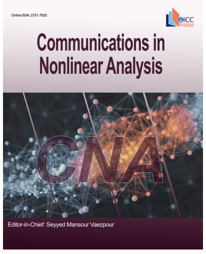

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Research Article

On distinguishing genuine from spurious chaos in planar singular and nonsmooth systems: A diagnostic approach

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Abstract:

We present a rigorous reassessment of chaotic behavior in two-dimensional autonomous systems with singular or nonsmooth dynamics. For the Cummings-Dixon-Kaus (CDK) model, we show that blow-up regularization restores smoothness and renders the hypotheses of the Poincaré-Bendixson theorem applicable, thereby excluding chaotic attractors away from the singular set. We prove topological equivalence between the original and regularized flows on annular domains, ensuring that no spurious invariant sets are introduced by desingularization. In contrast, for a nonsmooth system with a $|x|$ term, we recompute the entire period-doubling cascade, obtain a seven-term sequence of bifurcation values converging to Feigenbaum's constant, and confirm robust chaos through positive Lyapunov exponents, broadband spectra, and fractal dimension estimates. As a main outcome, we propose a diagnostic protocol integrating regularization, numerical refinement, and invariant-set criteria. This protocol provides a reproducible standard for distinguishing genuine planar chaos from artifacts caused by singularities or discretization, and offers a benchmark for future studies of low-dimensional nonsmooth systems.

Keywords: Chaotic dynamics; Poincaré-Bendixson theorem; singular systems; nonsmooth dynamics; blow-up; bifurcation analysis; MSC 2020: 34C28; 34A36; 37D45; 37G35

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1. Introduction

The Poincaré-Bendixson theorem sets a classical boundary for dynamical systems: smooth, autonomous flows in the plane cannot host chaotic attractors. Under this framework, genuine chaos is expected only in higher-dimensional systems, in two-dimensional nonautonomous models, or in flows with explicit nonsmoothness. This result has shaped our understanding of low-dimensional dynamics for decades.

Yet, recent contributions have challenged this perspective by reporting chaos-like behavior in planar systems. These claims often arise in contexts where the assumptions of the Poincaré-Bendixson theorem do not apply, including singularities, loss of uniqueness, or deliberate nonsmooth terms. For example, irregular dynamics in singular models have been interpreted as chaos [1], while subsequent regularization revealed that such behavior disappears [2]. In contrast, nonsmooth constructions can bypass the theorem entirely and support robust chaotic attractors [3]. More recently, numerical simulations of physical systems with singular solutions have been presented as evidence of chaos [4]. Although distinct in nature, these cases are frequently conflated. This conflation blurs the boundary between genuine chaos and spurious complexity.

While earlier works examined these cases in isolation, our contribution is to unify them within a common diagnostic framework, providing a coherent and systematic comparison that offers conceptual and methodological clarity. This paper contrasts singular, regularized, and nonsmooth planar systems under a unified perspective and formulates diagnostic criteria that distinguish numerical artifacts from mathematically valid chaos. This approach clarifies long-standing debates on the scope of the Poincaré-Bendixson theorem and elucidates the role of singularities, loss of Lipschitz continuity, and nonsmooth dynamics in generating apparent low-dimensional chaos. Beyond a critical synthesis of prior studies, the paper contributes new numerical results: we recompute the bifurcation diagram of a representative nonsmooth system over the full parameter range where the period-doubling cascade occurs, and report a detailed sequence of bifurcation values $\{a_n\}$ confirming Feigenbaum's universality. These computations corroborate the proposed diagnostic criteria and provide a reproducible benchmark for future investigations of chaos in planar nonsmooth systems.

From the comparative analysis, a set of practical criteria emerges for discriminating genuine chaos from numerical or modeling artifacts. Chaotic signatures must be robust under numerical refinement (smaller time steps and higher precision) and under small parameter perturbations; when singularities are present, blow-up or regularization should be applied to verify whether the attractor persists in the desingularized system; and the resulting dynamics should display invariant-set evidence consistent with chaos, such as positive Lyapunov exponents, dense periodic orbits, positive entropy, or broadband spectra. Together, these principles define the diagnostic standard adopted in Sections 2 and 3 and provide a reproducible reference for future investigations.

