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Original Research

Efficiency Analysis of Insurance Companies under Fuzzy Environment

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Abstract:

Performance evaluation is crucial for the sustainable growth and effective resource allocation of insurance companies. However, the uncertainty and imprecision inherent in operational and financial data limit the applicability of traditional methods. This study proposes a novel fuzzy two-stage data envelopment analysis (DEA) model to assess the efficiency of insurance companies under such conditions. The model employs trapezoidal fuzzy numbers to represent inputs, intermediate measures, and outputs, thereby providing a more robust and realistic efficiency measurement than conventional deterministic DEA. Applied to a dataset of 40 insurance companies, the model yields individual stage efficiencies, overall efficiency scores, and optimal improvement targets for each firm. The key findings reveal that this approach not only identifies significant inefficiencies but also offers actionable strategic insights. Specifically, it enables managers to enhance cost control, optimize policy issuance volumes, accelerate claims settlement, and improve customer satisfaction. These improvements are essential for achieving a sustainable competitive advantage in the highly dynamic insurance market. The study concludes that the fuzzy two-stage DEA model is a valuable tool for both managers and policymakers in making informed decisions for superior resource allocation and performance management in the insurance sector.

Keywords: Insurance performance; Fuzzy environment; Data envelopment analysis; Two-stage network; Efficiency

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1. Introduction

The insurance industry plays a vital role in the economic stability and risk management of societies. Evaluating the performance of insurance companies is therefore an essential task for regulators, managers, and policymakers. Traditional performance measurement methods, such as ratio analysis, often fail to capture the multidimensional nature of inputs and outputs in the insurance sector (Kamanda and Sibindi [1], Mergoni et al. [2]).

In real-world insurance operations, uncertainty and vagueness are inherent due to fluctuating market conditions, incomplete information, and subjective judgments in areas such as claims management or premium estimation. To address this challenge, fuzzy set theory provides an effective tool for modeling imprecise data. By integrating fuzzy concepts into DEA, a more flexible and realistic evaluation framework can be established (Zhao et al. [3]). Performance evaluation plays a critical role in guiding managerial decision-making, resource allocation, and policy formulation in service-oriented industries such as insurance. Traditional DEA, first introduced by Charnes et al. [4], provides a powerful non-parametric method for measuring the relative efficiency of *DMU*s with multiple inputs and outputs. However, conventional DEA models treat each *DMU* as a "black box" without considering the internal structures and interdependencies between sub-processes. This simplification often limits the ability to provide targeted managerial recommendations (Färe and Grosskopf, [5]). To overcome this limitation, Network DEA models explicitly represent the internal structure of (Decision Making Units) *DMU*s, decomposing the production process into multiple interconnected stages. Among these, the two-stage network DEA structure has received significant attention due to its ability to model sequential production processes, where the outputs of the first stage serve as inputs to the second stage (Kao and Hwang [6], Kao [7]). This approach not only enables the evaluation of overall efficiency but also provides insights into stage-specific performance, revealing inefficiencies that might be masked in traditional DEA. For example, in the context of insurance companies, a two-stage structure naturally emerges: (i) Underwriting/Sales stage, where human and financial resources are used to generate issued policies and premium revenues. (ii) Claims and Service stage, where the intermediate outputs from the first stage are used to settle claims, generate profits, and ensure customer satisfaction.

In real-world applications, particularly in the insurance sector, the data associated with costs, customer satisfaction, and service quality are often imprecise or vague due to market fluctuations, subjective assessments, and incomplete records. This uncertainty can be effectively represented using fuzzy set theory (Ostovan et al. [8], Gerami et al. [9, 10]), which allows variables to be expressed as fuzzy numbers rather than precise values. The integration of fuzzy data into DEA models has been extensively studied (Kao and Liu [11], Emrouznejad et al. [12]) and has proven valuable in improving the robustness of efficiency evaluations under uncertainty. Several researchers have extended two-stage DEA to handle fuzzy data. For example, Kao and Liu [11] introduced a fuzzy two-stage DEA framework, while Arana-Jimenez et al. [13] proposed a new approach for efficiency assessment and target setting by using a fully fuzzy DEA. However, many of these models are computationally complex, limiting their practical adoption. To address this, a common approach is to transform fuzzy DEA models into equivalent deterministic models using defuzzification techniques (Hatami-Marbini et al. [14]), enabling the use of conventional linear programming solvers while retaining the interpretability of fuzzy inputs and outputs.

Evaluating the operational efficiency of insurance companies has attracted significant attention in recent years due to the increasing complexity and competitiveness of the insurance market. DEA has been widely used as a non-parametric approach to assess the relative efficiency of insurance firms based on multiple inputs and outputs (Emrouznejad and Yang [15]). Early applications focused on conventional DEA models treating insurance companies as single-stage processes (Lu et al. [16]). However, this simplification often ignores the complex, multi-process nature of insurance operations. To address this limitation, network DEA models have been developed, allowing the decomposition of the production process into interconnected sub-processes or stages. The two-stage network DEA model is especially relevant for insurance companies, representing underwriting/sales activities in the first stage and claims handling/service in the second (Kao and Hwang [6]). This approach enables stage-specific efficiency measurement and deeper managerial insights into bottlenecks or inefficiencies at each operational phase (Chen et al. [17]). Moreover, real-world insurance data frequently contain uncertainty and vagueness due to market volatility, subjective judgments, and incomplete reporting. To handle such imprecision, researchers have integrated fuzzy set theory with DEA models. Fuzzy DEA enables the incorporation of imprecise data in inputs and outputs, providing a more realistic and flexible evaluation framework (Kao and Liu [11], Emrouznejad et al. [12]). Recent studies extend fuzzy DEA to network structures, allowing simultaneous analysis of multiple production stages under uncertainty (Hatami-Marbini et al. [14]). In the insurance domain, fuzzy DEA has been applied to assess financial performance, customer satisfaction, and operational efficiency with promising results (Chrysafis et al. [18]). Nonetheless, the computational complexity of fuzzy network DEA models remains a challenge. To overcome this, some studies propose transforming fuzzy DEA into equivalent crisp models through defuzzification methods, facilitating practical implementation (Gerami et al. [10]). Despite these advances, few studies have focused explicitly on developing two-stage fuzzy network DEA models tailored to insurance companies, which incorporate key inputs such as agents and operating costs, intermediate measures like policies issued and premiums collected, and outputs including claim settlement rates, net profit, and customer satisfaction. This gap motivates the current research to develop a robust and computationally efficient DEA model to provide actionable efficiency scores and performance targets for insurance firms under uncertainty.

However, in the insurance industry, data often contain ambiguity and uncertainty caused by market fluctuations, subjective judgments, incomplete information, and qualitative factors such as customer satisfaction and service quality (Kao and Liu, [11]).

Fuzzy DEA models address these challenges by incorporating fuzzy set theory, which allows the representation of imprecise, vague, or uncertain data through fuzzy numbers (Emrouznejad et al. [12]). This integration offers several key advantages over classical DEA approaches: (i) Flexibility in handling uncertainty and ambiguity: Fuzzy DEA enables decision-makers to model uncertain data as fuzzy numbers, thus capturing the range of possible values rather than a single crisp figure, leading to more realistic and robust efficiency evaluations (Soltanzadeh and Omrani [19]). (ii) Incorporation of both quantitative and qualitative data: Many important performance indicators in insurance, such as customer satisfaction and claim settlement rates, are inherently qualitative or subjective. Fuzzy DEA accommodates these types of data effectively, enhancing the comprehensiveness of the assessment (Zhou et al. [20, 21]). (iii) Capability to model multi-stage or network structures under uncertainty: Insurance operations often involve multiple interconnected stages, such as underwriting and claims processing. Fuzzy network DEA models allow simultaneous evaluation of these stages while accounting for data uncertainty, thus providing a

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more detailed and accurate picture of overall efficiency (Mozaffari et al. [22]). (iv) Reduction of managerial decision-making errors: By considering the fuzziness and vagueness in data, fuzzy DEA reduces the risk of misleading efficiency scores and enables managers to set realistic performance improvement targets and resource allocation plans (Mahapatra et al. [23]). (iiv) Suitability for incomplete or imprecise data: Given that precise and complete data are not always available in the insurance sector, fuzzy DEA provides a practical solution that maintains the integrity of the analysis despite data limitations (Peykani et al. [24, 25, 26]).

Compared with previous studies, the present research offers several important methodological and practical contributions. First, while conventional DEA models assume precise data, the proposed framework incorporates triangular fuzzy numbers to explicitly account for uncertainty and imprecision in insurance companies' operational and financial information, leading to more reliable and realistic efficiency assessments. Second, unlike traditional single-stage DEA approaches, the proposed model adopts a two-stage structure that reflects the internal operational process of insurance companies, enabling a more detailed identification of inefficiencies at each stage of performance. Third, the model simultaneously provides stage-specific efficiencies, overall efficiency scores, and optimal improvement targets within a unified fuzzy environment, thereby enhancing both analytical depth and managerial interpretability. Finally, by generating actionable improvement strategies under uncertainty, the proposed fuzzy two-stage DEA model extends the existing literature from mere efficiency measurement toward strategic performance management and sustainable resource allocation in the insurance industry.

Considering these benefits, it is evident that evaluating insurance companies' efficiency within a fuzzy environment is not only methodologically justified but also essential for capturing the true operational performance. Such evaluations support insurance managers in understanding complex market realities, optimizing resource utilization, and designing effective strategies for continuous improvement.

This paper contributes to the literature in three main ways:

1. Model development: We propose a novel fuzzy two-stage network DEA model that can handle fuzzy inputs, intermediate measures, and outputs simultaneously, tailored to the operational structure of insurance companies.

2. Defuzzification-based transformation: We present a systematic procedure to convert the fuzzy network DEA model into an equivalent deterministic form, enabling efficient computation of both overall and stage-specific efficiency scores.

3. Managerial application: We apply the proposed model to a dataset of 40 insurance companies, providing not only efficiency scores but also optimal target values for each input, intermediate measure, and output. These targets serve as actionable benchmarks for insurance managers to optimize cost structures, improve policy issuance, enhance claims settlement rates, and boost customer satisfaction. By bridging methodological innovation with managerial applicability, the proposed model offers a practical decision-support tool for performance benchmarking in the insurance industry under uncertainty. This approach allows decision-makers to pinpoint inefficiencies, allocate resources more effectively, and develop strategies for sustainable competitive advantage in a highly competitive and dynamic market.

The continuation of this paper is organized as follows. In the section 2, we examine some of the studies conducted in the fields in the field's network fuzzy DEA and insurance companies. In the section 3 present the methodology of the research. In the section 4, we use the proposed approach in this paper to evaluate a set of insurance companies in Iran. In the section 5, we present the results of the paper.

2. Literature review

In this section, we examine some of the studies conducted in the fields in the field's network fuzzy DEA and insurance companies' efficiency in DEA.

2.1 Network fuzzy DEA

Determining the exact value of some input and output data is one of the challenges in applications of DEA, where the observed data are often uncertain or inaccurate. Fuzzy sets are a suitable way to deal with this uncertainty in DEA. Then, imprecise or ambiguous data in DEA can be shown by linguistic expressions characterized by fuzzy numbers (Hatami-Marbini et al. [14]). Various fuzzy DEA (FDEA) methodologies have been developed to address situations in which the available data are of a fuzzy nature. A seminal reference in this area is Emrouznejad et al. [12], which expands upon the earlier review conducted by Hatami-Marbini et al. [14] and introduces a comprehensive taxonomy categorizing FDEA models into several groups: α-level set approaches (e.g., Kao and Liu [11], Saati et al. [27]), fuzzy ranking approaches (e.g., Guo and Tanaka [28]), possibility-based approaches (e.g., Lertworasirikul et al. [29]), fuzzy arithmetic-based approaches (e.g., Ostovan et al. [8]), and those employing fuzzy random variables or type-2 fuzzy sets (e.g., Tavana et al. [30]).

Mehdizadeh et al. [31] measure the efficiency of two-stage network processes and proposed a satisficing DEA approach. Peykani et al. [24] developed an adjustable fuzzy chance-constrained network DEA approach and applied it to ranking investment firms. Pourbabagol et al. [32] developed a fuzzy network DEA model to evaluate agile supply chain performance, incorporating possibility and necessity measures to address equality chance constraints, with empirical application to the Iranian dairy sector. Amirteimoori et al. [33] proposed a combined fuzzy DEA and artificial intelligence algorithms

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in the environmental performance analysis.

Amirteimoori et al. [34] developed a returns-to-scale and scale economies of two-stage production processes by using a fully fuzzy range-adjusted measure model with strong complementary slackness conditions. Hamidzadeh et al. [35] proposed a novel two-stage network DEA model for kidney allocation problem under medical and logistical uncertainty. Shojaie et al. [36] introduced a two-stage network DEA method incorporating fuzzy programming into the Malmquist Productivity Index (MPI) to analyse productivity changes in mutual funds, offering more robust insights into efficiency dynamics and technological progress over time. Rezakhanlou and Mirzapour [37] developed a dual-channel multi-objective green supply chain network design considering pricing and transportation mode choice under fuzzy uncertainty. Garrido et al. [38] proposed a two-stage fuzzy-possibility mixed-integer linear programming model for multi-echelon globalized agro-industrial supply chain under conditions of uncertainty. Peykani et al. [25, 26] proposed a novel adjustable intuitionistic fuzzy framework for two-stage DEA and developed an application in the banking sector.

Inverse DEA is widely utilized for resource allocation and target setting, yet conventional models largely depend on certain data, restricting their use in contexts with dual uncertainty. To bridge this gap, a recent study (Huang and Chen [39]) puts forward a new inverse DEA framework capable of operating under combined fuzzy and random uncertainties. This model aids managers in attaining desired efficiency levels across diverse production scales, without requiring strict presumptions about probability distributions or fuzzy number forms.

To address conventional DEA limitations, Sahil et al. [40] developed an integrated framework that evaluates internal network processes under uncertainty. By combining dual-frontier and two-stage DEA with type-2 fuzzy variables, their model provides both optimistic and pessimistic efficiency scores, enabling detailed performance differentiation. Tested across multiple sectors, this approach offers a robust tool for complex organizational assessment.

Recent advances in financial modelling seek to bridge behavioural psychology and systemic complexity. For instance, Zhou et al. [20] developed a high-order fuzzy portfolio model that integrates regret theory and a neural-network evaluation system. The model uses trapezoidal fuzzy numbers for uncertain returns and optimizes for mean, variance, and skewness, while a unique mutual evaluation framework assesses assets based on financial ratios. Tested with market data, it proves effective in aligning investment choices with behavioral biases and systemic features, offering a robust interdisciplinary tool for complex decision-making.

Recent research tackles the high uncertainty in group decision-making within social networks by synthesizing quantum theory and game theory. Yan et al. [41] introduce a method where psychological states influence attribute weights, trust evolves through a quantum fuzzy model, and interference effects are measured via Kullback-Leibler divergence to compute a final quantum-probabilistic ranking of options.

