

# Mathematical Model of the Diffusion and Concentration of a Dye

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## Original Research Abstract

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This paper examines the diffusion and concentration of a dye through a laboratory experiment involving a long thin glass tube filled with water and sealed at both ends. The tube is divided into two equal sections by a thin membrane placed down the center. Before the tube was sealed, different colors of dye were injected into each section. A mathematical model was developed to describe the diffusion of the dye within the tube, and a function  $C(x,t)$  representing the concentration of the dye per unit volume was evaluated. Subsequently, this concentration value was used to determine a critical point for the fastest rate of change of the diffusion and as well as its limiting value of dye concentration as time approaches infinity. The model was simulated using maple software to see the dye behavior under various conditions. Results shows that, a high rate of diffusion gradually decrease as the concentration gradient reduces, oscillation behaviors were also observed when  $t=0$ . However, when  $x=1$ , the dye concentration remain at steady rate, irrespective of the changes in the value of time, which shows the influence of temperature on medium properties. The findings from this paper have several real-life applications in various industries, including textiles, food processing, and pharmaceuticals.

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**Keywords:** Diffusion, Concentration of dye, Mathematical Model, Boundary Conditions, Diffusivity.

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## 1. Intoduction

Diffusion is the process by which matter is transported from one part of a system to another as a result of random molecular motions. This is usually demonstrated by the classical experiment in which a tall cylindrical vessel has its lower part filled with iodine solution, for instance, a column of clear water is poured on top, carefully and slowly, so that no convection currents are set up and the colored part is

separated from the clear water by a sharp, well-defined boundary, after sufficient time, the whole solution appears uniformly colored. The iodine is said to have diffused into the water [1].

The dye diffusion into fiber is described mathematically by Hill and Wilson's equations which compare the diffusion into a textile fiber with the diffusion into a cylinder of infinite length [2], [3]. Moreover, these models consider that the diffusion coefficient is constant during the entire dyeing

process and thus might the dye concentration and the dye bath temperature influence on the diffusion [4].

In a laser printer system [5], a thin layer, sand witched between two plastic sheets, was heated by a laser. The dye melts and diffuses into the sheets, the diffusion coefficient being a sensitive function of temperature. The model was used to predict the fraction of dye transferred to the receiver sheet at each radial position after a time which is long compared with the healing time.

### 2. Model formulation for diffusion of a dye

Consider the following laboratory experiment. As fig.1 suggests, a long thin glass tube, of length  $L$  is filled with water and sealed at both ends. The baffle, a thin membrane dividing the tube into equal separated sections, is placed at the exact Centre of the tube. Prior to the tube being sealed completely, two different colored dyes are injected into each section of the tube. Upon sealing the ends of the tube, the concentration of the dye is measured. The baffle then dissolves and the dye is free to diffuse throughout the tube. Where the function  $C(x, t)$  represents the concentration of dye per unit volume and  $k$  represents the diffusivity of the dye, the following partial differential equation

$$\frac{\partial^2 C(x,t)}{\partial x^2} = k \frac{\partial C(x,t)}{\partial t} \tag{1}$$

is satisfied.

As the tube is sealed at both ends, no dye can pass through the tube walls and the tube essentially “insulates” the dye satisfying the boundary conditions (2) for (1) as given by:

$$\frac{\partial C(0,t)}{\partial x} = 0 = \frac{\partial C(L,t)}{\partial x} \tag{2}$$

For this particular experiment, measurements indicate that at the instant the baffle dissolves in the tube, the dye concentration is zero at each end and a maximum value of 2 is observed at the exact Centre of the tube. This concentration is approximated analytically by

$$C(x, 0) = f(x) = 2 \sin^2\left(\frac{\lambda x}{L}\right) \tag{3}$$

Which is the initial condition for the model. We consider diffusivity in a long thin glass tube of constant cross section and homogeneous material, with the so diffusion only occurs along tube in the  $x$  direction. That is, we assume that at any location  $x$ , the concentration of dye is constant at every point in the cross section at  $x$ .

