_{(4)}| = 1$ (interior).
Edge types (2):
$|E_{2,3}| = 8$, $|E_{3,3}| = 0$, $|E_{3,4}| = 4$, $|E_{4,4}| = 0$.
Zagreb indices:

$$\begin{aligned} (i)\ \ M_1(P_m \times P_n) &= \sum_{v\in V(P_m\times P_n)} d(v)^2 \\ &= 4(3^2) + 4(2^2) + 1(4^2) = 68. \end{aligned}$$

$$\begin{aligned} (ii)\ \ M_2(P_m \times P_n) &= \sum_{uv\in E_{2,3}} (3\cdot 2) + \sum_{uv\in E_{3,3}} (3\cdot 3) \\ &+ \sum_{uv\in E_{3,4}} (3\cdot 4) + \sum_{uv\in E_{4,4}} (4\cdot 4) \\ &= 6|E_{2,3}| + 9|E_{3,3}| \\ &+12|E_{3,4}| + 16|E_{4,4}| \\ &= 6\cdot 8 + 9\cdot 0 + 12\cdot 4 + 16\cdot 0 = 96. \end{aligned}$$

Substituting m = n = 3 into the formula derived from Theorem 3.1:

$$\begin{aligned} (i)\quad M_1(P_m \times P_n) &= 16mn - 14(m+n) + 8 \\ &= 16(\cdot 3\cdot 3) - 14(3+3) + 8 = 68. \end{aligned}$$

$$\begin{aligned} (ii)\quad M_2(P_m \times P_n) &= 32mn - 38(m+n) + 36 \\ &= 32(3\cdot 3) - 38(3+3) + 36 = 96. \end{aligned}$$

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Reverse Zagreb indices: Based on (8):

$$\begin{aligned}
(i)\ \ RM_1(P_m \times P_n) &= \sum_{uv\in E(P_m\times P_n)} (c_u + c_v) \\
&= \sum_{uv\in E_{2,3}} (3+2) + \sum_{uv\in E_{3,3}} (2+2) \\
&\quad + \sum_{uv\in E_{3,4}} (2+1) + \sum_{uv\in E_{4,4}} (1+1) \\
&= |E_{2,3}|\cdot 5 + |E_{3,3}|\cdot 4 \\
&\quad +|E_{3,4}|\cdot 3 + |E_{4,4}|\cdot 2 \\
&= 8\cdot 5 + 0\cdot 4 + 4\cdot 3 + 0\cdot 2 = 52.
\end{aligned}$$

$$\begin{aligned}
(ii)\ \ RM_2(P_m \times P_n) &= \sum_{uv\in E(P_m\times P_n)} c_u \cdot c_v \\
&= \sum_{uv\in E_{2,3}} (3\cdot 2) + \sum_{uv\in E_{3,3}} (2\cdot 2) \\
&\quad + \sum_{uv\in E_{3,4}} (2\cdot 1) + \sum_{uv\in E_{4,4}} (1\cdot 1) \\
&= |E_{2,3}|\cdot 6 + |E_{3,3}|\cdot 4 \\
&\quad +|E_{3,4}|\cdot 2 + |E_{4,4}|\cdot 1 \\
&= 8\cdot 6 + 0\cdot 4 + 4\cdot 2 + 0\cdot 1 = 56.
\end{aligned}$$

Setting m = n = 3 in the expression derived from Theorem 3.5:

$$\begin{aligned}
(i)\quad RM_1(P_m \times P_n) &= 4mn + 4(m+n) - 8 \\
&= 4(3\cdot 3) + 4(3+3) - 8 = 52.
\end{aligned}$$

$$\begin{aligned}
(ii)\quad RM_2(P_m \times P_n) &= 2mn + 7(m+n) - 4 \\
&= 2(3\cdot 3) + 7(3+3) - 4 = 56.
\end{aligned}$$

3.3 Application of Degree-Based Indices on Grid Graphs in Chemistry

Two-dimensional grid networks P_mP_n serve as effective mathematical models for atomic arrangements in crystalline materials and polymeric networks, such as graphene sheets and two-dimensional covalent organic frameworks (2D-COFs). These materials exhibit regular lattice structures that are topologically equivalent to grid graphs, making degree-based topological indices particularly relevant for their characterization.

The Atom Bond Connectivity (ABC) index, along with Zagreb and Randić indices, provides quantitative descriptors that correlate with chemical stability, electronic properties, and mechanical strength within these molecular networks. The closed-form formulas developed in this work enable efficient computation of these indices for large-scale grid-like structures, eliminating the computational overhead of vertex-by-vertex calculations. This computational efficiency facilitates rapid screening of material properties and supports high-throughput analysis in materials design, demonstrating the practical advantage of our analytical approach in computational chemistry and nanotechnology applications.