2.2 Evaluation of insurance companies

In market economies, assessing the financial efficiency of insurance companies is critical for evaluating market position and guiding strategic improvement. Efficiency levels are shaped by operational scope and the range of insured risks (Lu et al. [16]). Literature categorizes evaluation methods into basic approaches, such as simplified and detailed ratio analyses Kamanda and Sibindi [1] and advanced techniques, notably DEA using CCR and BCC models, alongside Tobit regressions (Jaloudi [42]). Efficiency in this context is a non-monetary measure of core functions, including sales, customer service, underwriting, and claims settlement (Koc et al. [43]). However, accurate measurement is hindered by limited internal reporting, scarce benchmark data, and the sector's unique financial reporting practices. Given these challenges, this study critically reviews the application of DEA as a robust tool for assessing insurers' financial performance. Polyakov and Polyakova [44] proposed examples of the use of DEA to evaluate the financial effectiveness of insurance companies. Zhao et al. [3] proposed a method in the evolution and determinants of Chinese property insurance companies' profitability based on the DEA. Koc et al. [43] proposed a clustering based on the fuzzy classification with a noise cluster in detecting fraud in insurance. The new framework of them is robust against unbalanced class distributions. This method can predict fraud cases with high accuracy performance. Polyakov and Polyakova [44] determined the quality of financial management of insurance companies based on the DEA. They applied data from the financial statements of insurance companies for the period from 2017 to 2020 of the group of leaders in terms of insurance premiums of the Expert RA rating agency. Smętek et al. [45] proposed examples for the application of the dynamic financial analysis method to assess the financial situation and solvency of insurance companies. Liu et al. [46] developed an evaluation indicator system that includes negative data and proposes the dynamic two-stage improved base point slacks-based measure model in presence of negative data. They evaluated the performance of Chinese and foreign property insurance companies by considering negative data by the dynamic two-stage SBM model. For robust performance evaluation under uncertainty, recent methodological advancements integrate fuzzy weighting with enhanced ranking techniques. Özçalic et al. [47] developed a hybrid MCDM framework combining Fuzzy LBWA and a modified ARTASI method to assess leading insurance brokers. Ho and Hsu [48] found the effect of micro insurance on the insurance market in the evidence from Taiwan. They assessed insurance industry in this market. Jia-Ying et al. [49] proposed a performance evaluation and analysis of Chinese insurance companies under uncertain financial markets. Their research also increased

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understanding of how insurance businesses deal with uncertain economic situations and the tactics they use to ensure stability in the international financial markets environment. Seyed Esmaeili, and Mohammadi [50] proposed a Z-Number network DEA model and applied it for evaluating performance of the Iranian Insurance Industry. Lozano propoded a process efficiency of two-stage systems with fuzzy data for evaluation insurance companies. Nasseri and Khatir [51] reformulated two-stage DEA models with undesirable outputs and fuzzy stochastic data using a possibility–probability framework, yielding a deterministic linear structure and validating the approach in the banking sector. Nosrat et al. [52] extended two-stage DEA models to handle fuzzy inputs, outputs, and intermediate measures by leveraging credibility theory, accompanied by sensitivity and stability analyses to determine projections for inefficient units and stability radii for efficient ones. Ohene-Asare et al. [53] used the cost Malmquist productivity index in the insurance industry to find the change in the cost efficiency over time. They understand that a positive effect of an insurance-related act on the productivity of the insurance industry in Ghana. A key contribution is the novel use of Decision Stability Intervals (DSI) and Monte Carlo simulations to test ranking robustness, revealing that profitability and leverage indicators are the most rank-sensitive criteria. The model provides a transparent, stable evaluation tool applicable across various industries.

3. Preliminaries on fuzzy sets and fuzzy numbers

In this section, we briefly review the basic concepts and notations from fuzzy set theory that are used throughout the paper.

A **fuzzy set** φ is defined as a mapping $\varphi : R^n \to [0,1]$, where $\varphi(x)$ denotes the degree of membership of $x \in R^n$.

For a given fuzzy set φ and $\alpha \in [0,1]$, the α-Level (α-cut) is defined as $[\varphi]^\alpha = \{x \in R^n | \varphi(x) \geq \alpha\}$.

The **support** of φ is defined by $supp(\varphi) = \{x \in R^n | \varphi(x) \geq 0\}$.

The **0-level set** of φ is defined as $[\varphi]^0 = \overline{supp(\varphi)}$ where $\overline{A}$ denotes the closure of a set $A \in R^n$.

A fuzzy number is a special type of fuzzy set defined on R as follows (Zimmermann [54]).

Definition 1. (Fuzzy Number)

A fuzzy set $\varphi : R \to [0,1]$ is called a **fuzzy number** if it satisfies the following properties:

- **i. Normality:** There exists $\omega_o \in R$ such that $\varphi(\omega_o) = 1$.
- **ii. Upper semi-continuity:** The function φ is upper semi-continuous on R.
- **iii. Fuzzy convexity:** For all $\omega_1, \omega_2 \in R, \mu \in [0,1]$ then $\min\{\varphi(\omega_1), \varphi(\omega_2)\} \leq \varphi(\mu\omega_1 + (1-\mu)\omega_2)$.
- **iv. Compact support:** The 0-level set $[\varphi]^0$ is compact.

Definition 2. (Trapezoidal Fuzzy Number)

A fuzzy number as $\tilde{\tau} = (\tau^1, \tau^2, \tau^3, \tau^4)$ with $\tau^1 \leq \tau^2 \leq \tau^3 \leq \tau^4$, is called a trapezoidal fuzzy number if its membership function is defined as

$$\tilde{\tau}(\omega) = \begin{cases} \frac{\omega - \tau^1}{\tau^2 - \tau^1} & \text{if } \tau^1 \leq \omega \leq \tau^2 \\ 1 & \text{if } \tau^2 \leq \omega \leq \tau^3 \\ \frac{\tau^4 - \omega}{\tau^4 - \tau^3} & \text{if } \tau^3 \leq \omega \leq \tau^4 \\ 0 & \text{otherwise} \end{cases}$$

For a trapezoidal fuzzy number $\tilde{\tau}$, the corresponding α-Level set is given by $[\tau]^\alpha = [\tau^1 + \alpha(\tau^2 - \tau^1),\ \tau^4 - \alpha(\tau^4 - \tau^3)]$.

Let $TLF(R)$ denote the set of all trapezoidal fuzzy numbers on R. We denote by $TLF^+(R)$ the subset of **non-negative trapezoidal fuzzy numbers**, i.e., those satisfying $\tau^1 \geq 0$.

Definition 3. Let $\tilde{c} = (c^1, c^2, c^3, c^4) \in TLF(R)$ and $\tilde{d} = (d^1, d^2, d^3, d^4) \in TLF(R)$ be two trapezoidal fuzzy numbers, where $c^1 \leq c^2 \leq c^3 \leq c^4$ and $d^1 \leq d^2 \leq^3 \leq d^4$. The basic arithmetic operations are defined as follows:

(i) Addition:
$\tilde{c} + \tilde{d} = (c^1 + d^1, c^2 + d^2, c^3 + d^3, c^4 + d^4)$.

(ii) Multiplication by a scalar $\gamma \in R$,

$$\Gamma\tilde{c} = \begin{cases} (\gamma c^1, \gamma c^2, \gamma c^3, \gamma c^4) & if\ \gamma \geq 0 \\ (\gamma c^4, \gamma c^3, \gamma c^2, \gamma c^1) & if\ \gamma < 0 \end{cases}$$

(iii) Multiplication of two fuzzy numbers:
Let
$\tilde{c}\tilde{d} = \tilde{e} = (e^1, e^2, e^3, e^4)$,
where
$e^1 = \min\{c^1d^1, c^1d^4, c^4d^1, c^4d^4\}$,
$e^2 = \min\{c^2d^2, c^2d^3, c^3d^2, c^3d^3\}$,
$e^3 = \max\{c^2d^2, c^2d^3, c^3d^2, c^3d^3\}$,
$e^4 = \max\{c^1d^1, c^1d^4, c^4d^1, c^4d^4\}$.
In the particular where $\tilde{c} \in TLF^+(R)$, $\tilde{d} \in TLF^+(R)$ (i.e., all components are non-negative), multiplication simplifies to $\tilde{c}\tilde{d} = (c^1d^1, c^2d^2, c^3d^3, c^4d^4)$

(iv) Division
For $\tilde{c} \in TLF^+(R), \tilde{d} \in TLF^+(R)$, assuming that all components of $\tilde{d}$ are strictly positive, the division is defined as $\tilde{c}/\tilde{d} = (c^1/d^4, c^2/d^3, c^3/d^2, c^4/d^1)$.

In this study, the input variables, intermediate measures, output variables, and certain model parameters are considered as non-negative trapezoidal fuzzy numbers belonging to $TLF^+(R)$. The arithmetic operations among these quantities follow **Definition 3**. To compare trapezoidal fuzzy numbers, we adopt the LU-fuzzy partial order, which is well established in the literature (see, e.g., Arana-Jimenez et al. [13]).

Let $\tilde{u}$ and $\tilde{v}$ be two fuzzy numbers. Then, $\tilde{u} \leqq (\geqq)\tilde{v}$ if $\underline{u}_\alpha \leq (\geq)\underline{v}_\alpha$, $\overline{u}_\alpha \leq (\geq)\overline{v}_\alpha$, for $\alpha \in [0,1]$, where $[\underline{u}_\alpha, \overline{u}_\alpha]$ and $[\underline{v}_\alpha, \overline{v}_\alpha]$ denote the α-level sets of $\tilde{u}$ and

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$\tilde{v}$ respectively. In particular, for two trapezoidal fuzzy numbers $\tilde{c} = (c^1, c^2, c^3, c^4)$, $\tilde{d} = (d^1, d^2, d^3, d^4)$, we have $\tilde{c} \leqq (\geqq) \tilde{d}$ if only if $c^i \leqq (\geqq) d^i$, $i = 1,2,3,4$. We would like to emphasize that this component-wise approach is a sufficient but not necessary condition for comparability. That is, while it provides a practical way to compare trapezoidal fuzzy numbers in the model, there may exist some pairs of fuzzy numbers that are not comparable under this rule, which is consistent with the general properties of the LU-fuzzy partial order.

4. Fuzzy two-stage DEA

Now, we consider the two-stage process in which each *DMU* consume only the inputs from the first stage to product the final outputs in the second stage via intermediate measures as figure 1.

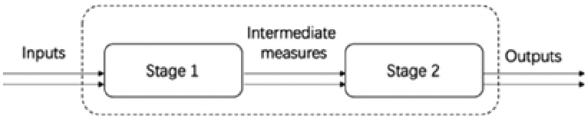

Figure 1. Two-stage network structure.

We present a novel fuzzy two-stage DEA model for calculating fuzzy efficiency score of *DMU*s. Let n *DMU*s as $DMU_j = (\tilde{X}_j, \tilde{Z}_j, \tilde{Y}_j)$, $j = 1, \ldots, n$, as shown in figure 1, any DMU_j with two-stage network structure, there has two sub-systems, i.e., stage 1 and stage 2. The fuzzy input vector $\tilde{X}_j = (\tilde{x}_{1j}, \ldots, \tilde{x}_{mj})^T$ are consumed into stage 1 to product h fuzzy intermediate measure vector $\tilde{Z}_j = (\tilde{z}_{1j}, \ldots, \tilde{z}_{mj})^T$ and then all fuzzy intermediate measures are regarded as the inputs of stage 2 for producing s fuzzy output vector $\tilde{Y}_j = (\tilde{y}_{1j}, \ldots, \tilde{y}_{mj})^T$. Then $\tilde{x}_{ij}$, $i = 1, \ldots, m$, $\tilde{z}_{fj}$, $f = 1, \ldots, f$, and $\tilde{y}_{rj}$, $r = 1, \ldots, s$, represent the i-th, f-th and r-th components of the fuzzy input, intermediate measure and output vector corresponding to DMU_j, respectively with two stage network structure. We consider the unit under evaluation with $DMU_o = (\tilde{X}_o, \tilde{Z}_o, \tilde{Y}_o)$.

Now we obtain the fuzzy two stage DEA model based on the directional distance functions (DDF) for measuring non-radial efficiency of $DMU_o = (X_o, Z_o, Y_o)$. Considering $(g_i^x, g_f^z, g_r^r : i = 1, \ldots, m, f = 1, \ldots, h, r = 1, \ldots, s)$ as a directional vector, shows the frontier projection of DMU_o and decision-makers are more concerned with the changes in inputs and outputs, rather than intermediate element. Then we select the projection direction as $(g_i^x = -x_{io}, g_f^z = 0, g_r^r = y_{ro} : i = 1, \ldots, m, f = 1, \ldots, h, r = 1, \ldots, s)$ if decision maker does not have a clear preference. By selecting this direction, DMU_o would try to increase outputs and reduce inputs as much as possible on the premise that remaining intermediate measure unchanged. We propose envelopment fuzzy two stage DEA model for measuring the technical efficiency score of DMU_o as follows.

$$
\begin{aligned}
IEFF_o^* = \max & \sum_{i=1}^{m} \tilde{\rho}_i + \sum_{r=1}^{s} \tilde{\delta}_r \\
s.t., & \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} \leq \tilde{x}_{io} - \tilde{\rho}_i \tilde{x}_{io}, \; i = 1, \ldots, m, \\
& \sum_{j=1}^{n} \lambda_j \tilde{z}_{fj} \geq \tilde{z}_{fo}, \; f = 1, \ldots, h, \\
& \sum_{j=1}^{n} \mu_j \tilde{z}_{fj} \leq \tilde{z}_{fo}, \; f = 1, \ldots, h, \\
& \sum_{j=1}^{n} \mu_j \tilde{y}_{rj} \geq \tilde{y}_{ro} + \tilde{\delta}_r \tilde{y}_{ro}, \; r = 1, \ldots, s, \\
& \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, \; j = 1, \ldots, n, \\
& \sum_{j=1}^{n} \mu_j = 1, \mu_j \geq 0, \; j = 1, \ldots, n, \\
& \tilde{\rho}_i \in TLF_+^m, \; i = 1, \ldots, m, \\
& \tilde{\delta}_r \in TLF_+^s, \; r = 1, \ldots, s.
\end{aligned}
\tag{1}
$$

$\tilde{\rho}_i \tilde{x}_{io}$ and $\tilde{\delta}_r \tilde{y}_{ro}$ play the same role as the input and output slacks in traditional DEA model. However, the inputs $\tilde{x}_{ij}$, intermediate measures $\tilde{z}_{dj}$ and outputs $\tilde{y}_{rj}$ assumed to be trapezoidal fuzzy numbers belong to $TLF^+(R)$. Then we put

$$
\begin{aligned}
\tilde{x}_{ij} &= \left(x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)}, x_{ij}^{(4)} \right)^T \in TLF_+^m, \\
\tilde{z}_{ij} &= \left(z_{fj}^{(1)}, z_{fj}^{(2)}, z_{fj}^{(3)}, z_{fj}^{(4)} \right)^T \in TLF_+^h, \\
\tilde{y}_{rj} &= \left(y_{rj}^{(1)}, y_{rj}^{(2)}, y_{rj}^{(3)}, y_{rj}^{(4)} \right)^T \in TLF_+^s, \\
\tilde{0} &= (0,0,0,0)^T \in TLF_+^s.
\end{aligned}
$$

The objective function of model (1) is also a trapezoidal fuzzy number, and only the intensity variables λ_j and μ_j, $j = 1, \ldots, n$, are non-negative real numbers.