In other words, the concentration of dye will only vary in  $x$  and we can hence consider the tube to be one-dimensional. Note then that the actual shape of the cross section is independent of the problem and does not matter.

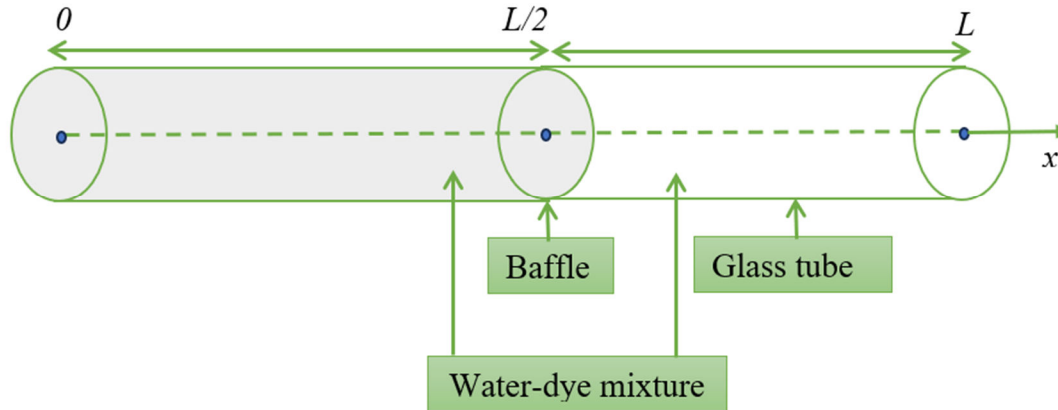


Figure 1. Water- dye mixture in a glass tube.

### 3. Solving the equation

Assuming the solution of the form,

$$C(x, t) = X(x)T(t) \tag{4}$$

Then  $C(x, t)$  is a multiplicatively separable function i.e given a two-variable function  $f(x, y)$  then we say that  $f$  is multiplicatively separable if  $\exists X(x), T(t)$  such that  $f(x, t) = X(x)T(t)$  is satisfied. If we calculate the relevant partial derivatives of  $C$ , those appearing in (1) and substitute them into the equation, we have the following results [6].

$$\frac{\partial^2 C(x,t)}{\partial x^2} = X''(x)T(t)$$

$$\frac{\partial C(x,t)}{\partial t} = X(x)T'(t)$$

Thus (1) becomes,  $\frac{d^2 X(x)}{dx^2} T(x) = kX(x) \frac{dT(t)}{dt}$ , and dividing through by  $X(x)T(t)$  yields,

$$\frac{X''(x)}{X(x)} = k \frac{T'(t)}{T(t)} = \lambda, \tag{5}$$

For some constant  $\lambda$ . If then  $\exists$  a solution to (1) of the form (4) the following two ordinary differential equations must be satisfied.

$$\frac{d^2x}{dx^2} - \lambda X(x) = 0 \tag{6}$$

$$\frac{dT}{dt} - \frac{\lambda}{k} T(t) = 0 \tag{7}$$

The boundary conditions for the original equation (2) given the following boundary conditions for  $X(x)$ ,

$$X'(0) = 0 = X'(L), T(t) \neq 0 \tag{8}$$

Which are then used to solve (6). Thus, we solve (6) for different  $\lambda$  as follows:

### 3.1. Case I: Solving (6) for $\lambda = 0$

Given the ordinary differential equation (6), with  $\lambda = 0$ , the solution obtained by indefinite double integration has the form  $X(x) = \alpha x + \beta$ .

Differentiating this solution once so that the boundary conditions (8) can be used to yields  $X'(x) = \alpha$ , some constant function. Substituting in the boundary conditions gives  $X'(0) = 0 = \alpha$ , and  $X'(L) = 0 = \alpha$ , hence  $\beta$  must be some arbitrary constant.