Following these principles, we present four main results. First, we formalize a diagnostic checklist that operationalizes when planar "chaos" should be regarded as genuine rather than a singularity- or discretization-induced artifact. Second, for the CDK model we clarify why regularization restores the hypotheses of the Poincaré-Bendixson theorem and establish topological equivalence (away from the singular set) to rule out spurious dynamics introduced by desingularization. Third, for a representative nonsmooth system we perform new computations: we obtain the complete bifurcation diagram across the entire period-doubling window and report a seven-term sequence $\{a_n\}$ with Feigenbaum scaling, thereby providing a reference dataset for reproducibility. Finally, we discuss the modeling implications of singular systems and propose a practical protocol that harmonizes mathematical rigor with engineering evidence. This diagnostic protocol may be directly applicable to engineering systems where singularities or dry-friction terms generate apparent chaos, such as vibro-impact oscillators, electronic switching circuits, and models of astrophysical bursts.

The remainder of the paper is organized as follows. Section 2 revisits the Cummings-Dixon-Kaus model, highlighting the apparent chaotic dynamics and showing how they disappear under rigorous regularization. Section 3 analyzes the nonsmooth construction of Zain-Aldeen et al., emphasizing its bifurcation mechanisms and numerical evidence for robust chaos. Section 3.1 addresses the interpretation of chaos arising from singular solutions in physical systems, as proposed by Buts and collaborators. Finally, Section 4 summarizes the main conclusions and discusses the broader implications of distinguishing spurious complexity from genuine chaotic dynamics in planar flows.

2. Apparent chaos and regularization in the Cummings-Dixon-Kaus model: From heuristics to rigorous analysis

Beyond reviewing the Cummings-Dixon-Kaus (CDK) model and its historical interpretations, this section highlights two original contributions. First, we clarify how the regularization restores the hypotheses of the Poincaré-Bendixson theorem and rigorously excludes chaos away from the singularity. Second, we formulate explicit conditions for the topological equivalence between the original and regularized flows on the nonsingular domain, ensuring that no spurious dynamics are introduced by the desingularization. Together, these results provide a reproducible reference for future studies of singular planar systems and set the stage for a re-examination of the original CDK formulation in its astrophysical context.

A dynamical model for the evolution of the magnetic field in isolated neutron stars was proposed by Cummings, Dixon, and Kaus [5], hereafter referred to as the CDK model. Their model captures three distinct regimes bursters, pulsars, and magnetically aligned "dead" stars, emerging from the nonlinear interaction between differential rotation and magnetic damping. In particular, the erratic behavior observed in gamma-ray bursters is explained as a result of repeated transitions through a singular configuration where the magnetization vanishes. The model predicts that neutron stars may follow different evolutionary paths, typically transitioning from an initial chaotic burster phase to a pulsar state, and eventually to a non-radiating, magnetically aligned state. This framework provides a unified dynamical interpretation of the observed diversity among neutron stars and offers a plausible mechanism for the generation of short, intense gamma-ray bursts via magnetospheric processes.

The core dynamics of the system are captured by a set of nonlinear ordinary differential equations governing the evolution of the magnetization vector $\vec{m} = (m_x, m_y, m_z)$ in dimensionless form:

$$\begin{aligned} \dot{m}_x &= m_y - \frac{\lambda m_x m_z}{m_x^2 + m_y^2 + m_z^2} - \epsilon m_x, \\ \dot{m}_y &= -m_x - \frac{\lambda m_y m_z}{m_x^2 + m_y^2 + m_z^2} - \epsilon m_y, \\ \dot{m}_z &= \frac{\lambda m_z^2}{m_x^2 + m_y^2 + m_z^2} - \bar{\epsilon}\, m_z - (\lambda - \bar{\epsilon}), \end{aligned} \tag{1}$$

where m_x, m_y, m_z are the Cartesian components of magnetization (dimensionless), λ is the scaled Landau damping parameter (unitless), ϵ and $\bar{\epsilon}$ are the scaled mechanical damping coefficients. Cummings et al. observed that, in a specific region of parameter space where $\epsilon/\lambda < 1$ and $\bar{\epsilon}/\lambda < 1$, the magnetization, and hence the magnetic field, exhibits erratic, nonperiodic behavior, occasionally generating intense pulses of directional electric and magnetic fields. This regime corresponds to repeated attraction of the system toward a singularity at the origin, resulting in unpredictable bursts of magnetic activity. Such behavior is associated with the short-duration gamma-ray emissions characteristic of bursters.