The proposed model is a Fuzzy DEA model with slacks (FDEA) that simultaneously adjusts inputs and outputs to project inefficient units onto the efficiency frontier. In its general form (Model (1)), the objective function maximizes the sum of two sets of slack variables $\tilde{\rho}_i$ (proportional reduction in inputs) and $\tilde{\delta}_r$ (proportional expansion in outputs). This represents a non-oriented (or hybrid) strategy. Interpreting "Direction" in model (1): the term "direction" refers to the orientation of the efficiency improvement. The three common orientations are: Input-Oriented Model: Seeks to minimize inputs while maintaining current output levels. Output-Oriented model: Aims to maximize outputs without requiring additional inputs. Non-Oriented (or Slack-Based) model: Simultaneously reduces inputs and increases outputs. Model (1) is inherently non-oriented because it incorporates both input contractions ($\tilde{\rho}_i$) and output expansions ($\tilde{\delta}_r$) in the objective function. Assume $\tilde{\rho}^* = (\tilde{\rho}_1^*, \ldots, \tilde{\rho}_m^*)$, $\tilde{\delta}* = (\tilde{\delta}_1^*, \ldots, \tilde{\delta}_r^*)$ and

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$\lambda^* = (\lambda_1^*, \ldots, \lambda_n^*)$, $\mu^* = (\mu_1^*, \ldots, \mu_n^*)$ be a feasible solution of model (1). Model (1) is multi-objective because it simultaneously optimizes multiple fuzzy improvement variables associated with individual inputs and outputs. Each $(\tilde{\rho}_i)$ and $(\tilde{\delta}_r)$ represents a distinct performance criterion. Since the objective function is fuzzy-valued, feasible solutions cannot be totally ordered and optimality is defined via fuzzy Pareto dominance (**Definition 4**). Therefore, the model is inherently multi-objective.

Definition 4. A feasible solution for model (1) as $(\tilde{\rho}^*, \tilde{\delta}^*, \lambda^*, \mu^*)$ is a fuzzy Pareto solution of model (1) if there does not exist a feasible solution as $(\tilde{\rho}, \tilde{\delta}, \lambda, \mu)$ for model (1) such that $\sum_{i=1}^m \tilde{\rho}_i + \sum_{r=1}^s \tilde{\delta}_r \geq \sum_{i=1}^m \tilde{\rho}_i^* + \sum_{r=1}^s \tilde{\delta}_r^*$ and $\sum_{i=1}^m \tilde{\rho}_i + \sum_{r=1}^s \tilde{\delta}_r \neq \sum_{i=1}^m \tilde{\rho}_i^* + \sum_{r=1}^s \tilde{\delta}_r^*$.

Theorem 1. If DMU_o is fuzzy Pareto solution then $IEFF_o^* = \tilde{0}$.

Proof: Suppose $(\tilde{\rho}^*, \tilde{\delta}^*, \lambda^*, \mu^*)$ is a fuzzy Pareto solution of model (1) such that the optimal objective function score from model (1) for this solution is unequal zero, i.e., $EFF_o^* \neq \tilde{0}$. Suppose the whole issue is disrupted. Hence $\tilde{\rho}^* \neq \tilde{0}$. In this case there exists an index $i_o \in \{1, \ldots, m\}$ such that $\tilde{\rho}_{i_o}^* \geq \tilde{0}, \tilde{\rho}_{i_o}^* \neq \tilde{0}$. Now, we define $(\tilde{X}^*, \tilde{Z}^*, \widetilde{ZZ}^*, \tilde{Y}^*) = (\sum_{j=1}^n \lambda_j^* \tilde{X}_j, \sum_{j=1}^n \lambda_j^* \tilde{Z}_j, \sum_{j=1}^n \mu_j^* \tilde{Z}_j, \sum_{j=1}^n \mu_j^* \tilde{Y}_j)$. According to the definition $(\tilde{X}^*, \tilde{Z}^*, \widetilde{ZZ}^*, \tilde{Y}^*)$, we will have $\tilde{x}_{i_o}^* \leq \tilde{x}_{i_o o}, \tilde{x}_{i_o}^* \neq \tilde{x}_{i_o o}$, then $\tilde{X}^* \leq \tilde{X}_o, \tilde{X}^* \neq \tilde{X}_o$. In addition, according to the output and intermediate constraints in model (1), we will have $\tilde{Z}^* \leq \tilde{Z}_o$, $\widetilde{ZZ}^* \geq \widetilde{ZZ}_o$ and $\tilde{Y}^* \geq \tilde{Y}_o$. Then, we conclude that $DMU_o = (\tilde{X}_o, \tilde{Z}_o, \tilde{Y}_o)$ is not a fuzzy Pareto solution and that this is a contradiction. Then, we will have $IEFF_o^* = \tilde{0}$, and the proof is completed. ■

The inequality $\geq$ is the fuzzy order relation defined in section 3 of the paper (component wise comparison of trapezoidal fuzzy numbers). This definition is a direct extension of the classical Pareto optimality to the fuzzy environment. It ensures that no other feasible solution can achieve a strictly greater total fuzzy improvement without being exactly equal in terms of the fuzzy sum. In DEA terms, a fuzzy Pareto solution corresponds to a fuzzy-efficient DMU that is non-dominated in the fuzzy sense: no convex combination of other DMUs can simultaneously reduce all inputs and expand all outputs (in the fuzzy order) while providing a strictly greater total fuzzy improvement. Meaning of "$DMU_o = (\tilde{X}_o, \tilde{Z}_o, \tilde{Y}_o)$ is a fuzzy Pareto solution", this statement means that the optimal solution $(\tilde{\rho}^*, \tilde{\delta}^*, \lambda^*, \mu^*)$ obtained for DMU_o from the FDEA model satisfies the fuzzy Pareto condition above. Hence, DMU_o is fuzzy-efficient in the sense that no other fuzzy combination of DMUs can outperform it in terms of total fuzzy improvement. **Theorem 1** states that if a DMU is efficient, its fuzzy inefficiency measure $IEFF_o^* = \tilde{0}$ implying that the corresponding fuzzy Pareto solution has zero total improvement $(\tilde{\rho}^* = 0, \tilde{\delta}^* = 0)$. This aligns with the classical DEA idea: an efficient DMU has no slack for improvement.

According to the **Definition 4** and **Theorem 1**, $IEFF_o^*$ is the set associated to the fuzzy Pareto solutions of model (1) and this set does not have to be a singleton necessarily. Then, we propose an new model for solving the previous fuzzy two stage DEA model namely model (1) for computing a crisp inefficiency measure for the DMU_o. Then, we obtain model (1) based on the definitions and arithmetical operations of fuzzy numbers in the third section and the order relationships between fuzzy numbers. Then, we can reformulate the inefficiency measure EFF_o^* as follows.

$$
\begin{aligned}
IEFC_o^* = \max \sum_{k=1}^{4}\sum_{i=1}^{m} \rho_i^k + \sum_{k=1}^{4}\sum_{r=1}^{s} \delta_r^k & \\
s.t., \ \sum_{j=1}^n \lambda_j x_{ij}^{(k)} \leq x_{io}^{(k)} - \rho_i^k x_{io}^{(k)}, \ & i = 1, \ldots, m, \ k = 1, \ldots, 4, \\
\sum_{j=1}^n \lambda_j z_{fj}^{(k)} \geq z_{fo}^{(k)}, \ & f = 1, \ldots, h, \ k = 1, \ldots, 4, \\
\sum_{j=1}^n \mu_j z_{fj}^{(k)} \leq z_{fo}^{(k)}, \ & f = 1, \ldots, h, \ k = 1, \ldots, 4, \\
\sum_{j=1}^n \mu_j y_{rj}^{(k)} \geq y_{ro}^{(k)} + \delta_r^k y_{ro}^{(k)}, \ & r = 1, \ldots, s, \ k = 1, \ldots, 4, \\
\sum_{j=1}^n \lambda_j = 1, \lambda_j \geq 0, \ & j = 1, \ldots, n, \\
\sum_{j=1}^n \mu_j = 1, \mu_j \geq 0, \ & j = 1, \ldots, n, \\
\rho_i^k \leq \rho_i^{k+1}, \ & i = 1, \ldots, m, \ k = 1,2,3, \\
\delta_r^k \leq \delta_r^{k+1}, \ & r = 1, \ldots, s, \ k = 1,2,3, \\
\rho_i^k \geq 0, \ & i = 1, \ldots, m, \ k = 1,2,3,4, \\
\delta_r^k \geq 0, \ & r = 1, \ldots, s, \ k = 1,2,3,4.
\end{aligned} \tag{2}
$$

Theorem 2. The model (2) is always feasible.

Proof: Let $\lambda_j = 0, \ j = 1, \ldots, n, \ j \neq o$ and $\lambda_o = 1$, $\rho_i^k = 0, i = 1, \ldots, m, k = 1, \ldots, 4, \delta_r^k = 0, r = 1, \ldots, s, k = 1, \ldots, 4, \mu_j = 0, j = 1, \ldots, n, j \neq o$ and $\mu_o = 1$, We obtain a feasible solution for model (2). ■

Theorem 3. If DMU_o is Pareto solution then$IEFC_o^* = 0$.

Proof: Assume that DMU_o is efficient. According to **Theorem 1**, this implies that $IEFF_o^* = \tilde{0}$. Then $\sum_{i=1}^m \tilde{\rho}_i^* + \sum_{r=1}^s \tilde{\delta}_r^* = \tilde{0}$, for every fuzzy Pareto optimal solution $(\tilde{\rho}^*, \tilde{\delta}^*, \lambda^*, \mu^*)$ of the model (1). Now argue by contradiction. Assume, contrary to the claim, that the optimal value of model (2) is strictly positive, i.e., $IEFC_o^* > 0$, then there exists a feasible solution $(\tilde{\rho}, \tilde{\delta}, \lambda, \mu)$ to model (2) such that $\sum_{k=1}^4 \sum_{i=1}^m \rho_i^k + \sum_{k=1}^4 \sum_{r=1}^s \delta_r^k > 0$. Then $(\tilde{\rho}, \tilde{\delta}, \lambda, \mu)$ is also feasible for model (1), it follows that that $\sum_{i=1}^m \tilde{\rho}_i + \sum_{r=1}^s \tilde{\delta}_r \geq \sum_{i=1}^m \tilde{\rho}_i^* + \sum_{r=1}^s \tilde{\delta}_r^*$ and $\sum_{i=1}^m \tilde{\rho}_i + \sum_{r=1}^s \tilde{\delta}_r \neq \sum_{i=1}^m \tilde{\rho}_i^* + \sum_{r=1}^s \tilde{\delta}_r^*$. Since this solution is also feasible for model (1), it follows that $IEFF_o^* \neq \tilde{0}$ for this solution which contradicts the result established in **Theorem 1**. Hence, the assumption is false and we must have $IEFC_o^* = 0$. ■

Model (2) can be regarded as a combination of two

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fuzzy two stage DDF DEA models, which are used to measure the inefficiency of two stages, respectively. Therefore, the fuzzy efficiencies of the overall system and two stages of model (2) can be defined as follows. Let $(\rho^*, \delta^*, \lambda^*, \mu^*)$ is a fuzzy Pareto solution of model (2), we define fuzzy efficiency of first, second and whole stage as follows.

$$EFC_o^{*1} = \Bigg(\frac{\sum_{i=1}^m (1-\rho_i^{4*})}{m}, \frac{\sum_{i=1}^m (1-\rho_i^{3*})}{m}, \frac{\sum_{i=1}^m (1-\rho_i^{2*})}{m}, \frac{\sum_{i=1}^m (1-\rho_i^{1*})}{m} \Bigg)$$

$$EFC_o^{*2} = \Bigg(\frac{\sum_{r=1}^s (\frac{1}{1+\delta_r^{4*}})}{s}, \frac{\sum_{r=1}^s (\frac{1}{1+\delta_r^{3*}})}{s}, \frac{\sum_{r=1}^s (\frac{1}{1+\delta_r^{2*}})}{s}, \frac{\sum_{r=1}^s (\frac{1}{1+\delta_r^{1*}})}{s} \Bigg)$$

$$EFC_o^* = EFC_o^{*1} * EFC_o^{*2} = \Bigg(\frac{\sum_{i=1}^m (1-\rho_i^{4*})}{m} * \frac{\sum_{r=1}^s (\frac{1}{1+\delta_r^{4*}})}{s}, \frac{\sum_{i=1}^m (1-\rho_i^{3*})}{m} * \frac{\sum_{r=1}^s (\frac{1}{1+\delta_r^{3*}})}{s}, \frac{\sum_{i=1}^m (1-\rho_i^{2*})}{m} * \frac{\sum_{r=1}^s (\frac{1}{1+\delta_r^{2*}})}{s}, \frac{\sum_{i=1}^m (1-\rho_i^{1*})}{m} * \frac{\sum_{r=1}^s (\frac{1}{1+\delta_r^{1*}})}{s} \Bigg).$$

Definition 5. $DMU_o = (\tilde{X}_o, \tilde{Z}_o, \tilde{Y}_o)$ is called fully fuzzy efficient in the first stage, if $EFC_o^{*1} = (1,1,1,1)$ else it is fuzzy inefficient.

Definition 6. $DMU_o = (\tilde{X}_o, \tilde{Z}_o, \tilde{Y}_o)$ is called fully fuzzy efficient in the second stage, if $EFC_o^{*2} = (1,1,1,1)$ else it is fuzzy inefficient.

Definition 7. $DMU_o = (\tilde{X}_o, \tilde{Z}_o, \tilde{Y}_o)$ is called fully fuzzy overall efficient in the, if $EFC_o^* = (1,1,1,1)$ else it is fuzzy inefficient.

5. Application to iranian insurance market

The Iranian insurance industry provides a suitable context for applying the proposed fuzzy two-stage network DEA model due to its structural diversity, competitive dynamics, and data uncertainty. The market consists of both large, established public insurers and smaller private companies, each operating under different resource constraints and strategic priorities. Regulatory requirements, macroeconomic volatility, and regional disparities contribute to variations in cost structures, policy issuance patterns, and service quality. Moreover, data in this sector are often imprecise (particularly in financial statements, customer satisfaction surveys, and claims records) making fuzzy modelling an appropriate tool for capturing inherent vagueness in performance indicators. This setting thus offers a robust tested for evaluating the interpretability and resilience of network DEA models under uncertain information.

5.1 Data collection and pre-processing

The dataset comprises operational and financial records from 40 active insurance companies in a single fiscal year, collected from annual reports, regulatory filings, and company surveys. Pre-processing involved normalization of monetary figures, cross-verification of reported data, and expert-based estimation of missing entries. Fuzzy variables—including operating costs, issued policies, collected premiums, claim settlement rates, net profits, and customer satisfaction—were represented using triangular or trapezoidal membership functions, while crisp variables, such as the number of sales agents, were retained in exact form. This ensured the dataset accurately reflected both measurable and uncertain aspects of performance, enabling realistic efficiency assessment through the proposed model.

An insurance companies can generally be considered as a two-stage network system as shown in figure 1. Each insurance companies are regarded as a two-stage system.

In this evaluation we consider two fuzzy inputs, two fuzzy intermediate measures and fuzzy three outputs.

In the two-stage network structure employed in this study, the first stage corresponds to the processes related to issuing insurance policies and selling insurance products. Two main inputs are considered in this stage:

1. **Number of Agents:** This variable is expressed as a crisp integer value representing the number of active agents within the company's sales network. It reflects the potential human resource capacity for attracting customers and selling insurance policies. From a DEA perspective, a higher number of agents may indicate a greater sales potential, although efficiency ultimately depends on the effective utilization of this capacity.