### 3.2. Case II: Solving (6) for $\lambda > 0$

Given the ordinary differential equation (6), with  $\lambda > 0$ , we define the auxiliary equation of (6) as,  $\zeta^2 - \lambda = 0$ , Which has real, unequal roots  $\zeta = \pm\sqrt{\lambda}$ , furthermore, the general solution of (6) is,  $X(x) = \alpha e^{\sqrt{\lambda}x} + \beta e^{-\sqrt{\lambda}x}$ .

Differentiating the solution once so that the boundary conditions (8) can be used to yields,  $X'(x) = \alpha\sqrt{\lambda}e^{\sqrt{\lambda}x} - \beta\sqrt{\lambda}e^{-\sqrt{\lambda}x}$ . Substituting the boundary conditions gives,

$$X'(0) = \alpha\sqrt{\lambda}e^{\sqrt{\lambda}0} - \beta\sqrt{\lambda}e^{-\sqrt{\lambda}0} = \alpha\sqrt{\lambda} - \beta\sqrt{\lambda} = 0$$

Which shows that  $\alpha = \beta$  and further,

$$\begin{aligned} X'(L) &= \alpha\sqrt{\lambda}e^{\sqrt{\lambda}L} - \beta\sqrt{\lambda}e^{-\sqrt{\lambda}L} = 0 \\ &= \alpha\sqrt{\lambda}e^{\sqrt{\lambda}L} - \alpha\sqrt{\lambda}e^{-\sqrt{\lambda}L} = 0 \\ &= \alpha\sqrt{\lambda}(e^{\sqrt{\lambda}L} - e^{-\sqrt{\lambda}L}) = 0 \end{aligned}$$

From the above, we defined  $\lambda > 0$ , and then have  $\alpha\sqrt{\lambda}(e^{\sqrt{\lambda}L} - e^{-\sqrt{\lambda}L}) \neq 0$ , and thus  $\alpha = 0 \Rightarrow \beta = 0$ .

Hence, the solution to (6) with  $\lambda > 0$ , is zero or trivial solution,  $X(x) = 0$ .

### 3.3. Case III: Solving (6) for $\lambda < 0$

Given the ordinary differential equation (6), with  $\lambda < 0$ , if we define  $\lambda = -\rho^2 < 0$ , then the auxiliary equation of (6) is  $\zeta^2 - \lambda = \zeta^2 + \rho^2 = 0$ .

Which has pure imaginary, complex roots  $\zeta = \pm\rho i$ , furthermore the general solution of (6) is,  $X(x) = \alpha \cos(\rho x) + \beta \sin(\rho x)$ .

Differentiating the solution once so that the boundary conditions (8) can be used to yields,  $X'(x) = \beta\rho \cos(\rho x) - \alpha\rho \sin(\rho x)$ .

Substituting the boundary conditions gives,

$$X'(0) = \beta\rho \cos(\rho \cdot 0) - \alpha\rho \sin(\rho \cdot 0) = 0 = \beta\rho = 0$$

Thus  $\beta = 0$  and further,

$$\begin{aligned} X'(0) &= \beta\rho \cos(\rho L) - \alpha\rho \sin(\rho L) = 0 \\ X'(0) &= 0 \cdot \rho \cos(\rho L) - \alpha\rho \sin(\rho L) = 0 \\ X'(0) &= -\alpha\rho \sin(\rho L) = 0 \end{aligned}$$

Now we have  $\alpha = 0$ , it will result to zero or trivial solution and thus we must have  $\sin(\rho L) = 0$ , which is true for  $\rho L = n\pi \Rightarrow \rho = \frac{n\pi}{L}, \forall n = 1,2,3, \dots$

Hence the solution  $X(x)$  is formed by a sequence of solutions,  $X_n(x) = a_n^* \cos\left(\frac{n\pi x}{L}\right), \forall n = 1,2,3, \dots$  where  $a_n^*$  are some constants.

Now we must solve the second, ordinary differential equation, (7), to complete our solution for  $C(x, t)$ .

Solving (7) for  $\lambda = 0$ .

Given the ordinary differential equation (7), with  $\lambda = 0$ , the solution obtained by one indefinite integration has the form  $T(t) = \gamma$ , the constant function.

In fact for  $\lambda = 0, T(t) = \gamma$ , is a trivial solution of (7).