4. Numerical Results and Graphical Comparisons

The computed values and graphical representations of the topological indices for $P_m \times P_n$, with m, n ranging from 3 to 10, are presented in the following tables and figures. These results illustrate how the indices grow as the size of the graph increases.

The figures and tables validate the closed-form formulas derived for both classical and inverse degree-based indices. They systematically capture growth trends, symmetry effects, and dependencies on $m + n$ and $m \cdot n$, while the tabulated data confirm the precision and generality of the formulas.

5. Analysis of Growth Trends

Based on the derived formulas and numerical results in the tables and figures, several key patterns emerge. The indices exhibit nonlinear growth relative to graph dimensions, where the dominant $m \cdot n$ term indicates growth proportional to grid area, while $m + n$ terms act as linear correction factors. This leads to rapid increase in index values as dimensions expand.

The GA and RGA indices show remarkably similar values across all grid sizes, which occurs because both indices use comparable edge-weighting schemes based on geometric means of adjacent vertex degrees. Additionally, square grids ($m \approx n$) consistently yield higher index values than rectangular grids due to greater structural symmetry and more interior vertices with maximum degree.

The results also demonstrate that classical indices grow faster than their reverse counterparts, with this difference increasing as grid size expands due to inverse degree weighting effects. These trends highlight the structural dependence of indices on graph geometry and provide insights for applications in mathematical chemistry and network analysis.

Conclusion

In this paper, we have derived closed-form expressions for classical and reverse degree-based topological indices of two-dimensional grid graphs $P_m \times P_n$.

The key contributions of this work include: - Development of explicit analytical formulas that eliminate the need for iterative calculations - Establishment of a unified computational framework for multiple topological indices - Provision of efficient tools for large-scale network analysis applications

These results provide a solid mathematical foundation for future research in computational chemistry, materials science, and network theory. Extensions to more complex graph structures such as irregular, weighted, or higher-dimensional networks represent promising directions for continued investigation.

https://doi.org/10.57647/IJM2C.2025.150425

Table 1. $M_1(P_m \times P_n)$ for $m, n \in [3, 10]$

$m\backslash n$	3	4	5	6	7	8	9	10	$m\backslash n$	3	4	5	6	7	8	9	10
3	68	102	136	170	204	238	272	306	3	52	68	84	100	116	132	148	164
4	102	152	202	252	302	352	402	452	4	68	88	108	128	148	168	188	208
5	136	202	268	334	400	466	532	598	5	84	108	132	156	180	204	228	252
6	170	252	334	416	498	580	662	744	6	100	128	156	184	212	240	268	296
7	204	302	400	498	596	694	792	890	7	116	148	180	212	244	276	308	340
8	238	352	466	580	694	808	922	1036	8	132	168	204	240	276	312	348	384
9	272	402	532	662	792	922	1052	1182	9	148	188	228	268	308	348	388	428
10	306	452	598	744	890	1036	1182	1328	10	164	208	252	296	340	384	428	472

