Volume 6, Issue 1, 062501 (1-7)

Fuzzy Optimization and Modeling Journal (FOMJ)

https://doi.org/10.57647/j.fomj.2025.16559

Optimizing bipolar fuzzy fractional programming problems using the variable neighborhood search algorithm

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Original Research

Received:
14 November 2024
Revised:
25 December 2024
Accepted:
19 January 2025
Published online:
23 April 2025

Abstract:

In this paper, we propose a new variable neighbourhood search algorithm for solving a bipolar fuzzy number fractional problem (FBVNS). The proposed algorithm utilises a modified version of Kerre's inequality, a tool employed to compare bipolar fuzzy numbers. The utilisation of the FBVNS algorithm enables the direct resolution of the bipolar fuzzy number fractional problem, obviating the necessity for conversion to a crisp number representation. The FBVNS algorithm is further enhanced by the introduction of a novel local search method, which is informed by ascending directions derived from the resolution of four distinct crisp mathematical programming problems. Finally, the efficacy of the proposed algorithm is demonstrated through its implementation on a range of problems.

Keywords: Fractional linear optimization; Bipolar fuzzy numbers; Modified Kerre's inequality; VNS algorithm

1. Introduction

Zhang [1] introduced the concept of bipolar-valued fuzzy sets in 1994 as an extension of fuzzy sets, expanding the membership degree range from $[0, 1]$ to $[-1, 0] \times [0, 1]$. These sets are also known as bipolar fuzzy sets, YinYang fuzzy sets, or YinYang bipolar fuzzy sets [1]. The theory combines polarity and fuzziness into a unified model, providing a basis for various applications like cognitive modeling and multi-agent decision analysis [2]. In a human decision making, there is a bipolar judgmental thinking on a negative side and a positive side; for instance, see [3]. In bipolar information, two types of information (as positive and negative) must be distinguished [4, 5]. Positive information is given by observation or experimentation. But, negative information represents impossibility. This domain has recently invoked several interesting research areas such as psychology [6], image processing [7], human reasoning [8] and graph theory [9]. Zhang [1] initiated the concept of bipolar fuzzy set as a generalization of fuzzy set. He defined bipolar fuzzy set as an extension of fuzzy set whose range of membership degree is $[-1, 1]$. Akram [9, 10] used the concept of bipolar fuzzy set in graph theory. Broumand [11] introduced the concept of bipolar-valued fuzzy sub-algebras of BCK/BCI-algebras and investigated some of their useful properties. Zhou and Li [3] presented the concepts of bipolar fuzzy h-ideals and normal bipolar fuzzy h-ideals. Then, they investigated characterizations of bipolar fuzzy h-ideals by means of positive t-cut, negative s-cut, homomorphism and equivalence relation. Some other works on bipolar fuzzy sets can be found in [12–14].

A type of fractional bipolar fuzzy number linear programming problem (FBFNLPP) can be described as follows:

$$\text{(FBFNLPP)} \quad \begin{aligned} &\max \frac{\tilde{a}x+\tilde{b}}{\tilde{c}x+\tilde{d}} \quad (1)\\ &s.t.\\ &Ax \leq h.\\ &x \geq 0. \end{aligned}$$

where, $\tilde{a} \in F(\mathbb{R}^n)$, $\tilde{c} \in F(\mathbb{R}^n)$, $\tilde{b} \in F(\mathbb{R})$, $\tilde{d} \in F(\mathbb{R})$, $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$ and $h \in \mathbb{R}^m$, with $F(\mathbb{R})$ being the set of all bipolar fuzzy numbers.

For solving (1), we propose a new fuzzy VNS algorithm with its local search intended to find a feasible solution by using increasing directions leading to the value of the objective function being bigger than the current solution. In our proposed algorithm, we compare the fuzzy value of the objective function using the modified Kerre's inequality. By using Kerre's inequality, we can solve the problem (1) directly without changing it to the crisp problem [15–17]. The rest of our research is structured in the following manner. In section 2, we provide some necessary definitions, properties of bipolar fuzzy ordering and basic aspects of modified Kerre's method. In section 3, we discuss about our proposed model. We present our VNS algorithm for solving BFFNLPP in section 4. Then, the numerical results are presented in section 5. We conclude in section 6.