If we define  $\lambda = -\rho^2 < 0$ , then the solution as with respect to  $X(x)$  of (6), can be defined as a sequence of functions.

As the general solution of the linear, separable ordinary differential equation of (7) is,  $T(t) = ce^{\frac{\lambda}{k}t}$ , we can form a sequence of solutions, substituting  $\lambda = -\rho^2$  and  $\rho = \frac{n\pi}{L}, T_n(t) = \partial_n e^{-\frac{n^2\pi^2}{kL^2}t} \forall n = 1,2,3, \dots$  Where  $\partial_n$  are some constants. Note then that there is no need to solve (7) for  $\lambda > 0$  as for this condition on  $\lambda$ , the solution of (6) gave the zero solution.

### 3.4. The solution $C(x, t) = X(x)T(t)$

Given the solutions to (6) and (7),  $X(x) = \beta$  and  $T(t) = \gamma$ , with  $\lambda > 0$ , the solution (4) is given by  $C(x, t) = X(x)T(t) = \beta\gamma = a_0$ , where  $a_0$  is some constant.

For  $\lambda < 0$ , we have a sequence of solutions for  $X(x)$  and  $T(t)$ , hence we have the following sequence of solutions for  $C(x, t)$ ,

$$\begin{aligned}
 C_n(x, t) &= X_n(x)T_n(t) \\
 &= a_n^* \cos\left(\frac{n\pi x}{L}\right) \cdot \partial_n e^{-\frac{n^2\pi^2}{kL^2}t} \\
 &= a_n \cos\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2}{kL^2}t} \tag{9}
 \end{aligned}$$

Where  $a_n = a_n^* \partial_n$  and  $n = 1, 2, 3, \dots$   
 Now since the partial differential equation of (1) is linear, by the principle of superposition, we can combine the solutions for  $\lambda = 0$ ,  $C(x, t) = a_0$  and  $\lambda < 0$ , with (9) to form the general solution to the equation,

$$C(x, t) = a_0 + \sum_{n=1}^{\infty} \left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2}{kL^2}t} a_n \cos \tag{10}$$

As stated in the introduction, the concentration is approximated analytically at  $t = 0$ , by the initial condition (3) and since we also have,

$$C(x, 0) = a_0 + \sum_{n=1}^{\infty} e^{\left(\frac{n\pi x}{L}\right)} a_n \cos,$$

at  $t = 0$ , a Fourier cosine series, we can use the initial condition (3),  $C(x, 0) = f(x) = 2 \sin^2\left(\frac{\pi x}{L}\right)$ , to calculate the coefficients  $a_0, a_n$ . Thus,  $a_0$  and  $a_n$  are given by Euler formulae:

$$\begin{aligned}
 a_0 &= \frac{1}{L} \int_0^L f(x) \cdot dx \\
 a_0 &= \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx
 \end{aligned}$$

Solving first for  $a_0$ .

$$a_0 = \frac{1}{L} \int_0^L 2 \sin^2\left(\frac{\pi x}{L}\right) \cdot dx = \frac{2}{L} \int_0^L \sin^2\left(\frac{\pi x}{L}\right) \cdot dx$$

Using the substitution  $u = \frac{\pi x}{L} \Rightarrow du = \frac{\pi}{L} dx$  and the trigonometric identity  $\cos(2\theta) = 1 - 2 \sin^2(\theta)$ , we have,

$$\begin{aligned}
 a_0 &= \frac{2}{L} \int_0^L \sin^2\left(\frac{\pi x}{L}\right) dx \\
 &= \frac{2}{L} \int_0^{\pi} \frac{1}{2} (1 - \cos(2u)) \frac{L}{\pi} \cdot du \\
 &= \frac{1}{\pi} \int_0^{\pi} 1 - \cos(2u) \cdot du \\
 &= \frac{1}{\pi} \left[ u - \frac{1}{2} \sin(2u) \right]_0^{\pi}
 \end{aligned}$$

$$= \frac{1}{\pi} [(\pi - 0) - (0 - 0)] = 1$$

And hence,  $a_0 = 1$ . Solving then for  $a_n$ ,

$$\begin{aligned}
 a_n &= \frac{2}{L} \int_0^L 2 \sin^2\left(\frac{\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) \cdot dx \\
 a_n &= \frac{4}{L} \int_0^L \sin^2\left(\frac{\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) \cdot dx
 \end{aligned}$$