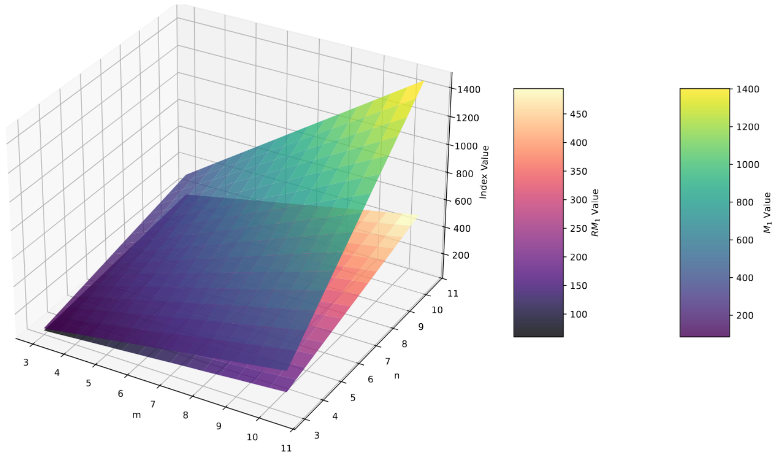

Figure 2. Comparison of M_1 and RM_1 of the Grid graph $P_m \times P_n$

Table 2. $M_2(P_m \times P_n)$ for $m, n \in [3, 10]$

$m\backslash n$	3	4	5	6	7	8	9	10	$m\backslash n$	3	4	5	6	7	8	9	10
3	96	154	212	270	328	386	444	502	3	56	69	82	95	108	121	134	147
4	154	244	334	424	514	604	694	784	4	69	84	99	114	129	144	159	174
5	212	334	456	578	700	822	944	1066	5	82	99	116	133	150	167	184	201
6	270	424	578	732	886	1040	1194	1348	6	95	114	133	152	171	190	209	228
7	328	514	700	886	1072	1258	1444	1630	7	108	129	150	171	192	213	234	255
8	386	604	822	1040	1258	1476	1694	1912	8	121	144	167	190	213	236	259	282
9	444	694	944	1194	1444	1694	1944	2194	9	134	159	184	209	234	259	284	309
10	502	784	1066	1348	1630	1912	2194	2476	10	147	174	201	228	255	282	309	336

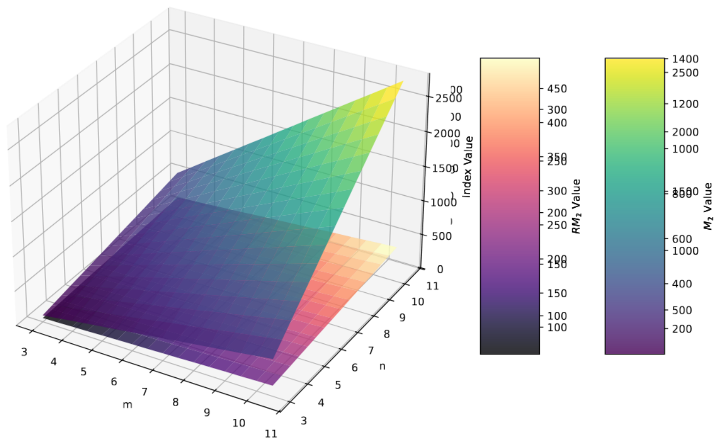

Figure 3. Comparison of M_2 and RM_2 of the Grid graph $P_m \times P_n$

https://doi.org/10.57647/IJM2C.2025.150425

Table 3. $R(P_m \times P_n)$ for $m, n \in [3, 10]$

$m\backslash n$	3	4	5	6	7	8	9	10	$m\backslash n$	3	4	5	6	7	8	9	10
3	4.4	5.9	7.4	8.9	10.4	11.9	13.4	14.9	3	6.1	9.5	12.9	16.3	19.8	23.2	26.6	30.0
4	5.9	7.9	9.9	11.9	13.9	15.9	17.9	19.9	4	9.5	14.9	20.3	25.8	31.2	36.6	42.0	47.4
5	7.4	9.9	12.4	14.9	17.4	19.9	22.4	24.9	5	12.9	20.3	27.8	35.2	42.6	50.0	57.4	64.8
6	8.9	11.9	14.9	17.9	20.9	23.9	26.9	29.9	6	16.3	25.8	35.2	44.6	54.0	63.4	72.8	82.2
7	10.4	13.9	17.4	20.9	24.4	27.9	31.4	34.9	7	19.8	31.2	42.6	54.0	65.4	76.8	88.2	99.7
8	11.9	15.9	19.9	23.9	27.9	31.9	35.9	39.8	8	23.2	36.6	50.0	63.4	76.8	90.2	103.7	117.1
9	13.4	17.9	22.4	26.9	31.4	35.9	40.3	44.8	9	26.6	42.0	57.4	72.8	88.2	103.7	119.1	134.5
10	14.9	19.9	24.9	29.9	34.9	39.8	44.8	49.8	10	30.0	47.4	64.8	82.2	99.7	117.1	134.5	151.9