2. **Operating Cost:** This variable is modelled as a fuzzy number and represents the total expenses associated with sales, marketing, and underwriting support activities, including personnel salaries, office rent, advertising, equipment, and other operational costs. The fuzzy representation is adopted due to the inherent uncertainty in cost data, which may arise from price fluctuations, regional cost differences, and variations in accounting practices. The data are represented using triangular or trapezoidal fuzzy numbers to capture the actual range of cost variations.

In the two-stage network DEA framework, the outputs of the first stage serve as the inputs of the second stage, commonly referred to as intermediate measures. In the context of insurance company evaluation, two key intermediate measures are identified:

1. **Number of Policies:** This variable represents the total number of insurance policies issued during the underwriting/sales stage. In the second stage, these policies serve as the basis for potential claims and post-sale service activities. Each policy may result in one or more claims during its coverage period.

2. **Premiums:** This variable reflects the total monetary amount collected from policyholders for insurance coverage during the observation period. From the perspective of the second stage, premiums constitute a potential financial source for claim payments.

Fuzzy Nature of intermediate measures: In practice, these data are not always recorded with complete precision, especially when part of the information is unavailable or subject to reporting delays and estimations. Therefore, representing these measures as fuzzy numbers (triangular or trapezoidal) is appropriate. This approach captures the uncertainty arising from incomplete or estimated records. Enhances the robustness of efficiency measurement against fluctuations in the data. In the fuzzy network DEA model, these two variables simultaneously act as outputs of stage 1 and inputs of stage 2, thereby establishing the structural linkage between the two main operational phases of the insurance company.

In the two-stage network DEA framework, final outputs represent the overall performance of the organization after both operational stages. In the evaluation of insurance companies, three key final outputs are considered, all modelled as fuzzy variables due to the uncertainty inherent in their measurement. They are as follows.

1. **Claim Settlement Rate:** This variable measures the percentage of claims paid relative to the total approved claims. A higher settlement rate typically indicates greater efficiency and responsiveness in customer service. Since the timing and amount of payments may be subject to delays or changes, this indicator is expressed as a fuzzy percentage.

2. **Net Profit:** Defined as the difference between total revenues and total expenses during the evaluation period, net profit reflects the company's ability to manage costs and generate income. Due to economic volatility, currency fluctuations, and variations in accounting practices, this variable is represented as a fuzzy number to capture the realistic range of profitability.

3. **Customer Satisfaction:** This variable reflects policyholders' satisfaction with the services received, claim handling processes, and quality of interactions with the company. It is usually obtained through surveys and qualitative indicators, which inherently involve uncertainty and subjectivity. Therefore, it is modelled as a fuzzy number (e.g., on a $0-1$ or $0-100$ scale).

These three outputs serve as the ultimate results of the interaction between stage 1 inputs, intermediate measures, and stage 2 resources. Modelling them as fuzzy variables enables the efficiency evaluation to better reflect the real-world conditions of the insurance market and the uncertainties involved.

We show two-stage structure of the insurance companies in figure 2.

Due to the insistence of the central insurance management, we refrained from mentioning the names of these insurance and only displayed the insurance companies (IC) with numbers C01 to C40. We show first two inputs as X1 and X2 and two intermediate measure as Z1 and Z2 and three inputs as Y1, Y2 and Y3 respectively. The data of insurance companies are in the Table 1 and 2.

The two-stage network structure used in this study is designed to reflect the real operational mechanism of insurance companies, which can be naturally divided into policy issuance activities and performance outcomes. The selection of variables follows both the production logic of the insurance industry and established practices in the DEA literature.

In the first stage, which represents policy issuance and sales activities, two inputs are considered. The number of agents is used as a proxy for labor input, as agents play a central role in marketing insurance products and acquiring policyholders. Operating cost captures administrative and marketing expenditures required to support underwriting and sales operations and is widely recognized as a core input in insurance efficiency analysis. The intermediate measures, namely the number of policies and premiums, represent the direct outputs of the first stage and simultaneously act as inputs to the second stage. The number of policies reflects business volume, while premium income measures the monetary value of issued contracts and serves as a key linkage between operational activities and financial performance.

The second stage evaluates how effectively insurance companies transform policies and premium income into final outcomes. Three outputs are considered. The claim settlement rate reflects operational reliability and service quality. Net profit represents overall financial performance and the ultimate economic objective of insurers. Customer satisfaction, which captures policyholders' perceptions of service quality, is inherently subjective and imprecise; therefore, it is appropriately modelled as

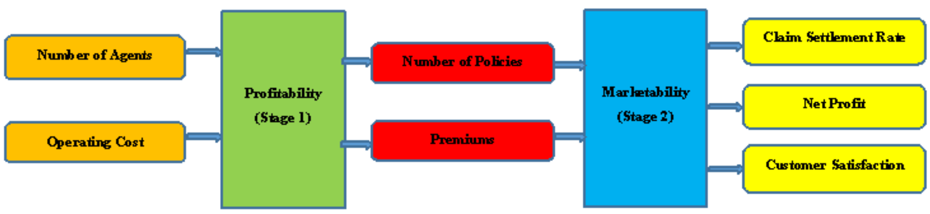

Figure 2. Two-stage network system of each insurance companies.

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Table 1. The fuzzy inputs and intermediates measures of insurance companies.

IC	I1	I2	Z1	Z2
C01	112	(355.18,417.03,443.64)	(1631,1806,2228)	(4416.09,4675.55,5383.21)
C02	445	(761.95,891.33,1025.75)	(2459,2947,3616)	(5708.75,6217.08,6637.47)
C03	280	(1490.88,1677.18,1762.3)	(2551,3014,3534)	(1948.1,2287.01,2457.29)
C04	116	(470.59,503.05,530.81)	(2185,2262,2534)	(1840.76,2027.63,2461.11)
C05	81	(256.46,295.43,329.82)	(1108,1295,1397)	(3013.06,3506.52,3702.77)
C06	30	(54.66,62.02,74.38)	(558,588,630)	(873.88,925.79,1167.61)
C07	131	(345.65,382.44,458.56)	(842,952,1019)	(1692.37,1834.43,2042.04)
C08	476	(1283.34,1462.29,1630.26)	(5359,6638,8289)	(12427.09,13078.48,16111.17)
C09	224	(1054.59,1096.7,1142.73)	(1596,1853,2012)	(2699.26,2858.56,3049.76)
C10	340	(1378.91,1441.78,1491.8)	(4227,4624,5096)	(5601.43,5942.41,7501.16)
C11	468	(2082.98,2405.48,2874.12)	(6091,7019,7899)	(16750.48,20148.85,21193.28)
C12	97	(243.39,259.58,269.48)	(1037,1264,1397)	(1122.09,1315.56,1702.95)
C13	382	(1292.88,1420.55,1593.43)	(5663,5897,7298)	(17376.62,18561.57,22512.13)
C14	109	(271.73,292.59,320.54)	(1379,1633,1860)	(4186.86,4364.82,4541.94)
C15	140	(637.78,684.63,791.28)	(1056,1186,1351)	(1779.36,2177.93,2274.72)
C16	318	(1160.16,1229.31,1289.9)	(6123,6317,6584)	(15550.07,16902.41,21830.08)
C17	353	(989.96,1035.91,1108.67)	(5700,6632,7864)	(12541.87,13634.48,14959.14)
C18	501	(1835.9,2134.9,2390.71)	(5758,5959,6866)	(8830.32,9225.26,11791.37)
C19	423	(771.03,859.23,900.61)	(3735,4105,4578)	(5067.52,5883.85,6473.33)
C20	395	(1941.27,2032.32,2283.86)	(2727,2925,3322)	(9355.52,10087.6,11460.59)
C21	201	(701.26,783.23,838.83)	(1340,1583,1703)	(4265.94,5279.81,5821.21)
C22	286	(976.03,1053.2,1228.94)	(5188,5367,6246)	(10955.85,13435,16473.02)
C23	170	(875.5,981.85,1046.21)	(2042,2291,2592)	(5036.74,5483.04,5914.11)
C24	469	(2062.44,2310.29,2730.8)	(2731,3196,3356)	(6166.08,6951.75,7454.95)
C25	323	(886.08,1021.15,1074.9)	(4743,5115,5302)	(13278.17,15305.93,16198.49)
C26	31	(125.61,133.85,150.56)	(372,430,475)	(1174.83,1396.05,1625.73)
C27	262	(950.72,1018.29,1221.48)	(4656,4870,5022)	(15298.9,16577.13,20748.47)
C28	570	(2908.86,3193.98,3795.86)	(4869,5644,6377)	(10693.87,12817.87,15571.59)
C29	484	(2388.36,2730.81,2988.78)	(7555,9349,11237)	(26634.13,28457.66,35651.35)
C30	68	(309.98,337.19,367.68)	(602,742,801)	(1424.99,1663.58,2041.63)
C31	520	(2008.46,2208.85,2558.49)	(2952,3415,3861)	(7377.57,8272.9,9607.45)
C32	485	(1906.33,2063.59,2445.38)	(2784,3300,3926)	(7576.96,7924.14,9852.26)
C33	199	(1073.83,1129.76,1227.44)	(1691,1872,2085)	(4901.31,5248.08,6260.99)
C34	572	(2090.9,2448.17,2651.25)	(7825,9344,10062)	(19258.7,21934.03,25892.52)
C35	576	(1701.14,1848.08,1995.9)	(8084,9074,10307)	(16266.51,19897.7,22290.96)
C36	253	(711.54,781.34,883.82)	(2444,2570,2962)	(6993.22,7223.78,8616.98)
C37	514	(1296.43,1387.34,1643.73)	(6832,7266,8064)	(5847.77,6810.17,8437.36)
C38	140	(276.18,295.74,324.85)	(1024,1175,1244)	(3493.42,3813.19,4774.48)
C39	494	(1647.45,1779.16,1982.03)	(6034,6945,8332)	(12363.87,12839.36,13863.68)
C40	30	(94.6,107.21,120.27)	(218,231,286)	(241.96,274.76,308.55)

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Table 2. The fuzzy outputs of insurance companies.

IC	Y1	Y2	Y3
C01	(69.2,71.83,73.5)	(0.613,0.685,0.776)	(92.34,97.14,100)
C02	(83.84,87.88,89.68)	(0.41,0.453,0.502)	(53.97,58.96,61.22)
C03	(82.27,85.36,87.69)	(0.143,0.171,0.202)	(62.46,68.06,70.44)
C04	(81.64,84.95,87)	(0.066,0.078,0.098)	(71.8,77.38,80.97)
C05	(92.46,96.28,98.36)	(0.51,0.611,0.718)	(73.51,80.58,84.58)
C06	(82.1,84.9,86.65)	(0.166,0.185,0.209)	(89.22,96.99,100)
C07	(82.61,85.81,88.17)	(0.258,0.292,0.372)	(67.38,71.29,74.38)
C08	(92.25,95.99,98.32)	(2.678,3.143,3.4)	(60.79,65.64,68.13)
C09	(87.02,91.04,93.39)	(0.189,0.232,0.24)	(70.29,77.67,80.61)
C10	(70.46,73.69,75.51)	(0.261,0.271,0.312)	(79.85,85.97,89.75)
C11	(79.87,83.97,85.73)	(1.999,2.45,2.645)	(87.68,93.66,97.7)
C12	(68.98,71.46,73.03)	(0.124,0.136,0.142)	(88.67,96.56,99.49)
C13	(80.59,83.41,85.23)	(1.299,1.555,1.717)	(56.69,62.8,65.46)
C14	(82.6,85.96,88.18)	(0.622,0.673,0.717)	(62.95,65.52,68.23)
C15	(96.85,101.19,100)	(0.344,0.39,0.41)	(55.64,59.27,61.3)
C16	(89.95,92.98,94.99)	(1.674,1.93,2.453)	(55.85,60.95,63.48)
C17	(75.5,79.06,81.11)	(1.992,2.141,2.68)	(79.11,86.3,89.58)
C18	(75.02,78.85,81.08)	(0.641,0.661,0.806)	(69.45,72.04,75.58)
C19	(75.74,79.26,81.31)	(0.768,0.934,0.981)	(86.4,91.85,96.29)
C20	(90.08,94.7,97.5)	(0.049,0.053,0.062)	(83.28,88.51,92.45)
C21	(67.77,70.63,72.16)	(0.465,0.482,0.508)	(81.65,87.45,90.81)
C22	(82.74,85.45,87.76)	(1.79,1.913,2.208)	(82.38,91.07,93.88)
C23	(86.49,89.4,91.62)	(1.143,1.364,1.708)	(61.13,67.75,70.53)
C24	(74.41,76.92,79.16)	(1.004,1.126,1.441)	(61.97,66.12,69.06)
C25	(85.06,88.48,90.39)	(0.53,0.632,0.709)	(68.91,72.89,75.77)
C26	(78.89,83.03,85.34)	(0.136,0.144,0.151)	(79.94,83.36,85.95)
C27	(87.46,91.15,93.12)	(1.074,1.167,1.264)	(80.42,84.7,88.54)
C28	(66.44,69.93,71.94)	(1.553,1.661,1.758)	(84.71,91.22,95.66)
C29	(70.67,73.41,74.95)	(2.379,2.597,2.992)	(82,86.89,90.44)
C30	(84.29,87.11,89.3)	(0.312,0.321,0.396)	(77.9,85.6,88.52)
C31	(79.9,84.09,86.1)	(1.144,1.268,1.589)	(84.17,91.89,95.43)
C32	(69.26,71.74,73.62)	(0.232,0.283,0.356)	(80.63,87.98,90.91)
C33	(70.31,72.69,74.2)	(0.656,0.686,0.815)	(53.79,58.64,61.07)
C34	(90.88,95.29,97.86)	(1.651,1.892,2.357)	(64.89,68.76,71.34)
C35	(70.72,74.02,75.57)	(2.723,2.818,3.541)	(83.4,88.85,91.97)
C36	(94.68,97.72,100)	(0.637,0.781,0.938)	(79.39,87.38,91.53)
C37	(79.62,83.43,85.12)	(1.132,1.263,1.637)	(66.77,71.5,74.37)
C38	(84.52,88.05,90.63)	(0.391,0.423,0.448)	(54.36,59.17,61.81)
C39	(70.96,74.59,76.2)	(1.51,1.636,1.955)	(88.96,92.01,95.02)
C40	(76.64,79.5,81.4)	(0.027,0.033,0.034)	(80.81,89.07,92.3)

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a fuzzy variable. Similarly, uncertainty in claim settlement performance and profit realization justifies the use of fuzzy modelling for all final outputs.

Overall, the selected inputs, intermediate variables, and outputs provide a coherent and realistic representation of insurance companies' internal processes within a two-stage network DEA framework.

5.2 Implementation of the fuzzy two-stage network DEA model

The model evaluates efficiency in two interconnected stages: operational and financial. The first stage assesses operational efficiency using inputs (e.g., sales agents, operating costs) and outputs (e.g., issued policies, collected premiums). The second stage captures financial efficiency, incorporating first-stage outputs and additional financial indicators such as net profits and claim settlement rates. Fuzzy representation of variables accounts for reporting imprecision, providing flexible and realistic performance evaluation. The model also identifies bottlenecks and improvement opportunities within the operational-financial network. The results of model (2) are in the Table 3.