Similarly, using the substitution  $u = \frac{\pi x}{L} \Rightarrow du = \frac{\pi}{L} dx$  and the trigonometric identity  $\cos(2\theta) = 1 - 2 \sin^2(\theta)$ , we have,

$$\begin{aligned}
 a_n &= \frac{4}{L} \int_0^L \sin^2\left(\frac{\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) \cdot dx \\
 &= \frac{2}{\pi} \int_0^{\pi} (1 - \cos(2u)) \cos(nu) \cdot du
 \end{aligned}$$

Applying integration by parts, we obtain,

$$\begin{aligned}
 a_n &= \frac{2}{\pi} \int_0^{\pi} (1 - \cos(2u)) \cos(nu) \cdot du \\
 &= \frac{2}{\pi} \left[ \left( u - \frac{1}{2} \sin(2u) \right) \cos(nu) \right]_0^{\pi} \\
 &\quad + \frac{2}{\pi} \int_0^{\pi} \left( u - \frac{1}{2} \sin(2u) \right) \sin(nu) \cdot du \\
 &= 2(-1)^n + \frac{2}{\pi} \int_0^{\pi} u \sin(nu) \cdot du \\
 &\quad - \frac{1}{\pi} \int_0^{\pi} \sin(2u) \sin(nu) \cdot du
 \end{aligned}$$

Repeating the method of integration by parts, we have,

$$\begin{aligned}
 a_n &= \frac{2}{\pi} \int_0^{\pi} (1 - \cos(2u)) \cos(nu) \cdot du \\
 &= \frac{2}{\pi} \left[ \left( u - \frac{1}{2} \sin(2u) \right) \cos(nu) \right]_0^{\pi} \\
 &\quad + \frac{2}{\pi} \int_0^{\pi} \left( u - \frac{1}{2} \sin(2u) \right) \sin(nu) \cdot du \\
 &= 2(-1)^n + \frac{2}{\pi} \int_0^{\pi} u \sin(nu) \cdot du
 \end{aligned}$$

$$-\frac{1}{\pi} \int_0^\pi \sin(2u) \sin(nu) . du$$

Repeating the method of integration by parts, we have,

$$\begin{aligned}
 a_n &= 2(-1)^n + \frac{2}{\pi} \times \\
 &\left\{ \left[ -\frac{1}{n} u \cos(nu) \right]_0^\pi + \frac{1}{n} \int_0^\pi \cos(nu) . du \right\} \\
 &-\frac{1}{\pi} \int_0^\pi \sin(2u) \sin(nu) . du \\
 a_n &= 2(-1)^n + \frac{2}{\pi} \times \\
 &\left\{ -\frac{\pi}{n} (-1)^n + \frac{1}{n^2} [\sin(nu)]_0^\pi \right\} \\
 &-\frac{1}{\pi} \int_0^\pi \sin(2u) \sin(nu) . du \tag{11} \\
 &= \frac{2(-1)^n(n-1)}{n} \\
 &-\frac{1}{\pi} \int_0^\pi \sin(2u) \sin(nu) . du
 \end{aligned}$$

Hence for the final integral in (11), we have two conditions to consider, when  $n \neq 2$ . for  $n = 2$ , we have the following result, again using the identity

$$\cos(2\theta) = 1 - 2 \sin^2(\theta)$$

$$\begin{aligned}
 \int_0^\pi \sin^2(2u) . du &= \int_0^\pi \frac{1}{2} (1 - \cos(4u)) . du \\
 &= \frac{1}{2} \left[ u - \frac{1}{4} \sin(4u) \right]_0^\pi \\
 &= \frac{1}{2} [(\pi - 0) - (0 - 0)] \tag{12} \\
 &= \frac{\pi}{2}
 \end{aligned}$$