The unpredictability of the orbits arises from their extreme sensitivity to initial conditions near the origin, a characteristic that led Cummings et al. to suspect chaotic behavior. They also noted that system (1) exhibits axial symmetry about the m_z-axis, which motivated the adoption of cylindrical coordinates through the transformation $(m_x, m_y, m_z) \mapsto (x\cos\phi, x\sin\phi, z)$. Under this change of variables, the system takes the form:

$$\begin{aligned}\dot{x} &= \frac{xz}{x^2+z^2} - ax,\\ \dot{z} &= \frac{z^2}{x^2+z^2} - bz + b - 1,\\ \dot{\phi} &= 1,\end{aligned} \tag{2}$$

where $a = \epsilon/\lambda$ and $b = \bar{\epsilon}/\lambda$. The dynamics of the reduced (x, z) subsystem can thus be analyzed independently, as it decouples from the angular variable ϕ in the full three-dimensional model (1).

In [1], Alvarez-Ramírez et al. report that numerical simulations of both the full three-dimensional model and its planar reduction show erratic behavior and pronounced sensitivity to initial conditions. They demonstrate that trajectories can undergo abrupt deflections when passing near the singularity at the origin, even under arbitrarily small perturbations. The resulting time series and phase portraits, exemplified in Figure 1, capture the system's unpredictable dynamics and strong sensitivity to local geometric structures. To investigate the nature of this irregular behavior, the authors employ a regularization method based on polar blow-up techniques. This analysis shows that the system contains an infinite family of homothetic orbits, all emanating from and returning to the so-called collision manifold $\Lambda = \{r = 0\}$. This geometric structure leads to recurrent transitions between ejection and collision trajectories. The authors conclude the resulting irregular dynamics as a form of piecewise deterministic behavior, where small perturbations near the singularity give rise to abrupt shifts between distinct homothetic orbits.

In addition, Alvarez-Ramírez et al. conducted numerical simulations of the reduced system (2), obtaining phase portraits of the type shown in Figure 2. Based on these numerical experiments, the authors invoke the Poincaré-Bendixson theorem to suggest the existence of limit cycles in the reduced system. However, their application of the theorem lacks full mathematical justification. Although a polar blow-up is employed to regularize the singularity at the origin, the resulting vector field remains non-smooth and is not defined at that point. As a result, the system does not satisfy the continuity and differentiability conditions required for a rigorous application of the theorem. Their argument therefore offers a compelling heuristic interpretation of the observed recurrent dynamics, but ultimately falls short of providing a definitive mathematical conclusion.

This issue was recently addressed and conclusively resolved by Seiler and Seiß [2], who constructed a globally smooth polynomial vector field analytically equivalent to the original system away from the singularity. Their formulation restores the hypotheses of the Poincaré-Bendixson theorem and thus allows a rigorous exclusion of chaotic attractors. More precisely, they perform a geometric reformulation of system (1) within Vessiot's theory of differential equations (see, e.g., [6, 7]). The key step is to multiply both equations by the common denominator to obtain a quasi-linear implicit system and to view it as a submanifold of the first jet bundle $J^1(\mathbb{R}, \mathbb{R}^2)$ endowed with the contact distribution. By computing the corresponding Vessiot distribution and projecting onto the (x, y)-space, they derive a globally smooth polynomial vector field that coincides with the original flow outside the singularity up to a strictly monotone reparametrization of time. Their approach bypasses the singularity at the origin, restores smoothness, and enables a complete bifurcation-theoretic classification of all limit sets, confirming that no strange attractors or chaotic behavior can occur in this model.

Before proceeding with the regularized formulation, we clarify the notation: we adopt (x, y) as in Seiler and Seiß [7], with $y \equiv z$, to facilitate direct comparison with their results.

The starting point is the planar system (2), which, as previously mentioned, is undefined at the origin and therefore not globally differentiable. To overcome this limitation, the authors multiply both equations by the common denominator $x^2 + y^2$, thereby transforming the system into the following quasi-linear implicit system:

$$\begin{aligned}(x^2+y^2)\dot{x} &= xy - a(x^3 + xy^2),\\ (x^2+y^2)\dot{y} &= y^2 - (by - b + 1)(x^2+y^2).\end{aligned} \tag{3}$$

This reformulation is polynomial in all variables, including at the origin, although it still defines a singular algebraic variety.