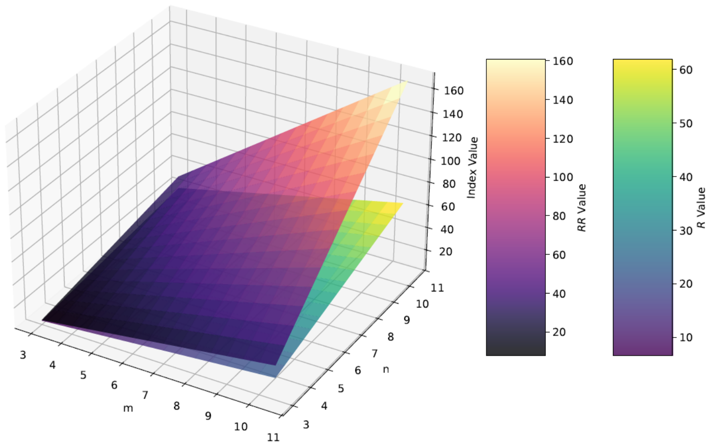

Figure 4. Comparison of R and RR of the Grid graph $P_m \times P_n$

Table 4. $ABC(P_m \times P_n)$ for $m, n \in [3, 10]$

$m\backslash n$	3	4	5	6	7	8	9	10	$m\backslash n$	3	4	5	6	7	8	9	10
3	8.2	11.5	14.7	18.0	21.2	24.4	27.7	30.9	3	8.5	11.3	14.1	17.0	19.8	22.6	25.5	28.3
4	11.5	15.9	20.4	24.9	29.3	33.8	38.2	42.7	4	11.3	14.1	17.0	19.8	22.6	25.5	28.3	31.1
5	14.7	20.4	26.1	31.8	37.5	43.1	48.8	54.5	5	14.1	17.0	19.8	22.6	25.5	28.3	31.1	33.9
6	18.0	24.9	31.8	38.7	45.6	52.5	59.4	66.3	6	17.0	19.8	22.6	25.5	28.3	31.1	33.9	36.8
7	21.2	29.3	37.5	45.6	53.7	61.9	70.0	78.1	7	19.8	22.6	25.5	28.3	31.1	33.9	36.8	39.6
8	24.4	33.8	43.1	52.5	61.9	71.2	80.6	90.0	8	22.6	25.5	28.3	31.1	33.9	36.8	39.6	42.4
9	27.7	38.2	48.8	59.4	70.0	80.6	91.2	101.8	9	25.5	28.3	31.1	33.9	36.8	39.6	42.4	45.3
10	30.9	42.7	54.5	66.3	78.1	90.0	101.8	113.6	10	28.3	31.1	33.9	36.8	39.6	42.4	45.3	48.1

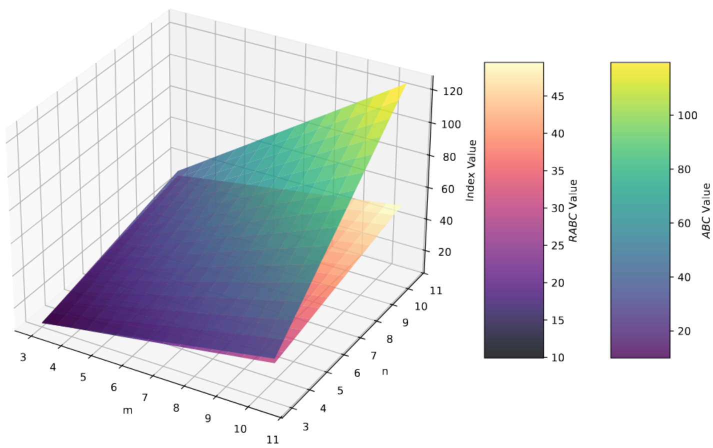

Figure 5. Comparison of ABC and $RABC$ of the Grid graph $P_m \times P_n$

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Table 5. $GA(P_m \times P_n)$ for $m, n \in [3, 10]$

$m\backslash n$	3	4	5	6	7	8	9	10	$m\backslash n$	3	4	5	6	7	8	9	10
3	11.8	16.8	21.8	26.7	31.7	36.7	41.7	46.7	3	11.6	16.5	21.4	26.3	31.2	36.0	40.9	45.8
4	16.8	23.8	30.7	37.7	44.7	51.7	58.6	65.6	4	16.5	23.4	30.3	37.2	44.0	50.9	57.8	64.7
5	21.8	30.7	39.7	48.7	57.7	66.6	75.6	84.6	5	21.4	30.3	39.2	48.0	56.9	65.8	74.7	83.6
6	26.7	37.7	48.7	59.7	70.6	81.6	92.6	103.6	6	26.3	37.2	48.0	58.9	69.8	80.7	91.6	102.5
7	31.7	44.7	57.7	70.6	83.6	96.6	109.6	122.6	7	31.2	44.0	56.9	69.8	82.7	95.6	108.5	121.4
8	36.7	51.7	66.6	81.6	96.6	111.6	126.6	141.5	8	36.0	50.9	65.8	80.7	95.6	110.5	125.4	140.2
9	41.7	58.6	75.6	92.6	109.6	126.6	143.5	160.5	9	40.9	57.8	74.7	91.6	108.5	125.4	142.2	159.1
10	46.7	65.6	84.6	103.6	122.6	141.5	160.5	179.5	10	45.8	64.7	83.6	102.5	121.4	140.2	159.1	178.0