Then for  $n \neq 2$ . we have the following results employing the trigonometric identity

$$\sin(\theta) \sin \varphi = \frac{1}{2} [\cos(\theta - \varphi) - \cos(\theta + \varphi)],$$

$$\begin{aligned}
 \int_0^\pi \sin^2(2u) . du &= \\
 \frac{1}{2} \left[ \int_0^\pi \cos((2+n)u) . du - \int_0^\pi \cos((2-n)u) . du \right] \\
 &= \frac{1}{2} \left[ \frac{\sin((2+n)u)}{2+n} - \frac{\sin((2-n)u)}{2-n} \right]_0^\pi \\
 &= \frac{1}{2} [(0 - 0) - (0 - 0)] = 0 \tag{13}
 \end{aligned}$$

Hence, for  $n = 2$ , equation (11) becomes,

$$a_n = \frac{2(-1)^2(2-1)}{2} - \frac{1}{2} = \frac{1}{2}$$

By (12) and  $n \neq 2$ , (11) becomes,

$$a_n = \frac{2(-1)^n(n-1)}{n}$$

By (13), thus we can substitute our evaluated coefficients,  $a_0$  and  $a_n$ , into the general solution to the partial differential equation (1) and yield the solution.

$$\begin{aligned}
 C(x, t) &= 1 + \sum_{n=1}^\infty a_n \cos\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2 \pi^2}{kL^2} t} \\
 a_n &= \begin{cases} \frac{1}{2} & n = 2 \\ \frac{2(-1)^n(n-1)}{n} & n \neq 2 \end{cases} \tag{14}
 \end{aligned}$$

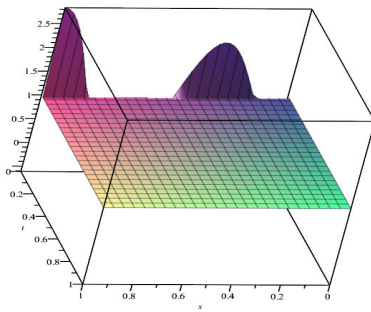
Which when extracting  $n = 1,2$  from the summation, we have the general solution to the problem as below:

$$\begin{aligned}
 C(x, t) &= 1 + \frac{1}{2} \cos\left(\frac{2\pi x}{L}\right) e^{-\frac{n^2 \pi^2}{kL^2} t} + \\
 &\sum_{n=3}^\infty \frac{2(-1)^n(n-1)}{n} \cos\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2 \pi^2}{kL^2} t} \tag{15}
 \end{aligned}$$

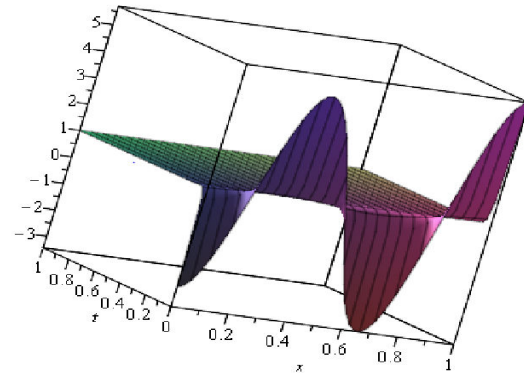
Subject to the boundary conditions (2) and initial condition(3).

The value of diffusivity of a dye constant  $k = 0.8 \times 10^{-5}$  also known as the diffusion coefficient, which is a measure of the rate which particles or molecules diffuse through a long thin glass tube of constant cross section and homogeneous material which has Water- dye mixture.