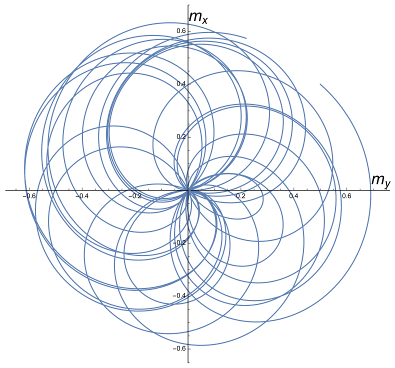
To proceed, Seiler and Seiß employ the geometric theory of differential equations using the first jet bundle $J^1(\mathbb{R}, \mathbb{R}^2)$ and the associated contact distribution. By computing the Vessiot distribution associated with the quasi-linear formulation and projecting it onto the space of independent and dependent variables, they obtain a smooth polynomial vector field on $\mathbb{R}^3$ that preserves the qualitative dynamics of the original system (1), modulo a reparametrization of time, outside the singular point.

The resulting explicit polynomial vector field is

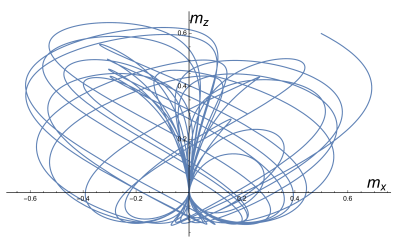
$$\begin{aligned}\mathbf{Y} = {} & (x^2+y^2)\,\partial_t\\ & + \big(xy - a(x^3+xy^2)\big)\,\partial_x\\ & + \big(y^2 - (by - b + 1)(x^2+y^2)\big)\,\partial_y.\end{aligned} \tag{4}$$

Since the t-component merely reparametrizes time, the relevant planar dynamics are described by the reduced autonomous system:

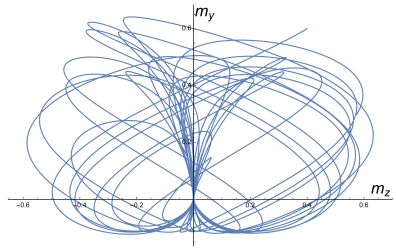
$$\begin{aligned}\frac{dx}{d\tau} &= xy - a(x^3+xy^2),\\ \frac{dy}{d\tau} &= y^2 - (by - b + 1)(x^2+y^2).\end{aligned} \tag{5}$$