2. Preliminaries

Here, we give some necessary definitions and new results on bipolar fuzzy set theory.

Definition 1. [1] Let X be a nonempty set. A bipolar fuzzy set $\tilde{A}$ in X is an object having the form, $\tilde{A} = \{(x, \mu^P_{\tilde{A}}(x), \mu^N_{\tilde{A}}(x))|x \in X\}$, where $\mu^P_{\tilde{A}}(x) : X \to [0,1]$ and $\mu^N_{\tilde{A}}(x) : X \to [-1,0]$.

Definition 2. [13] Let $\tilde{A} = (x, \mu^P_{\tilde{A}}(x), \mu^N_{\tilde{A}}(x))$ be a bipolar-valued fuzzy set and $(s,t) \in [-1,0] \times [0,1]$. The sets $A^P_t = \{x \in X|\mu^P_{\tilde{A}}(x) \geq t\}$ and $A^N_s = \{x \in X|\mu^N_{\tilde{A}}(x) \leq s\}$ are receptively called the positive t-cut of $\tilde{A}$ and the negative s-cut of $\tilde{A}$. For every $k \in [0,1]$, the set $A_k = A^P_k \cap A^N_{-k}$ is called the k-cut of $\tilde{A}$. We now define a bipolar triangular fuzzy number.

Definition 3. [13] A bipolar triangular fuzzy number is defined as a quadruple $\tilde{A} = (a^L, a^P, a^N, a^R)$ with positive and negative membership functions $\mu^P_{\tilde{A}}(x)$ and $\mu^N_{\tilde{A}}(x)$ as follows:

$$\mu^P_{\tilde{A}}(x) = \begin{cases} \frac{x-a^L}{a^P-a^L}, & a^L \leq x < a^P,\\ \frac{x-a^R}{a^P-a^R}, & a^P < x \leq a^R,\\ 0, & \text{otherwise}, \end{cases} \quad (2)$$

$$\mu^N_{\tilde{A}}(x) = \begin{cases} \frac{-(x-a^L)}{a^N-a^L}, & a^L \leq x < a^N,\\ \frac{-(x-a^R)}{a^N-a^R}, & a^N < x \leq a^R,\\ 0, & \text{otherwise}, \end{cases} \quad (3)$$

where $\mu^P_{\tilde{A}}(x)$ and $\mu^N_{\tilde{A}}(x)$ are respectively the membership functions of positive and negative polars (see figure 1).

Proposition 4. [18] Let $\tilde{a} = (a^L, a^P, a^N, a^R)$ and $\tilde{b} = (b^L, b^P, b^N, b^R)$ be two bipolar fuzzy numbers. We have the following results:

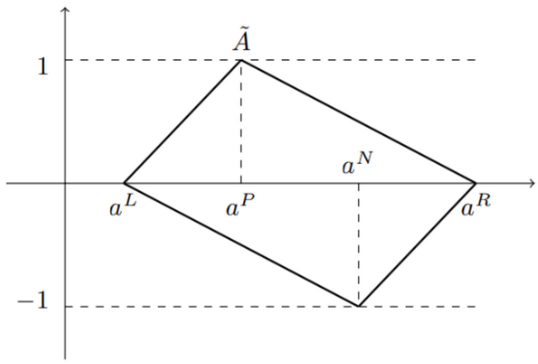

Figure 1. Triangular bipolar fuzzy number $a^P = a^N$.

$$\begin{cases} x > 0,\ x \in \mathbb{R},\ x\tilde{a} = (xa^L,\ xa^P,\ xa^N,\ xa^R)\\ x < 0,\ x \in \mathbb{R},\ x\tilde{a} = (xa^R,\ xa^N,\ xa^P,\ xa^L)\\ \tilde{a}+\tilde{b} = (a^L+b^L,\ a^P+b^P,\ a^N+b^N,\ a^R+b^R). \end{cases}$$

Remark 1. We denote the set of all bipolar triangular fuzzy numbers by $F(\mathbb{R})$.