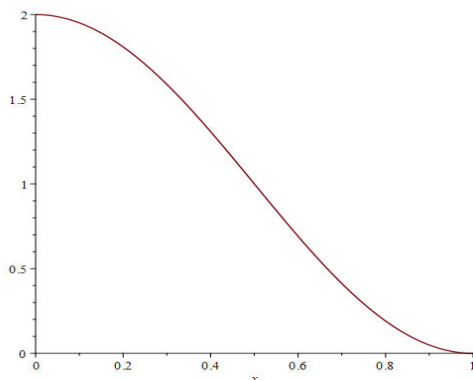
### 3.5. Graphs of the general equations



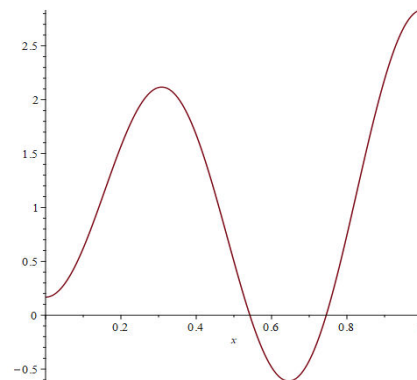
(i)  $(x, t, 0.8 \times 10^{-5})$



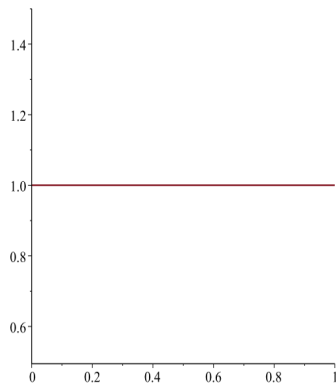
(ii)  $(x, t, 0.8 \times 10^{-5})$



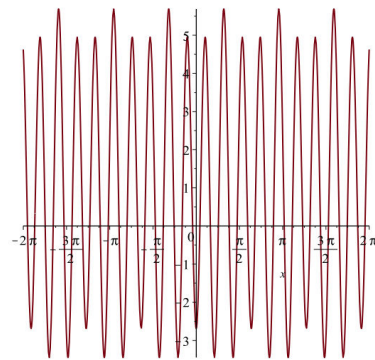
(iii)  $(0, 0, 0.8 \times 10^{-5})$



(iv)  $(0, 0, 0.8 \times 10^{-5})$



(v)  $(1, t, 0.8 \times 10^{-5})$



(vi)  $(1, 0, 0.8 \times 10^{-5})$

#### 4. Determination of critical point for the rate of change of dye

To determine where in the tube the dye concentration initially is changing at its fastest rate, we first need to calculate the rate of change for position. That is, we need to calculate  $C_x(x, t)$  for (15), (16),

$$\frac{\partial C(x, t)}{\partial x} = \frac{\pi}{L} \sin\left(\frac{2\pi x}{L}\right) e^{-\frac{n^2 \pi^2}{kL^2} t} - \frac{\pi}{L} \sum_{n=3}^{\infty} \frac{2(-1)^n(n-1)}{n} \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2 \pi^2}{kL^2} t} \quad (16)$$

The initial maximum value for  $C_x(x, t)$  can be obtained by finding the critical points of (16) at  $t = 0$ , that is, finding  $x$  satisfying  $C_{xx}(x, t) |_{t=0} = 0$ .

Differentiating (16) with respect to  $x$  and evaluating at  $t = 0$ , gives,

$$\frac{\partial^2 C(x, t)}{\partial x^2} |_{t=0} = -\frac{2\pi^2}{L^2} \cos\left(\frac{2\pi x}{L}\right) - \frac{\pi^2}{L^2} \sum_{n=3}^{\infty} 2(-1)^n(n-1) \cos\left(\frac{n\pi x}{L}\right) \quad (17)$$

Also,  $C_{xx}(x, t) |_{t=0} = 0, \forall \frac{n\pi x}{L} = \frac{\pi}{2}, n \geq 2 \Rightarrow x = \frac{L}{2n}$ .