According to Table 3, in the stage 1, insurance companies C04, C06, C13, C14, C16, C17, C22, C25, C26, C27, C29, C34, C35 and C37 are technical fuzzy efficient and the other banks are inefficient in the stage 2, banks B07, B08, B13, B15, B18, B20 and B22 are technical efficient and the other insurance companies are fuzzy inefficient. In the stage 2, insurance companies

Table 3. The technical efficiency scores of insurance companies of model (2).

IC	First stage	Second stage	Overall
C01	(0.8917,0.8917,0.902)	(1,1,1)	(0.8917,0.8917,0.902)
C02	(0.4296,0.4452,0.4742)	(0.7062,0.718,0.7295)	(0.3033,0.3197,0.3459)
C03	(0.424,0.424,0.4294)	(0.6996,0.703,0.703)	(0.2966,0.2981,0.3019)
C04	(1,1,1)	(0.6877,0.6877,0.6886)	(0.6877,0.6877,0.6886)
C05	(0.8322,0.8322,0.8322)	(1,1,1)	(0.8322,0.8322,0.8322)
C06	(1,1,1)	(1,1,1)	(1,1,1)
C07	(0.3446,0.3556,0.3583)	(0.8595,0.8595,0.8644)	(0.2962,0.3057,0.3097)
C08	(0.7989,0.8141,0.8454)	(1,1,1)	(0.7989,0.8141,0.8454)
C09	(0.3451,0.3451,0.3451)	(0.7794,0.7966,0.7966)	(0.269,0.2749,0.2749)
C10	(0.6355,0.6355,0.6355)	(0.6865,0.6865,0.6925)	(0.4362,0.4362,0.4401)
C11	(0.7014,0.7299,0.7459)	(1,1,1)	(0.7014,0.7299,0.7459)
C12	(0.6935,0.6935,0.6935)	(0.7886,0.8037,0.807)	(0.5469,0.5574,0.5597)
C13	(1,1,1)	(0.7889,0.7896,0.7913)	(0.7889,0.7896,0.7913)
C14	(1,1,1)	(0.8297,0.867,0.902)	(0.8297,0.867,0.902)
C15	(0.3343,0.3383,0.3383)	(1,1,1)	(0.3343,0.3383,0.3383)
C16	(1,1,1)	(0.9146,0.9153,0.9175)	(0.9146,0.9153,0.9175)
C17	(1,1,1)	(1,1,1)	(1,1,1)
C18	(0.5664,0.581,0.6077)	(0.7213,0.7213,0.7324)	(0.4086,0.4191,0.4451)
C19	(0.6622,0.6622,0.6783)	(0.9664,0.9999,0.9999)	(0.64,0.6622,0.6782)
C20	(0.3802,0.3802,0.3802)	(1,1,1)	(0.3802,0.3802,0.3802)
C21	(0.4697,0.4697,0.4697)	(0.8109,0.8426,0.8744)	(0.3809,0.3958,0.4107)
C22	(1,1,1)	(1,1,1)	(1,1,1)
C23	(0.5648,0.5648,0.5648)	(1,1,1)	(0.5648,0.5648,0.5648)
C24	(0.2792,0.2896,0.2963)	(0.8601,0.8601,0.8701)	(0.2401,0.2491,0.2578)
C25	(1,1,1)	(0.7503,0.7534,0.7642)	(0.7503,0.7534,0.7642)
C26	(1,1,1)	(1,1,1)	(1,1,1)
C27	(1,1,1)	(0.8914,0.903,0.9371)	(0.8914,0.903,0.9371)
C28	(0.3944,0.4106,0.4171)	(0.9039,0.9265,0.9376)	(0.3565,0.3804,0.391)
C29	(1,1,1)	(0.9445,0.9552,0.9591)	(0.9445,0.9552,0.9591)
C30	(0.4757,0.4757,0.4757)	(1,1,1)	(0.4757,0.4757,0.4757)
C31	(0.301,0.3047,0.3096)	(1,1,1)	(0.301,0.3047,0.3096)
C32	(0.321,0.3334,0.3367)	(0.6897,0.6913,0.6942)	(0.2214,0.2305,0.2337)
C33	(0.422,0.422,0.422)	(0.74,0.751,0.7823)	(0.3123,0.3169,0.3302)
C34	(1,1,1)	(0.9197,0.9204,0.9275)	(0.9197,0.9204,0.9275)
C35	(1,1,1)	(1,1,1)	(1,1,1)
C36	(0.6129,0.6129,0.6129)	(1,1,1)	(0.6129,0.6129,0.6129)
C37	(1,1,1)	(0.9394,0.9401,0.9431)	(0.9394,0.9401,0.9431)
C38	(0.7002,0.7002,0.7002)	(0.7882,0.8236,0.834)	(0.5519,0.5767,0.584)
C39	(0.7219,0.7338,0.7378)	(1,1,1)	(0.7219,0.7338,0.7378)
C40	(0.8092,0.8092,0.8092)	(1,1,1)	(0.8092,0.8092,0.8092)

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C01, C05, C06, C08, C11, C15, C17, C20, C22, C23, C26, C30, C31 C35, C36, C39 and C40 are technical fuzzy efficient and the other companies are inefficient. Insurance companies C06, C17, C22, C26 and C35 are only technical fully fuzzy overall efficient and the other companies are inefficient.

According to the results, if the fuzzy efficiency scores of companies include numbers related to the lower bound of fuzzy numbers close to the number one, then the companies have better performance in terms of performance than companies whose lower bound of fuzzy numbers is far from the number one and have a higher rank in terms of performance. To rank units in terms of fuzzy efficiency scores, we can compare the centre of the corresponding fuzzy numbers with the fuzzy numbers corresponding to the efficiency scores of the companies. For example, company C36 has a higher efficiency score than company C38. To rank efficient units, we can use fuzzy super efficiency DEA models.

The application of the proposed fuzzy two-stage network DEA model to the insurance sector provided valuable insights into the efficiency of companies operating under uncertainty. By modelling the first stage (underwriting/sales) with crisp and fuzzy inputs such as the number of agents and operating costs, and linking them to fuzzy intermediate outputs like the number of issued policies and collected premiums, we were able to capture the internal dynamics of insurance operations. In the second stage (claims and services), the use of fuzzy outputs, claim settlement rate, net profit, and customer satisfaction, allowed a realistic representation of managerial and market uncertainties.

The results highlighted the relative strengths and weaknesses of insurance companies across both stages, revealing that some firms perform well in policy issuance but face inefficiencies in claims handling and customer service. The fuzzy framework proved particularly useful in distinguishing borderline cases, where traditional crisp DEA might classify companies inaccurately. Moreover, the conversion of fuzzy data into a deterministic linear model ensured computational feasibility while preserving the richness of uncertainty representation.

From a managerial perspective, the findings provide senior executives with clear benchmarks and improvement targets under uncertain conditions. The stage-wise decomposition helps managers identify whether inefficiencies arise from sales and underwriting activities or from service and claims operations. This knowledge is critical for designing targeted strategies, such as optimizing operational costs, enhancing claims settlement processes, or improving customer satisfaction initiatives.

Overall, the fuzzy two-stage DEA approach demonstrates its practical relevance for insurance companies in competitive markets, where decision-making must account for vagueness and imprecision. By integrating fuzziness into efficiency analysis, this model enables more reliable performance evaluation and supports managers in making informed, uncertainty-aware strategic decisions.

Model (2) evaluates insurance companies within a two-stage network DEA framework under fuzzy data. The model provides not only the overall network efficiency score but also the decomposition into first-stage efficiency (operational efficiency) and second-stage efficiency (financial/service efficiency). This decomposition enables a deeper understanding of the source of inefficiency. Companies identified as fully efficient in model (2) achieve an efficiency score equal to one (under the fuzzy evaluation framework) and lie on the efficient frontier. These firms simultaneously satisfy: Stage 1 efficiency: Optimal transformation of inputs (number of agents and operating costs) into intermediate outputs (number of policies and premium income). Stage 2 efficiency: Effective conversion of intermediate measures into final outputs (claim settlement rate, net profit, and customer satisfaction). From an economic standpoint, these companies demonstrate balanced resource allocation, productive sales networks, effective underwriting policies, controlled claims expenses, and strong customer relationship management. Because efficiency is evaluated under fuzzy data, these firms also show robustness against uncertainty in performance indicators. Such companies serve as benchmark units, and their input–output structure provides reference targets for inefficient firms.

For inefficient companies, Model (2) reveals whether inefficiency originates from: Pure operational inefficiency (Stage 1): Excessive agents or operating costs relative to generated policies and premiums. Financial/service inefficiency (Stage 2): Inability to transform sufficient premium income into profit, high claim costs, or low customer satisfaction. Network imbalance: Acceptable performance in one stage but weakness in the other, leading to reduced overall efficiency.

The fuzzy modelling framework further indicates that variability in final outputs (especially customer satisfaction and claim settlement performance) contributes to efficiency dispersion, highlighting managerial uncertainty and operational risk. From an economic perspective, inefficient insurance companies can improve performance through targeted strategies depending on the source of inefficiency: If inefficiency arises in Stage 1, firms should improve agent productivity, optimize distribution channels, adopt digital sales platforms, and rationalize operating expenses. If inefficiency arises in Stage 2, firms should enhance underwriting accuracy, strengthen actuarial risk assessment, control claim ratios, and improve customer service processes. If inefficiency is due to inter-stage imbalance, strategic alignment between premium growth and profitability management is required. Importantly, the two-stage structure of model (2) provides managerial insights beyond traditional "black-box" DEA models, since it identifies the exact stage where corrective actions are needed. Therefore, the results of model (2) not only classify insurance companies into efficient and inefficient groups but also provide economically meaningful guidance for performance improvement under uncertainty.

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5.3 A comparative analysis of fuzzy two-stage DEA approaches: Kao and Liu [11], Lozano [55], and the proposed method

This study introduces a new methodological framework for two-stage systems operating under fuzzy conditions, followed by a structured comparison with two influential prior models. Kao and Liu [11] developed a foundational approach using extension principles to estimate fuzzy efficiency scores while maintaining the relational structure between stages. Lozano [55] introduces a method for determining the efficiency levels of individual stages within two-stage structures when information is imprecise or fuzzy. The procedure begins by establishing the potential range (minimum and maximum values) for the overall system's performance across different confidence levels, drawing on previously established frameworks. Following this, the efficiency boundaries for each internal process are derived through a sequential approach.

In this subsection, the efficiency status of individual companies is analysed and compared across the three approaches: the proposed fuzzy two stage DEA model (Table 3), the α-cut based approach of Kao and Liu [11] (Tables 4- 6), and the α-cut based approach of Lozano [55] (Tables 7- 9). The comparison is conducted at the first stage, second stage, and overall system level, empha-

Table 4. The efficiency scores of companies of Kao and Liu [11]'s approach for $\alpha = 0.5$.

IC	First stage		Second stage		Overall	
	Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound
C01	0.7863	0.792	0.8186	0.9103	0.6437	0.7209
C02	0.4528	0.5251	0.3459	0.3807	0.1566	0.1999
C03	0.504	0.504	0.1778	0.1988	0.0896	0.1002
C04	1	1	0.1514	0.1665	0.1514	0.1665
C05	0.8639	0.8988	0.8622	1	0.7449	0.8988
C06	0.983	1	0.9093	1	0.8938	1
C07	0.3218	0.3585	0.7854	0.9224	0.2527	0.3306
C08	0.6963	0.7841	0.8897	1	0.6194	0.7841
C09	0.4389	0.4389	0.3103	0.3375	0.1362	0.1481
C10	0.6655	0.6655	0.1542	0.1674	0.1026	0.1114
C11	0.7161	0.7161	0.661	0.757	0.4733	0.542
C12	0.71	0.7117	0.3464	0.3686	0.246	0.2624
C13	0.8318	0.8364	0.4473	0.5128	0.3721	0.4289
C14	0.9381	1	0.7521	0.8048	0.7056	0.8048
C15	0.4632	0.4632	0.7046	0.7538	0.3264	0.3492
C16	1	1	0.5643	0.6863	0.5643	0.6863
C17	0.9386	0.9936	0.6756	0.7881	0.6342	0.7831
C18	0.6292	0.6292	0.22	0.2456	0.1384	0.1545
C19	0.5398	0.5827	0.6015	0.6751	0.3247	0.3934
C20	0.4043	0.4043	0.1063	0.113	0.043	0.0457
C21	0.4175	0.4195	0.6481	0.6786	0.2706	0.2846
C22	0.943	0.9489	0.6833	0.7605	0.6444	0.7216
C23	0.7343	0.7343	0.8161	1	0.5992	0.7343
C24	0.323	0.323	0.7074	0.8449	0.2285	0.2729
C25	0.9329	1	0.2084	0.2378	0.1944	0.2378
C26	0.767	0.767	0.9547	1	0.7323	0.767
C27	0.9426	0.9494	0.462	0.4998	0.4355	0.4745
C28	0.4715	0.4715	0.5979	0.6359	0.2819	0.2998
C29	0.8928	0.8928	0.5734	0.644	0.5119	0.5749
C30	0.58	0.58	0.892	1	0.5174	0.58
C31	0.313	0.313	0.7572	0.889	0.237	0.2783
C32	0.3356	0.3356	0.2106	0.2499	0.0707	0.0838
C33	0.5083	0.5083	0.6907	0.7645	0.3511	0.3886
C34	0.8482	0.8482	0.3636	0.436	0.3084	0.3699
C35	0.7831	0.8481	0.6057	0.6952	0.4743	0.5896
C36	0.6763	0.7543	0.4265	0.5124	0.2884	0.3865
C37	0.4049	0.4573	0.6421	0.7775	0.26	0.3556
C38	0.9216	1	0.498	0.5313	0.4589	0.5313
C39	0.6259	0.6492	0.4964	0.5638	0.3107	0.366
C40	0.4416	0.4442	0.9269	1	0.4093	0.4442
Average	0.6687	0.6887	0.5661	0.6378	0.3801	0.4413

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sizing both numerical efficiency values and efficiency classification (efficient vs. inefficient).

The approach of Kao and Liu [11] transforms fuzzy data into multiple crisp DEA models using α-cuts, resulting in interval-valued efficiency scores for each α-level. As shown in Tables 4- 6, decreasing α leads to wider efficiency intervals, reflecting higher uncertainty, while increasing α produces narrower bounds but still yields two separate efficiency values (lower and upper bounds) for each stage and the overall system.

In contrast, the proposed approach directly computes triangular fuzzy efficiency scores for each companies and each stage (Table 3). These fuzzy efficiencies preserve the full uncertainty structure of the data without decomposing the problem into multiple deterministic models. A notable observation is that, for all companies, the fuzzy efficiency values obtained by the proposed approach lie within the ranges defined by the α-cut inter-

Table 5. The efficiency scores of companies of Kao and Liu [11]'s approach for $\alpha = 0.8$.