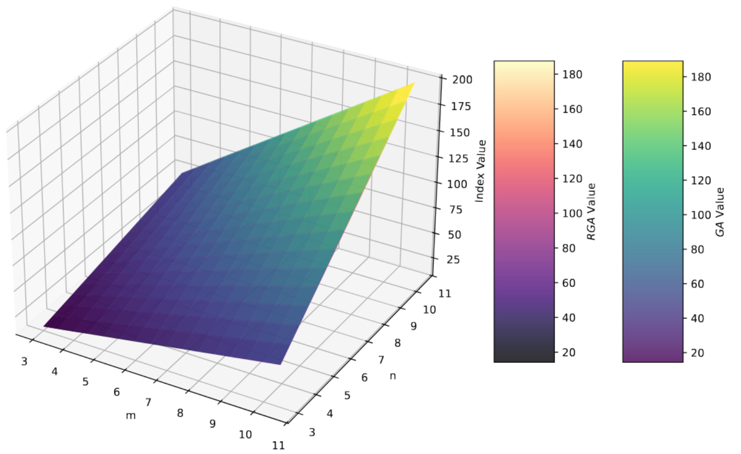

Figure 6. Comparison of GA and RGA of the Grid graph $P_m \times P_n$

Table 6. $H(P_m \times P_n)$ for $m, n \in [3, 10]$

$m\backslash n$	3	4	5	6	7	8	9	10	$m\backslash n$	3	4	5	6	7	8	9	10
3	4.3	5.8	7.3	8.8	10.3	11.8	13.3	14.8	3	5.9	9.2	12.5	15.9	19.2	22.5	25.9	29.2
4	5.8	7.8	9.8	11.8	13.8	15.8	17.8	19.8	4	9.2	14.5	19.9	25.2	30.5	35.9	41.2	46.5
5	7.3	9.8	12.3	14.8	17.3	19.8	22.3	24.7	5	12.5	19.9	27.2	34.5	41.9	49.2	56.5	63.9
6	8.8	11.8	14.8	17.8	20.8	23.8	26.7	29.7	6	15.9	25.2	34.5	43.9	53.2	62.5	71.9	81.2
7	10.3	13.8	17.3	20.8	24.3	27.7	31.2	34.7	7	19.2	30.5	41.9	53.2	64.5	75.9	87.2	98.5
8	11.8	15.8	19.8	23.8	27.7	31.7	35.7	39.7	8	22.5	35.9	49.2	62.5	75.9	89.2	102.5	115.9
9	13.3	17.8	22.3	26.7	31.2	35.7	40.2	44.7	9	25.9	41.2	56.5	71.9	87.2	102.5	117.9	133.2
10	14.8	19.8	24.7	29.7	34.7	39.7	44.7	49.7	10	29.2	46.5	63.9	81.2	98.5	115.9	133.2	150.5

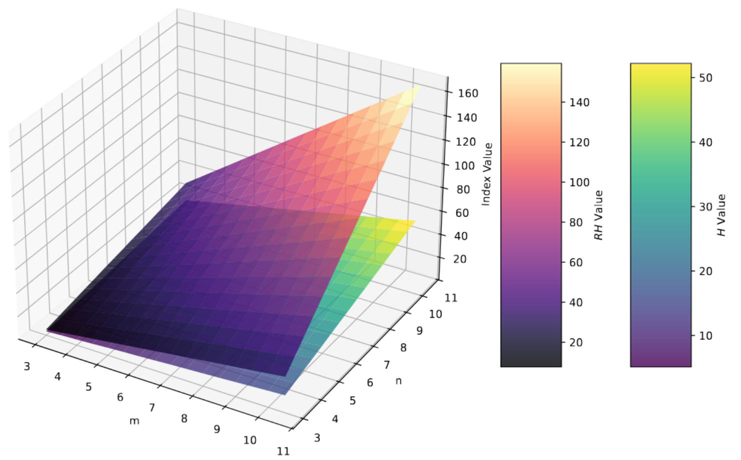

Figure 7. Comparison of H and RH of the Grid graph $P_m \times P_n$

https://doi.org/10.57647/IJM2C.2025.150425

Authors contributions
All the authors have participated sufficiently in the intellectual content, conception and design of this work or the analysis and interpretation of the data (when applicable), as well as the writing of the manuscript.
Availability of data and materials
The data that support the findings of this study are available from the corresponding author, upon reasonable request.

Conflict of interests
The author states that there is no conflict of interest.

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