Definition 4. [19] The fuzzy max between two fuzzy numbers $\tilde{M}$ and $\tilde{N}$ is:

$$\max(\tilde{M}, \tilde{N})(z) = sup\{\min(\tilde{M}(x), \tilde{N}(y))|\max(x,y) = z\}. \quad (4)$$

Definition 5. [19] The Hamming distance between two fuzzy numbers $\tilde{M}$ and $\tilde{N}$ is:

$$d(\tilde{M}, \tilde{N}) = \int_{-\infty}^{\infty} |\tilde{M}(x) - \tilde{N}(x)|dx. \quad (5)$$

2.1 Modified Kerre's inequality for comparison bipolar fuzzy numbers

Kerre's inequality is recognized as a highly efficient method for assessing fuzzy number comparisons [20]. Ghanbari et al. [18] introduced a modified version of Kerre's inequality and established straightforward formulas for comparing bipolar fuzzy triangular numbers (see formulas for comparing single polar fuzzy triangular numbers in [15]), as stated in the following theorem.

Remark 2. [18] Suppose $\tilde{M} = (m^L, m^P, m^N, m^R)$ and $\tilde{N} = (n^L, n^P, n^N, n^R)$ are two arbitrary bipolar LR fuzzy numbers and let:

$$\begin{aligned} r^b(\tilde{M}, \tilde{N}) =& (d(\tilde{M}, \tilde{O}^P) - d(\tilde{N}, \tilde{O}^P)) +\\ & (d(\tilde{M}, \tilde{O}^N) - d(\tilde{N}, \tilde{O}^N)). \end{aligned} \quad (6)$$

where,

$$\begin{aligned} \tilde{O}^P =& m\tilde{a}x^P(\tilde{M}, \tilde{N})(z) =\\ & \sup\{\min\left(\tilde{M}(x), \tilde{N}(y)\right)|\max(x,y) = z\},\\ \tilde{O}^N =& m\tilde{a}x^P(\tilde{M}, \tilde{N})(z) =\\ & \inf\{\max\left(\tilde{M}(x), \tilde{N}(y)\right)|\max(x,y) = z\}. \end{aligned} \quad (7)$$

$$\text{if } r^P(\tilde{M}, \tilde{N}) \geq 0 \text{ then } \tilde{M} \leq \tilde{N}, \text{ else } \tilde{M} > \tilde{N}.$$

Theorem 1. [18] Let $\tilde{M}=(m^L,m^P,m^N,m^R)$ and $\tilde{N}=(n^L,n^P,n^N,n^R)$ be two bipolar triangular fuzzy numbers. Then the following assertions hold:
1. If $m^R \leq n^L$, then,

$$r^b(\tilde{M},\tilde{N})=2(\frac{m^R-m^L}{2}+\frac{n^R-n^L}{2}). \tag{8}$$

2. If $m^R<n^P$ and $m^N=n^N$, where $\bar{y}^P=\tilde{M}^P_R(\bar{x}^P)=\tilde{N}^P_L(\bar{x}^P)$, with $\bar{x}^P$ as defined by (11), then

$$\begin{aligned}r^b(\tilde{M},\tilde{N})=&\frac{(n^R-n^L)}{2}+\frac{(m^R-m^L)}{2}-\bar{y}^P(m^R-n^L)+\\&\frac{(n^R+n^L)}{2}-\frac{(m^R+m^L)}{2}.\end{aligned} \tag{9}$$

3. If $m^R<n^P$ and $m^N<n^N$, where $\bar{y}^P=\tilde{M}^P_R(\bar{x}^P)=\tilde{N}^P_L(\bar{x}^P)$, with $\bar{x}^P$ as defined by (11) and $\bar{y}^N=\tilde{M}^N_R(\bar{x}^N)=\tilde{N}^N_L(\bar{x}^N)$. with $\bar{x}^N$ as defined by (12), then

$$\begin{aligned}r^b(\tilde{M},\tilde{N})=&\frac{(n^R-n^L)}{2}+\frac{(m^R-m^L)}{2}-\bar{y}^P(m^R-n^L)+\\&\frac{(n^R+n^L)}{2}-\frac{(m^R+m^L)}{2}+\bar{y}^N(m^R-n^L).\end{aligned} \tag{10}$$

where, $\bar{x}^P$ is the length of the intersection point of M_R and N_L for the positive polar, and $\bar{x}^N$ is the length of the intersection point of M_R and N_L for the negative polar.