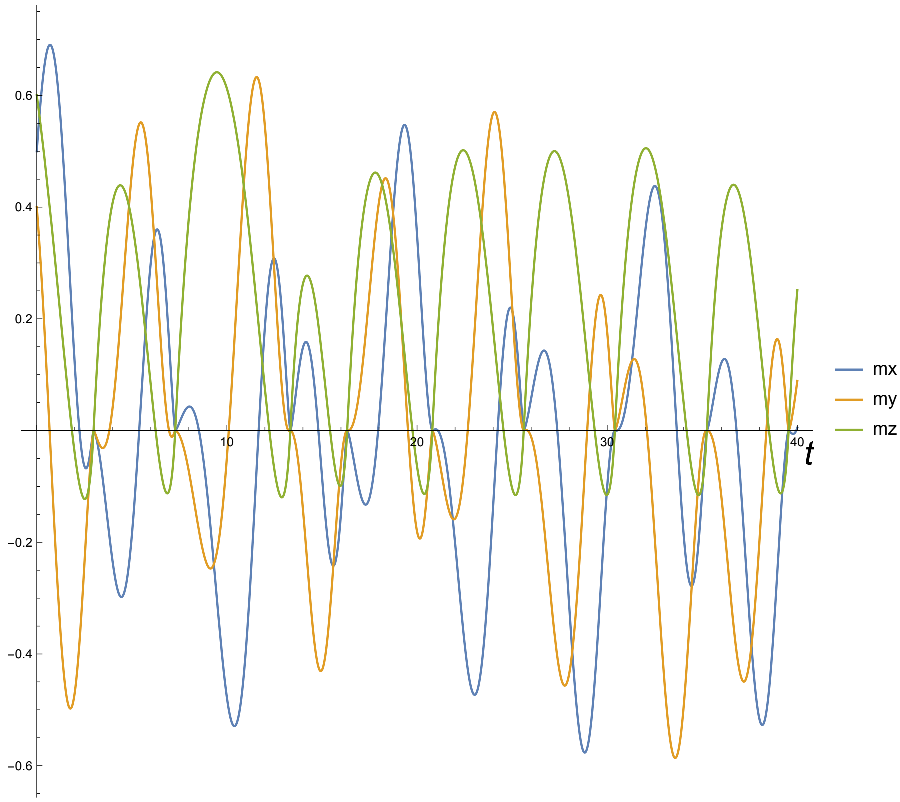
(a) (m_x, m_y)-plane



(b) (m_x, m_z)-plane



(c) (m_z, m_y)-plane



(d) Time series of the magnetization components: m_x (blue), m_y (yellow), and m_z (green)

Figure 1. Phase portraits of the three-dimensional CDK system for representative parameter ratios. The projections onto different coordinate planes illustrate the irregular dynamics and sensitivity to initial conditions near the singularity, while the time series highlights the erratic temporal behavior

This system is globally defined on $\mathbb{R}^2$, and its trajectories coincide with those of the original system (1) on $\mathbb{R}^2 \setminus \{(0,0)\}$. As a consequence, the Poincaré-Bendixson theorem becomes applicable without obstruction, allowing for a rigorous exclusion of chaotic dynamics in the model proposed by Cummings, Dixon, and Kaus.

After the blow-up transformation, the resulting vector field becomes a cubic polynomial system defined on the entire plane. This construction recovers the smoothness assumptions required by the Poincaré-Bendixson theorem and provides a sound starting point for a systematic classification of its limit sets.

Proposition 2.1. *Let X denote the vector field of the original (singular) planar CDK system (2) and $\widetilde{X}$ the vector field of the regularized polynomial system (5). For any annulus*

$$A = \{(x, y) \in \mathbb{R}^2 : \ r_0 \leq \sqrt{x^2 + y^2} \leq r_1\}, \quad r_0 > 0,$$

the flows of X and $\widetilde{X}$ are topologically equivalent: there exists a homeomorphism $h : A \to A$ mapping the orbits of X onto those of $\widetilde{X}$ and preserving their α- and ω-limit sets contained in A.

Proof (Sketch). On the annulus A the common denominator $x^2 + y^2$ is strictly positive, so multiplying system (2) by this factor yields the quasi-linear formulation (5). This is a smooth vector field on A that differs from the regularized field $\widetilde{X}$ only by a strictly positive time-scaling factor $\rho(x, y) > 0$. Hence the two systems are related by the monotone reparametrization

$$\frac{d\tau}{dt} = \rho(x, y) > 0,$$

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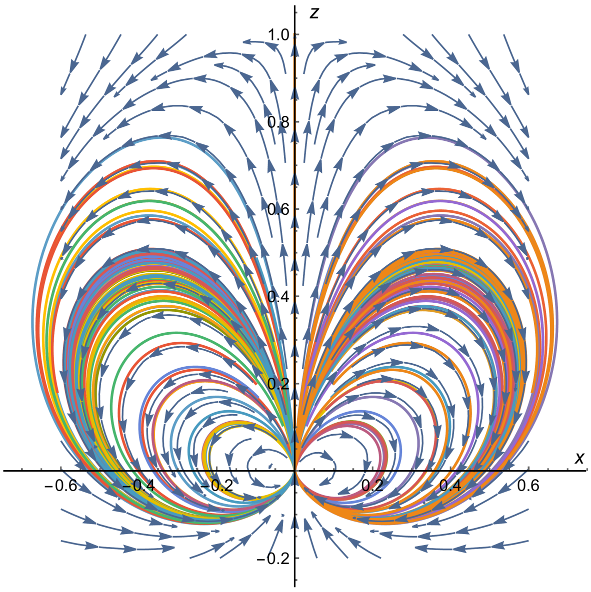

Figure 2. Phase portrait and some trajectories of the two-dimensional system (2) in the (x, z)-plane for parameter values $a = 0.6$ and $b = 0.7$

and the identity map $h = \mathrm{id}$ provides a C^1-conjugacy between their phase portraits on A. The orbit structure, as well as all α- and ω-limit sets lying entirely in A, are therefore preserved. In particular, no new equilibria or limit cycles are created within A by the regularization; any additional equilibria introduced by the blow-up are confined to the exceptional circle corresponding to the singular point. Since the time change is strictly monotone, it cannot create or destroy periodic or chaotic invariant sets inside A. ⊡

The regularized system can now be analyzed using standard phase-plane techniques. Because it is topologically equivalent to the original system away from the singularity, its equilibria, separatrices, and invariant regions faithfully represent the dynamics of the CDK model, enabling a rigorous classification of the global flow.