IC	First stage		Second stage		Overall	
	Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound
C01	0.8014	0.8046	0.7285	0.8102	0.5839	0.6519
C02	0.4317	0.5006	0.3158	0.3472	0.1364	0.1738
C03	0.5285	0.5285	0.1563	0.1748	0.0826	0.0924
C04	1	1	0.1387	0.1528	0.1387	0.1528
C05	0.8229	0.8266	0.8631	0.9965	0.7102	0.8237
C06	0.9087	1.0298	0.931	1.0317	0.846	1.0625
C07	0.3497	0.3627	0.7009	0.8135	0.2451	0.295
C08	0.5836	0.6574	0.9111	1.0241	0.5318	0.6733
C09	0.4262	0.4262	0.2948	0.3208	0.1257	0.1367
C10	0.677	0.677	0.1373	0.1491	0.093	0.1009
C11	0.7396	0.7396	0.5799	0.6641	0.4289	0.4912
C12	0.6819	0.6828	0.3235	0.3442	0.2206	0.235
C13	0.7754	0.7784	0.4345	0.4982	0.3369	0.3878
C14	0.9362	1.0172	0.6593	0.7058	0.6173	0.718
C15	0.441	0.441	0.6835	0.7305	0.3014	0.3222
C16	0.9986	1.0007	0.5113	0.6219	0.5106	0.6223
C17	0.8767	0.928	0.6211	0.7244	0.5445	0.6723
C18	0.6208	0.6208	0.2022	0.2258	0.1255	0.1402
C19	0.4877	0.5264	0.5752	0.6452	0.2805	0.3397
C20	0.3853	0.3853	0.1017	0.1082	0.0392	0.0417
C21	0.4044	0.4058	0.6065	0.6351	0.2453	0.2577
C22	0.9808	0.9851	0.5951	0.6623	0.5837	0.6524
C23	0.7005	0.7005	0.8749	1.072	0.6129	0.751
C24	0.3486	0.3486	0.5947	0.7103	0.2073	0.2476
C25	0.8626	0.8839	0.2095	0.2379	0.1807	0.2103
C26	0.7173	0.7173	0.9615	1.0069	0.6897	0.7222
C27	0.937	0.9418	0.4219	0.4564	0.3954	0.4299
C28	0.4878	0.4878	0.5243	0.5577	0.2558	0.272
C29	0.9408	0.9408	0.493	0.5538	0.4639	0.521
C30	0.5615	0.5615	0.9505	1.0509	0.5337	0.59
C31	0.3236	0.3236	0.6644	0.7801	0.215	0.2525
C32	0.347	0.347	0.1845	0.219	0.064	0.076
C33	0.4873	0.4873	0.6576	0.7272	0.3204	0.3544
C34	0.8401	0.8401	0.3326	0.3989	0.2795	0.3351
C35	0.8126	0.8301	0.5411	0.621	0.4397	0.5155
C36	0.5686	0.585	0.4865	0.5772	0.2766	0.3377
C37	0.3603	0.4069	0.6219	0.7502	0.224	0.3053
C38	0.8377	0.909	0.4774	0.5091	0.3999	0.4627
C39	0.6941	0.6964	0.418	0.474	0.2901	0.3301
C40	0.409	0.4108	0.9368	1.0104	0.3832	0.4151
Average	0.6524	0.6686	0.5356	0.6025	0.34899	0.4043

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vals of Kao and Liu [11]. This confirms the consistency and validity of the proposed method while avoiding repeated model solving for different α-levels. Moreover, the proposed approach provides a unique fuzzy efficiency representation, whereas the α-cut method requires separate interpretations for each α-level. From a practical perspective, this significantly improves interpretability and reduces computational burden.

Lozano [55] also employs an α-cut based framework but allows a higher degree of flexibility, which results in considerably wider efficiency intervals, especially for the first stage (Tables 7- 9). In several cases, the upper bounds of efficiencies reach unity for most *DMU*s across all α-levels, reducing the discriminatory power of the model. Compared to this, the proposed approach generates more informative fuzzy efficiency scores, as the spread of the triangular fuzzy numbers reflects realistic uncertainty while maintaining discrimination among

Table 6. The efficiency scores of companies of Kao and Liu [11]'s approach for $\alpha = 0.3$.

IC	First stage		Second stage		Overall	
	Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound
C01	0.7852	0.8039	0.8556	0.9527	0.6718	0.7659
C02	0.4624	0.5362	0.3664	0.4037	0.1694	0.2165
C03	0.4889	0.4889	0.1903	0.2123	0.093	0.1038
C04	1	1	0.1591	0.1744	0.1591	0.1744
C05	0.8912	0.9663	0.8315	0.9714	0.7411	0.9387
C06	0.9764	0.9969	0.8993	0.9821	0.8781	0.979
C07	0.3209	0.368	0.8051	0.9575	0.2583	0.3524
C08	0.7665	0.8584	0.8769	0.9847	0.6721	0.8452
C09	0.4481	0.4481	0.3144	0.3416	0.1409	0.1531
C10	0.655	0.655	0.1619	0.1757	0.1061	0.1151
C11	0.7002	0.7002	0.7225	0.8274	0.5059	0.5793
C12	0.709	0.7141	0.367	0.3906	0.2602	0.2789
C13	0.8521	0.898	0.4462	0.5115	0.3802	0.4594
C14	0.9367	0.9902	0.8134	0.8705	0.7619	0.862
C15	0.4783	0.4783	0.7014	0.7526	0.3355	0.3599
C16	1	1	0.6031	0.7334	0.6031	0.7334
C17	0.9392	0.9914	0.7448	0.8688	0.6995	0.8613
C18	0.6123	0.618	0.2366	0.2645	0.1449	0.1635
C19	0.5781	0.624	0.6126	0.688	0.3542	0.4293
C20	0.4172	0.4172	0.1091	0.1158	0.0455	0.0483
C21	0.4498	0.459	0.6193	0.6483	0.2786	0.2976
C22	0.8822	0.9357	0.7425	0.8263	0.6551	0.7732
C23	0.7571	0.7571	0.7811	0.9571	0.5913	0.7246
C24	0.3255	0.3255	0.7424	0.8947	0.2417	0.2913
C25	0.9279	0.983	0.2292	0.2616	0.2127	0.2572
C26	0.7982	0.7982	0.9475	0.9924	0.7563	0.7921
C27	0.9255	0.9336	0.5001	0.5412	0.4628	0.5052
C28	0.4604	0.4604	0.6533	0.6949	0.3008	0.32
C29	0.8603	0.8603	0.636	0.7143	0.5471	0.6145
C30	0.5926	0.5926	0.8493	0.9605	0.5033	0.5692
C31	0.3059	0.3086	0.8158	0.9592	0.2495	0.296
C32	0.3141	0.3141	0.2326	0.2755	0.0731	0.0865
C33	0.5225	0.5225	0.7086	0.7894	0.3702	0.4125
C34	0.825	0.8318	0.3965	0.4755	0.3271	0.3955
C35	0.792	0.8546	0.6636	0.7615	0.5255	0.6508
C36	0.7307	0.8108	0.4333	0.5231	0.3166	0.4241
C37	0.4416	0.4987	0.6453	0.7814	0.2849	0.3897
C38	0.909	0.9863	0.5454	0.5823	0.4958	0.5743
C39	0.619	0.6481	0.5331	0.6085	0.33	0.3944
C40	0.4638	0.4672	0.9205	0.9932	0.4269	0.464
Average	0.67302	0.69753	0.585315	0.6605025	0.3982525	0.4663025

units. Again, the fuzzy efficiencies reported in Table 1 are fully compatible with the lower and upper bounds reported in Tables 7- 9, indicating that the proposed approach captures the same information as Lozano [55]'s method but in a more compact and analytically coherent form.

Several companies, such as C06, C17, C22, C26, and C35, are identified as fully efficient in all stages under the proposed approach, with fuzzy efficiency scores equal to (1, 1, 1) for both stages and the overall system (Table 3). For these companies, the α-cut based approaches of Kao and Liu [11] and Lozano [55] consistently report efficiency intervals whose upper bounds equal one across all α-levels. For instance, for company C06, Kao and Liu [11] report overall efficiency bounds of [0.8938,1] at $\alpha = 0.5$ and [0.8781,0.979] at $\alpha = 0.3$, while Lozano [55] also reports an upper bound of one for all α-levels. This strong agreement indicates that,

Table 7. The efficiency scores of companies of Lozano [55]'s approach for $\alpha = 0.5$.

IC	First stage		Second stage		Overall	
	Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound
C01	0.669	1	0.4603	0.912	0.4603	0.7209
C02	0.3017	1	0.0942	0.394	0.0942	0.1999
C03	0.1959	1	0.0686	0.4713	0.0686	0.1002
C04	0.5096	1	0.1148	0.2912	0.1148	0.1665
C05	0.6287	1	0.5501	1	0.5501	0.8988
C06	0.8419	1	0.698	1	0.698	1
C07	0.2917	0.692	0.2	0.9297	0.2	0.3306
C08	0.4986	1	0.4222	1	0.4222	0.7841
C09	0.2913	1	0.099	0.4417	0.099	0.1481
C10	0.3726	1	0.0723	0.2531	0.0723	0.1114
C11	0.5845	1	0.3153	0.757	0.3153	0.542
C12	0.3376	1	0.1787	0.6573	0.1787	0.2624
C13	0.6432	1	0.2524	0.5513	0.2524	0.4289
C14	0.6033	1	0.4606	0.961	0.4606	0.8048
C15	0.3015	1	0.2326	0.994	0.2326	0.3492
C16	0.8311	1	0.3827	0.6863	0.3827	0.6863
C17	0.6924	1	0.4056	0.8373	0.4056	0.7831
C18	0.3448	1	0.0954	0.3768	0.0954	0.1545
C19	0.3114	1	0.185	0.8102	0.185	0.3934
C20	0.2926	1	0.0326	0.1249	0.0326	0.0457
C21	0.3074	1	0.1889	0.7629	0.1889	0.2846
C22	0.7837	1	0.4382	0.7681	0.4382	0.7216
C23	0.5993	0.7343	0.5993	1	0.5993	0.7343
C24	0.2534	0.3878	0.1538	0.8901	0.1538	0.2729
C25	0.6528	1	0.1405	0.286	0.1405	0.2378
C26	0.5539	1	0.5539	1	0.5539	0.767
C27	0.7682	1	0.2992	0.5091	0.2992	0.4745
C28	0.3845	1	0.1895	0.6359	0.1895	0.2998
C29	0.7587	1	0.355	0.644	0.355	0.5749
C30	0.414	0.6408	0.414	1	0.414	0.58
C31	0.2576	0.4502	0.1614	0.889	0.1614	0.2783
C32	0.2622	0.5777	0.0497	0.2615	0.0497	0.0838
C33	0.3728	1	0.2407	0.849	0.2407	0.3886
C34	0.6335	1	0.2078	0.482	0.2078	0.3699
C35	0.6421	1	0.3296	0.7219	0.3296	0.5896
C36	0.4237	1	0.2145	0.7105	0.2145	0.3865
C37	0.233	1	0.1646	1	0.1646	0.3556
C38	0.361	1	0.2659	0.902	0.2659	0.5313
C39	0.4706	1	0.2193	0.6479	0.2193	0.366
C40	0.3592	0.5706	0.3592	1	0.3592	0.4442
Average	0.4759	0.9263	0.2716	0.7102	0.2716	0.4413

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for clearly efficient units, all three approaches lead to the same qualitative conclusion, confirming the robustness of the proposed method.

To illustrate the differences and similarities among the three approaches at the company level, the efficiency results of Company C04 are examined in detail for the first stage, second stage, and overall system. According to Table 1, Company C04 is fully efficient in the first stage, with a triangular fuzzy efficiency score equal to (1, 1, 1). However, its second-stage efficiency is significantly lower, with a fuzzy score of approximately (0.6877, 0.6877, 0.6886). As a consequence, the overall system efficiency of C04 is also fuzzy inefficient and equals (0.6877, 0.6877, 0.6886). These results clearly indicate that the inefficiency of C04 originates entirely from the second stage, while the first stage operates on the efficient frontier. Using the α-cut based approach of Kao and Liu [11], the efficiency of C04 is reported

Table 8. The efficiency scores of companies of Lozano [55]'s approach for $\alpha = 0.8$.

IC	First stage		Second stage		Overall	
	Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound
C01	0.7515	1	0.0002	0.7834	0.5249	0.6292
C02	0.3817	1	0.1168	0.3377	0.1168	0.1613
C03	0.2144	1	0.0762	0.4008	0.0762	0.0894
C04	0.5682	1	0.1284	0.2508	0.1284	0.1479
C05	0.7378	0.8739	0.6515	0.9997	0.6515	0.789
C06	0.8853	1	0.8264	1	0.8264	1
C07	0.3333	1	0.2289	0.7893	0.2289	0.2762
C08	0.5501	1	0.5037	1	0.5037	0.6327
C09	0.3227	1	0.1162	0.4002	0.1162	0.1355
C10	0.4162	1	0.0829	0.2218	0.0829	0.0975
C11	0.6928	1	0.3926	0.6489	0.3926	0.4799
C12	0.3793	1	0.1998	0.5747	0.1998	0.2323
C13	0.7288	1	0.3074	0.4834	0.3074	0.3759
C14	0.7049	1	0.5315	0.8295	0.5315	0.6896
C15	0.3522	1	0.2736	0.8604	0.2736	0.319
C16	0.9394	1	0.4613	0.5773	0.4613	0.5773
C17	0.7777	1	0.4757	0.7073	0.4757	0.6188
C18	0.3852	1	0.1106	0.3255	0.1106	0.1326
C19	0.3813	1	0.2389	0.7095	0.2389	0.3241
C20	0.3438	1	0.0362	0.1104	0.0362	0.0409
C21	0.3624	1	0.2174	0.6552	0.2174	0.2536
C22	0.887	1	0.5156	0.659	0.5156	0.6239
C23	0.656	1	0.656	1	0.656	0.7005
C24	0.2932	0.5686	0.1857	0.7402	0.1857	0.2304
C25	0.7434	1	0.1656	0.2505	0.1656	0.2017
C26	0.638	0.9787	0.6256	1	0.6256	0.7139
C27	0.876	1	0.3532	0.4494	0.3532	0.4196
C28	0.4567	1	0.2259	0.5483	0.2259	0.2675
C29	0.881	1	0.4106	0.5303	0.4106	0.4989
C30	0.497	0.7071	0.4762	1	0.4762	0.5615
C31	0.3035	0.4887	0.1924	0.7309	0.1924	0.2365
C32	0.3122	0.6108	0.059	0.2156	0.059	0.072
C33	0.4372	0.6982	0.2827	0.7237	0.2827	0.3382
C34	0.7506	1	0.2514	0.3914	0.2514	0.3131
C35	0.6701	1	0.3803	0.6521	0.3803	0.4772
C36	0.4794	1	0.2552	0.61	0.2552	0.3189
C37	0.2684	1	0.1958	0.8543	0.1958	0.2765
C38	0.4625	1	0.3358	0.792	0.3358	0.446
C39	0.5224	1	0.2577	0.5638	0.2577	0.3132
C40	0.3803	0.5589	0.3803	1	0.3803	0.4103
Average	0.5431	0.9371	0.3045	0.6344	0.3176	0.3856

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as interval values for different α-levels: For all α-levels ($\alpha = 0.3$, 0.5, 0.8), the lower and upper bounds are exactly equal to 1, confirming full efficiency of the first stage. At $\alpha = 0.5$, the efficiency interval is approximately [0.1514,0.1665], and similar low intervals are observed for $\alpha = 0.3$ and $\alpha = 0.8$. The overall efficiency interval coincides with the second-stage interval, remaining well below unity for all α-levels. Thus, the α-cut results of Kao and Liu [11] fully agree with the proposed approach in terms of efficiency classification C04 is efficient in the first stage but inefficient in the second stage and overall system. Under the α-cut based approach of Lozano [55], the following patterns are observed: The upper bound of the efficiency interval equals 1 for all α-levels, while the lower bound remains below 1, indicating potential efficiency but with high uncertainty. The efficiency intervals are relatively wide; for example, at $\alpha = 0.5$, the second-stage efficiency lies ap-

Table 9. The efficiency scores of companies of Lozano [55]'s approach for $\alpha = 0.3$.