$$\bar{x}^P=\frac{m^Rn^P-n^Lm^P}{n^P-n^L-m^P+m^R}, \tag{11}$$

$$\bar{x}^N=\frac{m^Rn^N-n^Lm^N}{n^N-n^L-m^N+m^R}. \tag{12}$$

So, we have:

$$\bar{y}^P=\frac{m^R-n^N}{n^P-n^L-m^P+m^R}, \tag{13}$$

$$\bar{y}^N=\frac{n^L-m^R}{n^N-n^L-m^N+m^R}. \tag{14}$$

Remark 2. Throughout our work, $<$, $>$, and $=$ on fuzzy numbers are defined based on our modified Kerre's inequality for bipolar fuzzy numbers, and so we show them by $<^{bK}$, $>^{bK}$, $=^{bK}$ respectively [15].

3. Problem statement

We consider the following bipolar fuzzy fractional programming problem:

$$\begin{aligned}&&&\max \frac{\tilde{a}^Tx+\tilde{\alpha}}{\tilde{c}^Tx+\beta}\\(\text{BFFNLPP})&&&s.t.,\\&&&Ax\leq b.\\&&&x\geq 0.\end{aligned} \tag{15}$$

where, $\tilde{a}\in F(\mathbb{R}^n)$, $\tilde{\alpha}\in F(\mathbb{R})$, $c\in\mathbb{R}^n$, $\beta\in\mathbb{R}$, $\tilde{a}$ and $\tilde{\alpha}$ LR bipolar fuzzy numbers and $c^Tx+\beta>0$ for any feasible solution x. Let $z=\frac{1}{c^Tx+\beta}$ and $y=zx$. Inspiring from [21], the problem (15) can be converted to the following problem:

$$\begin{aligned}&\max \tilde{a}^Ty+\tilde{\alpha}z,\\&s.t.\\&Ay-bz\leq 0,\\&c^Ty+\beta z=1,\\&y,z\geq 0.\end{aligned} \tag{16}$$

We can rewrite the problem (16) to the vector form as follows:

$$\begin{aligned}&\max[\tilde{a},\tilde{\alpha}]\begin{bmatrix}y\\z\end{bmatrix},\\&s.t.,\\&[A,-b^T]\begin{bmatrix}y\\z\end{bmatrix}\leq 0,\\&[c^T,\beta]\begin{bmatrix}y\\z\end{bmatrix}=1,\\&[y,z]\geq 0.\end{aligned} \tag{17}$$

If we suppose, $X=\begin{bmatrix}y\\z\end{bmatrix}$, $\tilde{D}=[\tilde{a},\tilde{\alpha}]$, $B=[A,-b^T]$ and $H=[c^T,\beta]$. Then we have,

$$\begin{aligned}&\max \tilde{f}(X)=\tilde{D}X,\\&s.t.,\\&BX\leq 0,\\&X\geq 0.\end{aligned} \tag{18}$$

By solving the problem (18), we can find the solution of the problem (15).

Example 1. Consider the following programming problem:

$$\max \frac{(-6,-4,4,10)x_1+(-14,-4,3,5)x_2+(-22,-10,1,3)}{x_1+x_2+3} \tag{19}$$

$$\begin{aligned}&s.t.,\\&4x_1+x_2\leq 10,\\&x_1+x_2\leq 5,\\&x_1,x_2\geq 0.\end{aligned}$$

Let, $z=\frac{1}{x_1+x_2+3}$ and $y=zx$. So problem (19) converts to the following problem:

$$\max(-6,-4,4,10)y_1+(-14,-4,3,5)y_2+(-22,0,5,12)z, \tag{20}$$

$$\begin{aligned}&s.t.,\\&4y_1+y_2-10z\leq 0,\\&y_1+y_2-5z\leq 0,\\&y_1+y_2+3z=1,\\&y_1,y_2,z\geq 0.\end{aligned}$$

Finally, we can rewrite the problem (20) to the following problem:

$$\max\left((-6,-4,4,10),(-14,-4,3,5),(-22,0,5,12)\right)\begin{pmatrix} y_1 \\ y_2 \\ z \end{pmatrix}, \quad (21)$$

$s.t.,$

$$[1,1,3]\begin{pmatrix} y_1 \\ y_2 \\ z \end{pmatrix}=1,$$

$$\begin{pmatrix} 4 & 1 & -10 \\ 1 & 1 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ z \end{pmatrix}\leq 0,$$

$$y_1,y_2,z\geq 0.$$

In the next section we propose VNS algorithm for solving the problem (18).