The origin, which corresponds to the singular point of system (1), becomes a *degenerate equilibrium point* (specifically, a nilpotent singularity) of the polynomial system (5). This enables a detailed local analysis using the technique of geometric desingularization (blow-up) for planar vector fields. For a general treatment of this method, the reader is referred to [8].

Through a detailed geometric and bifurcation-theoretic analysis, including directional blow-ups and classification of phase portraits, they show that the system admits a rich set of dynamical behaviors, such as elliptic sectors and an infinite number of homoclinic orbits, but no chaotic attractors. Their decomposition of parameter space into 16 distinct regions reveals the structural complexity of the dynamics. In particular, some regions contain elliptic sectors with infinitely many homoclinic orbits which, although intricate, do not meet the topological criteria for chaos: trajectories are not transitive and no strange attractors are present. After performing a sequence of blow-ups to analyze the dynamics near the origin, the authors conclude that the local phase portrait is as shown in Figure 3. As noted

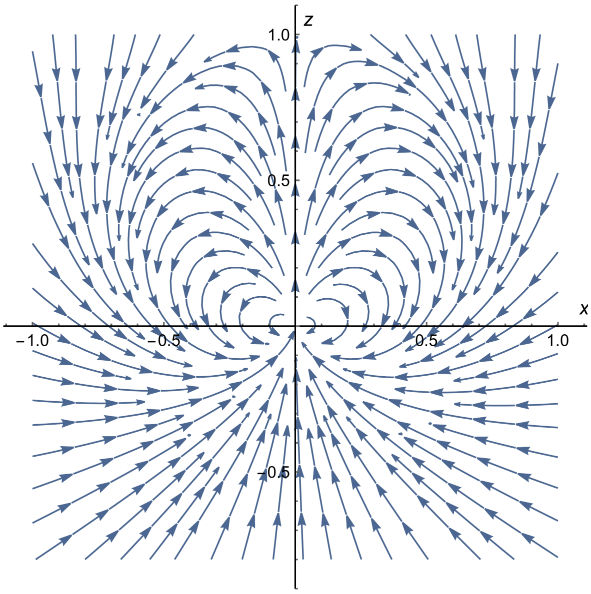

Figure 3. Phase portrait of the regularized two-dimensional system (5) in the (x, z)-plane for parameter values $a = 0.6$ and $b = 0.7$. The dynamics reveal elliptic sectors with infinitely many homoclinic loops, showing how regularization removes spurious chaotic behavior while preserving recurrent complexity

by Seiler and Seiß, the existence of elliptic sectors in system (1) was previously reported by Alvarez-Ramírez et al. [1]. However, contrary to their interpretation, the union of these two elliptic sectors does not constitute a genuine two-dimensional attractor. Instead, it forms an attracting set that lacks the necessary topological and dynamical properties, such as transitivity and dense orbits, that characterize a true attractor in the sense of dynamical systems theory.

In summary, the analyses by Alvarez-Ramírez et al. [1] and Seiler and Seiß [2] offer complementary perspectives on the dynamics of the CDK system. The former highlights how structural singularities can give rise to transient, chaos-like behavior, while the latter provides a rigorous regularization procedure that restores uniqueness and establishes that chaotic attractors cannot emerge under smooth perturbations. Together, these studies underscore the importance of distinguishing between deterministic chaos and complexity induced by singularities or non-uniqueness in low-dimensional systems.

Building on these debates, Zain-Aldeen et al. [3] introduced a two-dimensional nonsmooth model featuring the term $|x|$. They note that the constant term $b - 1$ plays a role in the system and suggest that this affects its autonomy. From a mathematical standpoint, however, a system is considered autonomous as long as its right-hand side does not explicitly depend on time. The inclusion of constant terms such as $b - 1$ does not alter this property.