IC	First stage		Second stage		Overall	
	Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound
C01	0.6106	1	0.001	0.994	0.4168	0.7883
C02	0.2422	1	0.004	0.4308	0.0769	0.2278
C03	0.1805	1	0.0628	0.5256	0.0628	0.106
C04	0.4879	1	0.1099	0.3216	0.1099	0.1779
C05	0.562	1	0.4902	1	0.4902	0.984
C06	0.7209	1	0.6358	1	0.6358	1
C07	0.2612	1	0.1809	1	0.1809	0.3752
C08	0.459	1	0.3693	1	0.3693	0.8801
C09	0.2717	1	0.089	0.4723	0.089	0.154
C10	0.3444	1	0.0658	0.2765	0.0658	0.1176
C11	0.5222	0.9848	0.2728	0.8401	0.2728	0.5882
C12	0.3084	1	0.1642	0.7202	0.1642	0.2815
C13	0.5863	1	0.2203	0.602	0.2203	0.4728
C14	0.538	1	0.4156	1	0.4156	0.881
C15	0.2717	1	0.2095	1	0.2095	0.3626
C16	0.7601	1.0001	0.3356	0.7685	0.3356	0.7685
C17	0.6872	1.0419	0.3649	0.8712	0.3649	0.9077
C18	0.3179	1	0.086	0.4152	0.086	0.1698
C19	0.2719	1	0.1562	0.8871	0.1562	0.4431
C20	0.264	1	0.0305	0.1348	0.0305	0.0489
C21	0.2729	1	0.1709	0.8259	0.1709	0.302
C22	0.6924	1	0.3904	0.8666	0.3904	0.8013
C23	0.5476	1	0.5476	1	0.5476	0.7571
C24	0.2304	0.5977	0.1363	1	0.1363	0.3056
C25	0.5926	1	0.1249	0.3099	0.1249	0.2672
C26	0.5084	1	0.5084	1	0.5084	0.7982
C27	0.6985	1	0.2663	0.5513	0.2663	0.5135
C28	0.3433	0.9333	0.1692	0.7028	0.1692	0.3236
C29	0.6739	1	0.3173	0.7345	0.3173	0.6319
C30	0.3778	0.6415	0.3778	1	0.3778	0.5926
C31	0.2293	1	0.1429	1	0.1429	0.309
C32	0.2316	1	0.0441	0.2872	0.0441	0.0895
C33	0.3357	1	0.217	0.9437	0.217	0.4258
C34	0.5613	1	0.1821	0.554	0.1821	0.4137
C35	0.5801	1	0.2969	0.8205	0.2969	0.6913
C36	0.386	0.8647	0.1893	0.7792	0.1893	0.4472
C37	0.2122	1	0.1472	1	0.1472	0.4154
C38	0.3139	1	0.2341	0.9673	0.2341	0.5888
C39	0.4348	1	0.1955	0.7112	0.1955	0.4114
C40	0.3425	0.5784	0.3425	1	0.3425	0.4678
Average	0.4308	0.9661	0.2315	0.7579	0.2438	0.4822

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proximately in [0.1148,0.2912]. The overall efficiency intervals mirror the second-stage behaviour and remain far below unity in their lower bounds. Although Lozano [55]'s approach yields wider intervals, the qualitative conclusion remains the same, is inefficient at the second stage and therefore inefficient overall. From the above comparison, the following conclusions can be drawn for Company C04: (1) All three approaches unanimously identify C04 as fully efficient in the first stage and inefficient in the second stage and overall system. (2) The α-cut based approaches provide interval efficiencies that depend on the selected α-level, while the proposed approach delivers a single triangular fuzzy efficiency score. (3) The proposed approach directly reveals the source of inefficiency (second stage) through compact fuzzy numbers, whereas α-cut methods require consulting multiple tables to reach the same insight. (4) Robustness of the proposed approach. The fuzzy efficiency obtained for C04 lies within the bounds implied by the α-cut results, confirming numerical consistency with established methods.

The comparative analysis of Company C06 demonstrates a complete consensus among the three fuzzy DEA approaches in classifying this company as fully efficient across both stages and the overall system. The proposed triangular fuzzy representation yields crisp unity scores of (1, 1, 1) for the first stage, second stage, and overall system, indicating consistent operation on the efficient frontier under uncertainty. The Kao and Liu [11] α-cut based method produces interval-valued efficiencies that consistently include unity across all α-levels, with representative intervals at $\alpha = 0.5$ of [0.983,1] for the first stage and [0.9093,1] for the second stage. Similarly, the Lozano [55] approach generates wider intervals that nevertheless maintain unity as the upper bound across all α-levels for both stages and the overall system. This example highlights that for clearly efficient decision-making units, the proposed methodology successfully reproduces the conclusions of established α-cut based models while offering a more parsimonious representation and computational simplicity, eliminating the need for multiple α-level analyses without compromising the robustness of efficiency classifications.

The comparative analysis of Company C24 reveals consistent agreement among all three fuzzy DEA approaches in identifying this company as inefficient across both stages and the overall system, while the proposed method offers clearer diagnostic insight. According to the proposed triangular fuzzy representation, Company C24 exhibits efficiency scores of (0.2792, 0.2896, 0.2963) in the first stage, (0.8601, 0.8601, 0.8701) in the second stage, and (0.2401, 0.2491, 0.2578) overall, clearly indicating that the severe inefficiency in the first stage dominates and drives the low system performance. The Kao and Liu [11] α-cut approach corroborates these findings, with interval efficiencies at $\alpha = 0.5$ of [0.323,0.323] for the first stage, [0.7074,0.8449] for the second stage, and [0.2285,0.2729] overall, consistently remaining below unity across all α-levels. The Lozano [55] method produces substantially wider intervals, such as [0.1538,0.8901] for the second stage at $\alpha = 0.5$, yet the lower bounds still confirm inefficiency throughout. This comparison demonstrates that while all approaches classify C24 as inefficient, the proposed methodology provides a more compact representation with enhanced discriminatory power, precisely identifying the first stage as the primary source of overall inefficiency without the interpretational ambiguity introduced by excessively wide intervals in alternative α-cut based approaches.

Companies such as C24, C31, and C32 are identified as inefficient in both stages and the overall system under the proposed approach. For instance, C24 has fuzzy efficiencies of approximately (0.2792, 0.2896, 0.2963) in the first stage and (0.8601, 0.8601, 0.8701) in the second stage, resulting in a low overall efficiency. The α-cut based results confirm this conclusion. Kao and Liu [11] report overall efficiency bounds for C24 that remain well below unity across all α-levels (e.g., [0.2285,0.2729] at $\alpha = 0.5$), and Lozano [55] similarly reports low lower bounds, despite large upper bounds caused by the relaxed structure of the model. The main difference is that Lozano [55]'s approach yields very wide efficiency intervals, which may obscure the degree of inefficiency, whereas the proposed approach offers a more precise fuzzy assessment.

For companies such as C08, C11, and C14, the proposed approach reports high fuzzy efficiencies close to unity in one or more stages, while remaining slightly inefficient overall. For example, C14 achieves full efficiency in the first stage and high second-stage efficiency, leading to an overall fuzzy efficiency of approximately (0.8297, 0.867, 0.902). Under the α-cut based approaches, these companies exhibit efficiency intervals that include high upper bounds but lower bounds that remain noticeably below one, particularly at lower α-levels. This demonstrates that the proposed approach captures both optimistic and pessimistic efficiency scenarios within a single fuzzy number, whereas α-cut methods require examining multiple tables to reach the same conclusion.

At the company level, the main similarities and differences among the approaches can be summarized as: (i) For all companies, the fuzzy efficiency values obtained by the proposed approach lie within the lower and upper bounds reported by the α-cut based methods, confirming numerical consistency. (ii) The α-cut based approaches provide interval efficiencies that vary across α-levels, while the proposed approach delivers a single fuzzy efficiency score that preserves uncertainty without repeated model runs. (iii) Lozano [55]'s approach often produces very wide intervals with upper bounds equal to one for many companies, reducing discrimination, whereas the proposed approach maintains clearer differentiation among units. (iv) Stage-wise inefficiencies (e.g., inefficiency driven by the second stage) are more directly observable under the proposed approach.

Overall, the company-level comparison demonstrates that the proposed approach not only reproduces the efficiency classification obtained by existing α-cut based models but also offers a more concise, interpretable, and computationally efficient framework for analysing two-stage systems with fuzzy data.

5.4 Comparison with existing fuzzy network methods

The proposed model is specifically designed for the serial two-stage insurance process (Premium Acquisition to profit and service realization), unlike general fuzzy network DEA models for more complex networks. By preserving trapezoidal fuzzy arithmetic throughout all stages, it offers practical and interpretable results for managers while maintaining mathematical consistency.

We propose a comparison with existing fuzzy network methods as follows.

(1) α-Cut Based Sequential Approach:

Standard methods solve the network DEA at discrete α-cuts and then aggregate results. This can lead to inconsistent λ weights across α-levels, high computational cost, and approximate fuzzy efficiencies. Our integrated approach computes the full trapezoidal fuzzy efficiency in a single optimization, ensuring coherence and adherence to fuzzy arithmetic.

(2) Possibility programming/Chance-constrained approach:

These methods transform fuzzy constraints into crisp equivalents using thresholds or probabilities, often resulting in loss of fuzzy information, subjective parameter choices, and complex non-linear models. In contrast, our model maintains the full fuzzy efficiency outputs, avoids arbitrary thresholds, and retains a linear programming structure.

(3) Fuzzy ranking function approach:

Ranking-based methods convert fuzzy inputs/outputs into crisp scores before optimization. This early defuzzification ignores data uncertainty, introduces bias from the ranking function, and produces only crisp efficiencies. Our model propagates uncertainty through the network, yielding stage-specific fuzzy efficiency scores.

(4) Direct fuzzy arithmetic approaches:

Some advanced models use direct fuzzy arithmetic like ours. However, most either handle single-stage networks or aggregate multi-stage efficiencies multiplicatively, complicating stage-specific interpretation. Our model applies additive aggregation in a two-stage network with clear fuzzy slack decomposition for each stage.

Table 10 summarizes the advantages of our model compared to these existing methods in solving fuzzy network DEA model.

5.5 Managerial implications

The practical utility of the proposed fuzzy two-stage DEA model extends beyond theoretical efficiency measurement, offering actionable strategic guidance for insurance management. The key managerial implications derived from the empirical analysis are summarized as follows.

First, the model provides diagnostic precision by decomposing overall inefficiency into its stage-specific sources (Premium Acquisition vs. Profit & Service Realization). This allows managers to prioritize investments and interventions exactly where they are needed, for example, enhancing claims processing capability rather than expanding sales channels, enabling targeted resource reallocation.

Second, the model yields fuzzy improvement targets (e.g., reduce administrative costs by $4-7\%$, or increase claims settlement ratio by $10-15\%$) rather than single-point estimates. These interval-based benchmarks acknowledge operational uncertainty and support flexible, realistic goal setting in volatile market environments.

Third, the approach facilitates robust peer benchmarking under uncertainty. Insurers can identify best-practice peers for each sub-process, allowing for focused organizational learning and adoption of stage-specific strategies rather than generic imitation.

Finally, the fuzzy efficiency intervals provide insight into performance robustness. A *DMU* with a narrow, high-efficiency range exhibits stable performance, while one with a wide, lower range signals vulnerability and a need for strategic realignment. This supports risk-aware strategic planning and resilience building.

In essence, the model equips insurance managers with a structured, uncertainty-sensitive framework to enhance decision-making, optimize processes, and sustain competitive advantage in a dynamic industry landscape.

Table 10. Comparative analysis of solution methods for fuzzy network DEA models.

Method / Feature	α-Cut Sequential	Possibilistic Programming	Ranking Function (Defuzzify First)	Our Integrated Fuzzy LP
Preserves Full Fuzzy Output	Approximate only	No (interval/crisp)	No (crisp only)	Yes (Trapezoidal)
Model Consistency	Poor	Good	Good (but on defuzzified data)	Excellent
Computational Burden	High	Very High	Low	Moderate (Single LP)
Network Linkage Handling	Fragile	Complex	Simple (but after defuzzification)	Coherent & Integrated
Managerial Interpretability	Medium	Low (due to thresholds)	High (but loses uncertainty)	High (with uncertainty)

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6. Conclusions and future research directions

This study introduced a novel non-radial two-stage network DEA model based on the directional distance function for evaluating the efficiency of insurance companies under fuzzy data conditions. By representing inputs, intermediate products, and outputs as trapezoidal or triangular fuzzy numbers and transforming the fuzzy model into an equivalent deterministic linear form, the proposed framework ensures both computational tractability and practical feasibility. The model was applied to a sample of insurance companies operating in a competitive market. In the first stage, crisp inputs such as the number of sales agents and fuzzy operational costs were transformed into fuzzy intermediates, number of policies issued and collected premium income. These intermediates were then used in the second stage to produce fuzzy final outputs, including claim settlement rate, net profit, and customer satisfaction. The results provided fuzzy efficiency scores for each stage and the overall system, revealing diverse patterns of strengths and weaknesses across companies.

For senior insurance executives, the model offers a valuable diagnostic tool to detect inefficiencies at specific operational stages and set realistic improvement targets under uncertainty. Managers can use these insights to optimize sales networks, improve claims processing efficiency, and enhance customer satisfaction, leading to sustainable competitive advantages.

The main limitations of the study are as follows. First, the two-stage DEA model assumes a sequential structure and fully observable linkages between stages, which may not capture all operational complexities in insurance companies. Second, the fuzzy data, particularly customer satisfaction and claim settlement performance, are based on subjective assessments and may involve imprecision. Third, financial indicators such as net profit and premiums may be affected by reporting delays or estimation errors. Finally, the analysis is limited to the available sample, so the results may not be fully generalizable. These limitations suggest avenues for future research, including incorporating stochastic uncertainty and expanding the dataset.

In the current formulation of model (1), we have adopted a two-stage network structure but focused solely on the efficiency improvement of initial inputs and final outputs without explicitly optimizing intermediate measures between the two stages. This design was intentional for the reasons: The primary aim of this paper is to evaluate the overall system efficiency and to provide managerial insights into how inputs can be reduced and final outputs can be increased without restructuring the internal process flow. The intermediate measures in our application represent linking activities that are often fixed or non-discretionary in the short term. Modeling focus on system-wide Improvement: By allowing only input reduction and output expansion in the objective, we emphasize global optimization rather than sub process reallocation. Introducing improvement in intermediate measures could shift the focus toward internal reallocation rather than whole-system performance enhancement, which is beyond the scope of this study.

The company-level comparison conducted in the case study demonstrates that the proposed approach not only replicates the efficiency classifications of existing α-cut based models but also provides a more concise, interpretable, and computationally efficient framework for analyzing two-stage systems under fuzzy uncertainty.