4. The proposed fuzzy bipolar VNS

The VNS method employs a systematic rotation of neighborhood structures to effectively avoid local optima. It begins with an initial solution and progresses through three key stages - Shaking, Local Search, and Neighborhood Change - until a specified stopping condition is reached. To implement VNS successfully, certain parameters must be established: a method for generating the initial solution, a perturbation strategy (Shaking) to break free from local optima, a selection of neighborhood structures, and a plan for altering the neighborhood.

The subsequent sections will elaborate on these parameters for the proposed VNS algorithms[13]. We explore the last neighborhood structure kmax, the search goes back to the first neighborhood. This process continues until a stop condition is reached. Some successful examples of application of VNS can also be found in [8, 22, 23]. Based on the definition of the objective function in the (FFNLP), the value of the objective function for each feasible solution is a fuzzy number. Thus, to compare the solutions in the proposed algorithm (e.g., Step 1-2-3), we use modified Kerre's inequality in Theorem 1. By using our modified Kerre's inequality, the fuzzy optimization problem is solved directly without changing it to a crisp problem.

Details of the steps involved in algorithm 1 are described below.

(1) Neighboring structure [15]: We define neighboring structure as follows:

$$\|x'-x_0\|<t\varepsilon, \quad (22)$$

For $t=1,\cdots,t_{\max}$, where x' is in the t-th neighborhood of x_0 and ε is an arbitrary parameter.

(2) Selection of a point in the neighborhood of x_0: Let x' be a point in the neighborhood of x_0. Then, x' is to satisfy in the following conditions [15]:

1. $(t-1)\varepsilon<\|x'-x_0\|\leq t\varepsilon$,
2. $Bx'\leq 0$,
3. $Hx'=1$,
4. $x'\geq 0$.

(3) Local search [15]: Let x_k represent a feasible solution during iteration k, we want to find a feasible solution with larger value of the objective function with respect to x_k.

$$\tilde{D}^T(x_k+s_k)=((D^{L^T}(x_k+s_k),D^{P^T}(x_k+s_k),$$
$$D^{N^T}(x_k+s_k),D^{R^T}(x_k+s_k))$$
$$\tilde{D}^T x_k=(D^{L^T}x_k,D^{P^T}x_k,D^{N^T}x_k,D^{R^T}x_k)$$

Since the given results for the bipolar modified Kerre's method depend on the mean value of fuzzy numbers, to find s_k we consider the following two cases:

Case 1: $D^{P^T}x_k\leq D^{P^T}(x_k+s_k),D^{N^T}x_k\leq D^{N^T}(x_k+s_k)$,

Case 2: $D^{P^T}(x_k+s_k)\leq D^{P^T}x_k,D^{N^T}(x_k+s_k)\leq D^{N^T}x_k$,

Case 1: Let $D^{P^T}x_k\leq D^{P^T}(x_k+s_k),D^{N^T}x_k\leq D^{N^T}(x_k+s_k)$

Algorithm 1. BFVNS for solving FFNLPP

1. Input Neighboring structures $Nt(t=1,2,\cdots,t_{\max})$, an initial feasible solution x_0 and maxiter (maximum number of iterations).
2. **While do** $k<$ maxiter:
3. $t=1$
4. **if** $t<t_{\max}$ **then**:
5. Select one point in the neighborhood of x_0 and name it $x_1(x_1\in N_t(x_0))$.
6. Apply local search on x_1 and name the new point as x_2.
7. **if** $\tilde{f}(x_2)\leq^{bK}\tilde{f}(x_0)$ **then**:
8. let $t=t+1$ and go to 4.
9. **else:**
10. go to 14
11. **endif**
12. **enfif**
13. $x_0=x_2$
14. $k=k+1$
15. **end while**
16. **Output**: Return $x*$