Hence  $x = \frac{L}{2n}$ , is a maximum or minimum if

$$\frac{\partial^3 C}{\partial x^3} |_{x=\frac{L}{2n}} < 0 \text{ or } \frac{\partial^3 C}{\partial x^3} |_{x=\frac{L}{2n}} > 0 \quad \text{respectively.}$$

Differentiating (17), we obtain  $\frac{\partial^3 C}{\partial x^3} |_{x=\frac{L}{2n}}$ ,

$$\frac{\partial^3 C(x, t)}{\partial x^3} = \frac{4\pi^3}{L^3} \sin\left(\frac{2\pi x}{L}\right) e^{-4\frac{\pi^2}{kL^2} t} + \frac{\pi^3}{L^3} \sum_{n=3}^{\infty} 2(-1)^n(n-1) \sin\left(\frac{n\pi x}{L}\right) e^{-4\frac{\pi^2}{kL^2} t} \quad (18)$$

##### 4.1. The limiting value of dye concentration

Finally, to calculate the limiting value of the dye concentration for very large values of time we need (15) as  $t \rightarrow \infty$ , in fact, for  $t \rightarrow \infty, C(x, t) \rightarrow 1$  and thus the limiting value of the dye concentration for very large values of time is 1 by (15).

#### 5. Discussion

In this study, we investigated the diffusion of dye in a controlled environment using a long thin glass tube divided by a measure. The model was analyzed to see the concentration of dye over time and space. The graphs were generated using the Maple software as presented in the graphs of (i) - (vi) showing the concentration changes and the behaviors of the dye.

The result initially shows the concentration of the dye in each section of the tube, leading to a steep concentration generated at the beginning of the tube. The graphs of (i) and (ii) using  $(x, t, 0.8 \times 10^{-5})$ , shows different surfaces been seen as the general equation of (10) and that of equation (15) are similar showing two surfaces and three surfaces respectively with their behaviors when the values of  $x$  and  $t$  are between  $0 \leq 1$ .

The surfaces of the two graphs (i) and (ii), indicates a high rate of diffusion which gradually decreased as the concentration gradient reduces.

Meanwhile, in graphs (iii) and (iv), using  $(x, 0, 0.8 \times 10^{-5})$  shows oscillation behaviors with remaining constant when the value of  $t = 0$ . However, when the value of  $x = 1$ , its generated graphs of (v) -  $(1, t, 0.8 \times 10^{-5})$  and (vi), showing that the dye concentration remain steady all through irrespective of the change in the value of time.

As time progresses, the concentration in both sections approaches a limiting value, showing the dye is moving towards steady state of the equilibrium with minimal changes after a certain period of time. This behavior agreed with the prediction of model which suggested that, the limiting concentration can be calculated as the average of the initial concentration of the two dyes.

Similarly, the graphs of (i) to (iv) shows the rate of change in concentration over time  $t$ , reaching its peaks or highest point, indicating the fastest rate of change in dye concentration, showing how fast the dye spreads through the medium.

The findings from this paper have several real-life applications in various industries, including textiles, food processing, and pharmaceuticals.

In the textiles industry, controlling dye diffusion is essential for achieving uniform collaboration in fabrics, while in food processing, it is to enhance flavor infusion in products and, in pharmaceuticals is to control the release of drugs which can be modelled using similar diffusion principles for allowing the design of more effective delivery systems.

#### 6. Limitations and future work

This paper provides valuable insights into the diffusion process, it is necessary to acknowledge its limitations. The model assumes ideal conditions such as uniform temperature and pressure, which may not always be present in real-world scenarios.

Future research could explore the effects of varying environmental conditions on dye-diffusion and extend the model to account other factors such as temperature and presence of other solutes.

#### 7. Conclusion

In conclusion, the concentration of dye per unit volume with a function  $C(x, t)$  and  $k$  which represents the diffusivity of the dye diffusion presented in this paper has enhances our understanding of the diffusion principles with different boundary conditions using partial differential equations.

The graphs generated using the Maple software has shown various surface behaviors of the dye concentrations, the diffusivity of the material varying with the location in the

tube, when the change in total diffusion is assumed negligible.

#### Authors Contribution

All the authors have participated sufficiently in the intellectual content, conception and design of this work or the analysis and interpretation of the data (when applicable), as well as the writing of the manuscript.

#### Availability of data and materials

The data that support the findings of this study are available from the corresponding author, upon reasonable request.

#### Conflict of interests

The author states that there is no conflict of interest.

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