Future research could explore several extensions, potential extensions include developing multi-stage or dynamic fuzzy DEA models, integrating stochastic elements to handle both randomness and vagueness, and applying the approach to longitudinal datasets to monitor performance trends over time. Additionally, combining the method with multi-criteria decision-making (MCDM) techniques could incorporate strategic qualitative factors into the evaluation. Also, potential directions include: (1) extending the model to a multi-stage or more complex network structure to capture additional operational interactions within insurance companies; (2) incorporating stochastic or probabilistic uncertainty alongside fuzzy data to better represent real-world risk; (3) expanding the dataset to include more insurance companies or cross-country comparisons for greater generalizability; and (4) considering additional performance indicators, such as digital service efficiency or customer retention, to enhance the comprehensiveness of the efficiency evaluation. These directions aim to provide both methodological improvements and practical managerial insights for future studies. In conclusion, the proposed fuzzy two-stage network DEA model provides a robust, realistic, and managerially relevant framework for performance evaluation in uncertain environments, empowering insurance companies to make data-driven decisions and achieve operational excellence.

Authors contributions
All authors contributed equally to the conception, design, execution, and writing of this work. All authors read and approved the final manuscript.

Availability of data and materials
The authors declare that the data supporting the findings of this study are available within the paper.

Conflict of interests
The authors assert that they do not have any identifiable conflicting financial interests or personal relationships that might be perceived to influence the work presented in this paper.

References

1. Kamanda M and Sibindi AB. Determinants of Financial Performance on Insurance Companies: Empirical Evidence Using Kenyan Data. Risk and Financial Management 2021; 14(12):566. DOI: 10.3390/jrfm14120566

2. Mergoni A, Emrouznejad A, and Witte KD. Fifty Years of Data Envelopment Analysis. European Journal of Operational Research 2025 :1–25. DOI: 10.1016/j.ejor.2024.12.049
3. Zhao T, Pei R, and Pan J. The evolution and determinants of Chinese property insurance companies' profitability: A DEA-based perspective. Journal of Management Science and Engineering 2021; 6:449–66. DOI: 10.1016/j.jmse.2021.09.005
4. Charnes A, Cooper WW, and Rhodes E. Measuring the efficiency of decision making units. European Journal of Operational Research 1978; 2(6):429–44. DOI: 10.1016/0377-2217(79)90229-7
5. Färe R and Grosskopf S. Network DEA. Socio-economic planning Science 2000; 34(1):35–49. DOI: 10.1016/S0038-0121(99)00012-9
6. Kao C and Hwang SN. Efficiency decomposition in two-stage data envelopment analysis: An application to non-life insurance companies in Taiwan. European Journal of Operational Research 2008; 185(1):418–29. DOI: 10.1016/j.ejor.2006.11.041
7. Kao C. Maximum slacks-based measure of efficiency in network data envelopment analysis: A case of garment manufacturing. Omega 2024; 123:102989. DOI: 10.1016/j.omega.2023.102989
8. Ostovan S, Mozaffari MR, Jamshidi A, and Gerami J. Evaluation of Two-Stage Networks Based on Average Efficiency Using DEA and DEA-R with Fuzzy Data. International Journal of Fuzzy Systems 2020; 22:1665–78. DOI: 10.1007/s40815-020-00896-9
9. Gerami J, Mozaffari MR, Wanke P, and Tan Y. Fuzzy cost, revenue efficiency assessment and target setting in fuzzy DEA: a fuzzy directional distance function approach. Journal of Modelling in Management 2024; 19(1):240–87. DOI: 10.1108/JM2-05-2022-0121
10. Gerami J, Mozaffari MR, Wanke P, and Tan Y. Fully fuzzy DEA: a novel additive slacks-based measure model. Soft Computing 2023. DOI: 10.1007/s00500-023-08247-8
11. Kao C and Liu ST. Efficiencies of two-stage systems with fuzzy data. Fuzzy Sets and Systems 2011; 176:20–35. DOI: 10.1016/j.fss.2011.02.017
12. Emrouznejad A, Tavana M, and Hatami-Marbini A. The state of the art in fuzzy data envelopment analysis. Studies in Fuzziness and Soft Computing 2014; 309:1–45. DOI: 10.1007/978-3-319-04711-7_1
13. Arana-Jimenez N, Sanchez-Gill MC, and Lozano SE. Efficiency assessment and target setting using a fully fuzzy DEA approach. International Journal of Fuzzy Systems 2020; 22:1056–72. DOI: 10.1007/s40815-020-00821-0A
14. Hatami-Marbini A, Tavana M, and Ebrahimi A. A fuzzy fuzzified data envelopment analysis model. International Journal of Information and Decision Sciences 2011; 3(3):252–64. DOI: 10.1504/IJIDS.2011.041586
15. Emrouznejad A and Yang GL. A survey and analysis of the first 40 years of scholarly literature in DEA: 1978 – 2016. Socio-Economic Planning Sciences 2018; 61:4–8. DOI: 10.1016/j.seps.2017.01.008
16. Lu WM, Wang WK, and Kweh QL. Intellectual capital and performance in the Chinese life insurance industry. Omega 2014; 42(1):65–74. DOI: 10.1016/j.omega.2013.03.002
17. Chen Y, Cook WD, Li N, and Zhu J. Additive efficiency decomposition in two stage DEA. European Journal of Operational Research 2009; 96:1170–6. DOI: 10.1016/j.ejor.2008.05.011
18. Chrysafis KA, Papadopoulou GC, and Theotokas IN. Measuring financial performance through operating business efficiency in the global cruise industry: A fuzzy benchmarking study on the 'big three'. Tourism Management 2024; 100:104830. DOI: 10.1016/j.tourman.2023.104830
19. Soltanzadeh E and Omrani H. Dynamic network data envelopment analysis model with fuzzy inputs and outputs: An application for Iranian Airlines. Applied Soft Computing 2021; 63:268–88. DOI: 10.1016/j.asoc.2021.107648
20. Zhou X, Pedrycz W, Kuang Y, and Zhang Z. Type-2 fuzzy multiobjective DEA model: An application to sustainable supplier evaluation. Applied Soft Computing 2016; 46:424–40. DOI: 10.1016/j.asoc.2016.04.038
21. Zhou Y, Yan C, and Wang X. Higher-order moment fuzzy portfolio selection model based on neural networks: Integrating behavioral finance and complex system analysis. Chaos, Solitons & Fractals 2025; 201:117195. DOI: 10.1016/j.chaos.2025.117195
22. Mozaffari MR, Mohammadi S, Wanke PF, and Correa HL. Towards greener petrochemical production: Two-stage network data envelopment analysis in a fully fuzzy environment in the presence of undesirable outputs. Expert Systems With Applications 2021; 164:113903. DOI: 10.1016/j.eswa.2020.113903
23. Mahapatra BS, Ghosh D, Pamucar D, and Mahapatra GS. Dynamic group decision-making for enterprise resource planning selection using two-tuples Pythagorean fuzzy MOORA approach. Expert Systems with Applications 2025; 263:125675. DOI: 10.1016/j.eswa.2024.125675

24. Peykani P, Mohammadi E, and Emrouznejad A. An Adjustable Fuzzy Chance-Constrained Network DEA Approach with Application to Ranking Investment Firms. Expert Systems with Applications 2021; 166:113938. DOI: 10.1016/j.eswa.2020.113938

25. Peykani P, Sargolzaei M, Hamidzadeh F, Seyed-Esmaeili FS, and Takaloo A. Possibilistic Network DEA Approach for Performance Evaluation of Two-Stage Decision Making Units Under Uncertainty. Analytical Decision Making and Data Envelopment Analysis: Advances and Challenges 2024 :59–79. DOI: 10.1007/978-981-97-6972-8_3

26. Peykani P, Mahmoodirad A, and Amirteimoori A. A novel adjustable intuitionistic fuzzy framework for two-stage data envelopment analysis: An application in the banking sector. Information Sciences 2025; 718:122372. DOI: 10.1016/j.ins.2025.122372

27. Saati S, Memariani A, and Jahanshahloo GR. Efficiency analysis and ranking of DMUs with fuzzy data. Fuzzy Optimization and Decision Making 2002; 1:255–67. DOI: 10.1023/A:1019648512614

28. Guo P and Tanaka H. Fuzzy DEA: a perceptual evaluation method. Fuzzy Sets and Systems 2001; 119(1):149–60. DOI: 10.1016/S0165-0114(99)00106-2

29. Lertworasirikul S, Fang SC, Joines JA, and Nuttle HLW. Fuzzy data envelopment analysis (DEA): a possibility approach. Fuzzy Sets and Systems 2003; 139(2):379–94. DOI: 10.1016/S0165-0114(02)00484-0

30. Tavana M, Khanjani-Shiraz R, Hatami-Marbini A, Agrell PJ, and Paryab K. Fuzzy stochastic data envelopment analysis with application to base realignment and closure (BRAC). Expert System with Applications 2012; 39:12247–59. DOI: 10.1016/j.eswa.2012.04.049

31. Mehdizadeh S, Amirteimoori A, Charles V, Behzadi MH, and Kordrostami S. Measuring the Efficiency of Two-Stage Network Processes: A Satisficing DEA Approach. Journal of the Operational Research Society 2021; 72(2):354–66. DOI: 10.1080/01605682.2019.1654933

32. Pourbabagol H, Amiri M, Taghavifard MT, and Hanafizadeh P. A New Fuzzy DEA Network Based on Possibility and Necessity Measures for Agile Supply Chain Performance Evaluation: A Case Study. Expert Systems with Applications 2023. DOI: 10.1016/j.eswa.2023.119552

33. Amirteimoori A, Allahviranloo T, Zadmirzaei M, and Hasanzadeh F. On the environmental performance analysis: A combined fuzzy data envelopment analysis and artificial intelligence algorithms. Expert Systems with Applications 2023; 224:119953. DOI: 10.1016/j.eswa.2023.119953

34. Amirteimoori A, Allahviranloo T, and Mousavi SMF. Returns-to-scale and scale economies of two-stage production processes using a fully fuzzy range-adjusted measure model with strong complementary slackness conditions. Expert Systems with Applications 2024; 262:125606. DOI: 10.1016/j.eswa.2024.125606

35. Hamidzadeh F, Pishvaee MS, and Zarrinpoor N. A Novel Two-Stage Network Data Envelopment Analysis Model for Kidney Allocation Problem under Medical and Logistical Uncertainty: A Real Case Study. Management Science 2024; 27:555–79. DOI: 10.1007/s10729-024-09683-6

36. Shojaie SE, Sadjadi SJ, and Tavakkoli-Moghaddam R. Malmquist Productivity Index under Network Structure and Negative Data: An Application to Banking Industry. Journal of System Management 2024; 10(3):107–18. Available from: https://sanad.iau.ir/Journal/sjsm/Article/918428

37. Rezakhanlou M and Mirzapour S. A dual-channel multi-objective green supply chain network design considering pricing and transportation mode choice under fuzzy uncertainty. Environment, Development and Sustainability 2025; (27):10341–72. DOI: 10.1007/s10668-023-04312-8

38. Garrido A, Cárdenas-Barrón LE, Buitrago OY, and Pomar L Álvarez. A Multi-Echelon globalized Agro-Industrial supply chain under conditions of Uncertainty: A Two-Stage Fuzzy-Possibilistic Mixed-Integer linear programming Model. Expert Systems with Applications 2025; 270:126569. DOI: 10.1016/j.eswa.2025.126569

39. Huang L and Chen L. Fuzzy random multi-objective optimization using a novel mixed fuzzy random inverse DEA model in input-output production. Journal of Computational and Applied Mathematics 2025; 470:116717. DOI: 10.1016/j.cam.2025.116717

40. Sahil MA, Tiwari A, and Lohani QMD. Two-stage type-2 fuzzy parabolic double frontier data envelopment analysis. Engineering Applications of Artificial Intelligence 2025; 144:110154. DOI: 10.1016/j.engappai.2025.110154

41. Yan S, Xu Y, Liu J, and Cabrerizo FG. A fuzzy group decision-making method integrating cooperative game theory and quantum cognition for dynamic trust network. Expert Systems with Applications 2026; 307:131045. DOI: 10.1016/j.eswa.2025.131045

42. Jaloudi MM. The efficiency of Jordan insurance companies and its determinants using DEA, slacks, and logit models. Journal of Asian Business and Economic Studies 2019; 26(1):153–66. DOI: 10.1108/JABES-06-2018-0022

43. Koc O, Baser F, and Selcuk-Kestel AS. Clustering based fuzzy classification with a noise cluster in detecting fraud in insurance. Applied Soft Computing 2024; 167:112430. DOI: 10.1016/j.asoc.2024.112430

44. Polyakov K and Polyakova M. Determinants of the quality of financial management of insurance companies. Procedia Computer Science 2024; 242:1125–32. DOI: 10.1016/j.procs.2024.08.185

45. Smętek K, Strzelecka A, and Zawadzka D. Examples of the application of the Dynamic Financial Analysis (DFA) method to assess the financial situation and solvency of insurance companies. Procedia Computer Science 2024; 246:4787–95. DOI: 10.1016/j.procs.2024.09.344

46. Liu P, Sun H, and Xu H. Performance evaluation of Chinese and foreign property insurance companies considering negative data: Based on the dynamic two-stage IBP-SBM model. Socio-Economic Planning Sciences 2025; 98:102169. DOI: 10.1016/j.seps.2025.102169

47. Özçalici M, Pamucar D, Gurler HE, Karyağdi NG, and Kaya A. A robust hybrid MCDM framework with emphasis on decision stability intervals: Performance evaluation of global insurance brokers using fuzzy LBWA and modified ARTASI. Applied Soft Computing 2026; 190:114557. DOI: 10.1016/j.asoc.2026.114557

48. Ho SJ and Hsu HH. The effect of microinsurance on the insurance market: evidence from Taiwan. The Geneva Papers on Risk and Insurance - Issues and Practice 2021; 46(1):130–45. DOI: 10.1057/s41288-020-00175-6

49. Jia-Ying J, Zhou LJ, Shen W, Li X, Huizhen J, and Wentsao P. Performance evaluation and analysis of Chinese insurance companies under uncertain financial markets. Heliyon 2023; 9(11):e21487. DOI: 10.1016/j.heliyon.2023.e21487

50. Esmaeili FSS and Mohammadi E. Z-Number Network Data Envelopment Analysis Approach: A Case Study on the Iranian Insurance Industry. Plos One 2024; 19(7):e0306876. DOI: 10.1371/journal.pone.0306876

51. Nasseri SH and Khatir MA. Fuzzy Stochastic Undesirable Two-Stage Data Envelopment Analysis Models with Application to Banking Industry. Journal of Intelligent & Fuzzy Systems 2019; 37(5):7047–57. DOI: 10.3233/JIFS-181684

52. Nosrat A, Sanei M, Payan A, Hosseinzadeh-Lotfi F, and Razavyan S. Using Credibility Theory to Evaluate the Fuzzy Two-Stage DEA; Sensitivity and Stability Analysis. Journal of Intelligent & Fuzzy Systems 2019; 37(4):5777–96. DOI: 10.3233/JIFS-181519

53. Ohene-Asare K, Asare JKA, and Turkson C. Dynamic cost productivity and economies of scale of Ghanaian insurers. The Geneva Papers on Risk and Insurance - Issues and Practice 2019; 44:148–77. DOI: 10.1057/s41288-018-0111-6

54. Zimmermann HJ. Introduction to Fuzzy Sets. Fuzzy Set Theory-and Its Applications, Springer, Dordrecht 2001. DOI: 10.1007/978-94-010-0646-0-1

55. Lozano S. Process efficiency of two-stage systems with fuzzy data. Fuzzy Sets and Systems 2014; 243:36–49. DOI: 10.1016/j.fss.2013.05.012