based on Note 2, we must have $r(\tilde{D}^T x_k, \tilde{D}^T(x_k+s_k)) \geq 0$, and thus we propose problem (P_1) as follows:

$$
(P_1)\quad
\begin{aligned}
&\max\ r(\tilde{D}^T(x_k).\tilde{D}^T(x_k+s_k)), \qquad (23)\\
&s.t.\\
&D^{P^T} s_k \geq 0.\\
&D^{N^T} s_k \geq 0.\\
&Bs_k \leq -Bx_k.\\
&Hs_k = 1 - Hx_k.\\
&-s_k \leq x_k.
\end{aligned}
$$

(i) Let $\tilde{D}^T(x_k)^P_R$ and $(\tilde{D}^T(x_k+s_k))^P_L$ have an intersection point and also $\tilde{D}^T(x_k)^N_R$ and $(\tilde{D}^T(x_k+s_k))^N_L$ have an intersection point. In this case, we have $D^{P^T} x_k < D^{P^T}(x_k+s_k)$, $D^{N^T} x_k < D^{N^T}(x_k+s_k)$ and $D^{L^T}(x_k+s_k) \leq D^{R^T} x_k$ and (23) turn into $(P_1{}')$ as follows:

$$
(P_1{}')\quad
\begin{aligned}
&\max\ 2(D^R-D^L)^T x_k + (D^R-D^L)^T s_k \qquad (24)\\
&-\frac{((D^R-D^L)^T x_k - D^{L^T} s_k)^2}{(D^P-D^L)^T s_k + (D^R-D^L)^T x_k}\\
&-\frac{((D^L-D^R)^T x_k + D^{L^T} s_k)^2}{(D^N-D^L)^T s_k + (D^R-D^L)^T x_k},\\
&s.t.,\\
&-D^{P^T} s_k \leq 0,\\
&-D^{N^T} s_k \leq 0,\\
&D^{L^T} s_k \leq (D^R-D^L)^T x_k,\\
&Bs_k \leq -Bx_k,\\
&Hs_k = 1 - Hx_k,\\
&-s_k \leq x_k.
\end{aligned}
$$

Note that $(P_1{}')$ is a quadratic fractional programming problem. By using the method proposed in [24] we solve problem $(P_1{}')$.

(ii) $D^{R^T} x_k \leq D^{L^T}(x_k+s_k)$, that means two numbers do not have any intersection point (see (8)). Nowusing (8) we have the following program:

$$
(P_2{}')\quad
\begin{aligned}
&\max\ (D^R-D^L)^T s_k, \qquad (25)\\
&s.t.,\\
&-D^{P^T} s_k \leq 0,\\
&-D^{N^T} s_k \leq 0,\\
&-D^{L^T} s_k \leq (D^L-D^R)^T x_k,\\
&Bs_k \leq -Bx_k,\\
&Hs_k = 1 - Hx_k,\\
&-s_k \leq x_k.
\end{aligned}
$$

Case1: Let $D^{P^T} x_k \geq D^{P^T}(x_k+s_k)$ and $D^{N^T} x_k \geq D^{N^T}(x_k+s_k)$ in this case if $r(\tilde{D}^T(x_k+s_k), \tilde{D}^T x_k) \leq 0$ then $\tilde{D}^T(x_k+s_k) >^{bK} \tilde{D}^T x_k$. So, we propose the following problem:

$$
(P_3)\quad
\begin{aligned}
&\min\ r(\tilde{D}^T(x_k+s_k), \tilde{D}^T x_k), \qquad (26)\\
&s.t.\\
&D^{P^T} s_k \leq 0,\\
&D^{N^T} s_k \leq 0,\\
&Bs_k \leq -Bx_k,\\
&Hs_k = 1 - Hx_k,\\
&-s_k \leq x_k.
\end{aligned}
$$

(i) Let $(\tilde{D}^T(x_k+s_k))^P_R$ and $(\tilde{D}^T x_k)^P_L$ have an intersection point and $(\tilde{D}^T(x_k+s_k))^N_R$ and $(\tilde{D}^T x_k)^N_L$ have an intersection point. In this case, we have $D^{P^T} x_k > D^{P^T}(x_k+s_k), D^{N^T} x_k > D^{N^T}(x_k+s_k)$ and $D^{L^T} x_k \leq D^{R^T}(x_k+s_k)$. The problem (26) turns into $(P_3{}')$. As follow:

$$
(P_3{}')\quad
\begin{aligned}
&\min\ 2(D^R-D^L)^T x_k + (D^R-D^L)^T s_k \qquad (27)\\
&-\frac{((D^R-D^L)^T x_k + D^{R^T} s_k)^2}{(D^R-D^L)x_k + (D^R-D^P)^T s_k}\\
&-\frac{((D^R-D^L)^T x_k + D^{R^T} s_k)^2}{(D^R-D^L)^T x_k + (D^R-D^N)^T s_k},\\
&s.t.\\
&D^{P^T} s_k \leq 0,\\
&D^{N^T} s_k \leq 0,\\
&-D^{R^T} s_k \leq (D^R-D^L)^T x_k,\\
&Bs_k \leq -Bx_k,\\
&Hs_k = 1 - Hx_k,\\
&-s_k \leq x_k.
\end{aligned}
$$

By using a method that is proposed in [24] (Theorem and section 3) we solve problem (P_3).

(ii) $D^{R^T}(x_k+s_k) \leq D^{L^T} x_k$, that means two numbers do not have any intersection point (see (8). Since $D^{R^T}(x_k+s_k) < D^{L^T} x_k$ and $\tilde{D}^T(x_k+s_k)$ is located completely on the left side of $\tilde{D}^T x_k$ and according to (8) $r(\tilde{D}^T(x_k+s_k).\tilde{D}^T(x_k)) > 0$, so there is no increasing direction.

(4) Stopping condition [15]: We stop the proposed FVNS algorithm when reach maxiter successive iterations without improvement.

Finally, to find s_k, we need to solve for $P_1{}'$, $P_2{}'$ and $P_3{}'$. It is clear that $P_1{}'$, $P_2{}'$ and $P_3{}'$ can be solved in parallel. After solving these problems, the best s_k is chosen. That is, the one given by $P_1{}'$, $P_2{}'$ and $P_3{}'$ that causes the largest increase in the objective function.

Example 2. Suppose the fuzzy fractional linear programming problem as follows:

$$
\begin{aligned}
&\max\ \tilde{f}(x) = \qquad (28)\\
&\frac{(1,7,10,23)x_1 + (-10,0,14,30)x_2 + (-20,5,30,42)x_3 + (-10,3,14,25)}{64x_1 + 3x_2 + 12x_3 + 82},\\
&s.t.\\
&86x_1 + 11x_2 + 86x_3 \leq 280,\\
&73x_1 + 90x_2 + 17x_3 \leq 343,\\
&x_1, x_2, x_3 \geq 0.
\end{aligned}
$$

Table 1. Objective function values obtained by Algorithm 1 on the examples.

dim	Obj(init)	Obj (Al1)	r(Al1,)
10 × 15	[-0.1601, 0.0598, 0.0576, 0.1653]	[-0.7218, 0.2820, 0.5585, 0.9982]	0.7691
10 × 20	[-0.0924, 0.0247, 0.0411, 0.0909]	[-0.1820, 0.0615, 0.0469, 0.1614]	0.019
15 × 45	[-0.1574, 0.467, 0.0392, 0.1401]	[-0.0882, 0.0212, 0.0307, 0.1321]	-0.0296
20 × 50	[-0.1344, 0.0435, 0.0443, 0.1337]	[-0.1279, 0.0560, 0.0571, 0.1367]	0.0337
25 × 75	[-0.0905, 0.0203, 0.0163, 0.1058]	[-0.1455, 0.0505, 0.0478, 0.1513]	0.0453
25 × 200	[-0.1542, 0.0510, 0.0505, 0.1542]	[-0.1773, 0.0669, 0.0632, 0.2066]	0.0567
40 × 300	[-0.1502, 0.0497, 0.0499, 0.1495]	[-0.1566, 0.0576, 0.0650, 0.1911]	0.0572
300 × 600	[-0.1539, 0.0494, 0.0491, 0.1528]	[-0.1503, 0.0520, 0.0540, 0.1673]	0.0255

with x^0 as the starting point (which can be generated randomly in the interval [1, 5]), using algorithm 1, after one iteration we obtain:

$$\begin{aligned}
&x^0 = [2,2,1], \qquad (29)\\
&\tilde{f}(x^0) = [-0.2105, 0.0965, 0.4035, 0.7588],\\
&x^* = [0, 3.2752, 2.8369],\\
&\tilde{f}(x^*) = [-0.7904, 0.1365, 1.1517, 1.9259],\\
&r(\tilde{f}(x^0), \tilde{f}(x^*)) = 1.1307.
\end{aligned}$$

According to Remark 2, we can conclude that $\tilde{f}(x^*) >^{bk} \tilde{f}(x^0)$.

5. Numerical results

In this section, we demonstrate the effectiveness of the algorithm 1 we have developed. To evaluate its performance, we generated numerous test problems for the FFLPP. Problems were created using a random generation process in the MATLAB 2017 programming environment, executed on a notebook equipped with an Intel(R) Core(TM) i5-3210M CPU running at 2.5 GHz, with 4.00 GB of RAM. To introduce fuzzy random coefficients into the objective function, we initiated the process by generating three numbers within the range of [0, 100]. Subsequently, these numbers were sorted in ascending order and employed as the components of the fuzzy number.

In Table 1, the dim column shows the dimensions of the test problems, the column entitled Obj (init) shows the value of the objective function corresponding to the initial solution of algorithm 1, the column entitled Obj (Al1) shows the values of the objective functions obtained by algorithm 1 and the column entitled r(Al1) shows the comparison of the value of objective function of initial solution corresponding to algorithm 1 with the ones obtained by algorithm 1 using Theorem 1. According to the obtained results shown in column labeled as r(Al1), it is observed (based on Note 2) that the algorithm 1 can improve the initial solution. In Table 2, we compare the result of algorithm 1 with the method of Ghanbari et al. [12].

In Table 2, the column entitle dim shows the dimensions of the test problems, the column entitled Obj (Gh) shows the value of the objective function corresponding to the ranking function is defined in [12], the column entitled r(Al1, Gh) shows the comparison of the value of objective function of algorithm 1 and the method is introduced in [12]. According to the obtain results in column labeled as r(Gh, Al1), it is observed that the objective function of algorithm 1 is more high than the one due to other method on all test problems.

6. Conclusion

In this paper, we considered a fraction bipolar fuzzy linear programming problem, with the assumption that the parameters of the objective function are fuzzy numbers. We

Table 2. Objective function values obtained by various methods on the examples.

dim	Obj(Gh,)	r(Gh, Al1)
10 × 15	[-0.1521, 0.0283, 0.0345, 0.1210]	0.8256
10 × 20	[-0.1730, 0.0102, 0.0302, 0.0712]	0.1395
15 × 45	[-0.2012, 0.312, 0.0263, 0.1230]	0.0012
20 × 50	[-0.2315, 0.0105, 0.0236, 0.1030]	0.2046
25 × 75	[-0.1812, 0.0111, 0.0121, 0.0819]	0.1695
25 × 200	[-0.2312, 0.0312, 0.0411, 0.1040]	0.2087
40 × 300	[-0.1801, 0.0311, 0.0401, 0.1302]	0.1316
300 × 600	[-0.1812, 0.0402, 0.0481, 0.1812]	0.0342

presented a new model, which we have termed FBFNLPP. The paper then goes on to propose an FBVNS algorithm based on modified Kerre's inequality for solving FBFNLPP. The efficacy of the proposed algorithm was demonstrated through the resolution of a series of test cases, characterised by triangular fuzzy coefficients. The analysis revealed that the direction introduced by Kerre's inequality for the comparison of bipolar fuzzy numbers can be enhanced through the utilisation of the proposed algorithm, thereby leading to significant improvements in the initial solution.

Authors contributions
Authors have contributed equally in preparing and writing the manuscript.

Availability of data and materials
The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflict of interests
The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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