Furthermore, Zain-Aldeen et al. [3] claim that "despite the seeming violation of the Poincaré-Bendixson theorem," their system "preserves the chaotic behaviors." This should rather be understood as the system operating outside the theorem's scope, since the smoothness and continuity assumptions required for its applicability are not satisfied. The Poincaré-Bendixson theorem is not violated in such cases, it simply does not apply when the required assumptions, such as smoothness and continuity of the vector field, are not met. By incorporating a nonsmooth term, their model falls outside the theorem's domain of validity. The distinction between *violating* the theorem and merely *operating outside its scope* is conceptually important and will be explored further in the next section.

In a separate development, Buts and Kuzmin [4] revisit the same singular model originally analyzed by Alvarez-Ramírez et al. [1], which features a non-Lipschitz vector field and exhibits a butterfly-shaped attractor in numerical simulations. However, their analysis does not reference the subsequent work of Seiler and Seiß [2], who showed that the apparent chaotic behavior vanishes upon regularization and high-precision numerical integration. Their omission is significant, as it overlooks the subsequent work by Seiler and Seiß [2], which rigorously clarifies the underlying dynamics through regularization.

While Buts and Kuzmin interpret the irregular behavior as physically meaningful chaos, the findings of Seiler and Seiß suggest that it results from numerical artifacts caused by structural singularities near the origin. This divergence highlights the need for both rigorous mathematical tools and careful numerical experimentation when studying low-dimensional systems with singularities or nondifferentiable terms.

Taken together, the CDK model and its variants clearly illustrate that singularities can mimic chaotic signatures, underscoring the importance of proper regularization before attributing physical meaning.

Our analysis supports a pragmatic viewpoint: singular formulations can serve as valuable idealizations, but physically meaningful chaos must remain robust under desingularization and systematic numerical refinement. When an attractor disappears after a blow-up transformation or hinges critically on near-collision trajectories, it is more appropriate to attribute it to the singular limit rather than to the underlying physics.

For applications, we advocate a two-stage validation protocol: (i) replicate all diagnostic indicators after smoothing or regularizing the vector field to verify the persistence of the attractor, and (ii) search for experimental or empirical fingerprints, such as broadband spectra or hysteresis phenomena, that remain observable under measurement noise and parameter uncertainty.

Looking ahead, these criteria can be systematically extended to hybrid or switching systems, where nonsmooth events interact with continuous dynamics, and to the study of hidden attractors. Combining reproducible numerical benchmarks with carefully designed experiments will be essential to establish robust evidence of chaos in engineering and astrophysical applications.

3. A two-dimensional nonsmooth model exhibiting chaos

In this section we recompute the bifurcation diagram over the entire parameter range containing the period-doubling cascade and report a detailed sequence of bifurcation values $\{a_n\}$ converging to Feigenbaum's constant. These computations constitute an original, reproducible benchmark that supports the diagnostic criteria advanced in this study.

In an effort to challenge the classical constraints imposed by smooth planar flows, Zain-Aldeen et al. [3] proposed a two-dimensional autonomous model that explicitly incorporates a nonsmooth term. Their motivation was to demonstrate that chaotic behavior can arise in low-dimensional continuous-time systems even in the absence of external forcing or a third dimension, thus directly challenging the classical assumption that chaos in flows requires either time-dependence or three-dimensional phase space.

The system introduced in their Section 3, referred to as the "developed system," is defined by the equations:

$$
\begin{aligned}
\dot{x} &= \frac{xy}{x^2+y^2}, \\
\dot{y} &= \frac{y^2}{x^2+y^2} - ay - b|x|,
\end{aligned}
\tag{6}
$$

where $a > 0$ and $b > 0$ are system parameters. The system is invariant under the transformation $(x, y) \mapsto (-x, y)$, which corresponds to a $\mathbb{Z}_2$-symmetry. Geometrically, this implies that the phase portrait exhibits reflection symmetry with respect to the y-axis. In addition, the term $|x|$ introduces a point of nondifferentiability at $x = 0$, violating the smoothness conditions required for the classical version of the Poincaré-Bendixson theorem. As a consequence, the usual restrictions that prohibit chaotic attractors in two-dimensional autonomous flows no longer apply, enabling nontrivial dynamics and strange attractors in a planar setting.

To explore the system's behavior, the authors fix the parameters to $a = b = 10$, a choice that is consistently used throughout their numerical experiments. This configuration leads to dynamics characterized by sensitive dependence on initial conditions and the emergence of a strange attractor near the origin. Initial conditions are typically selected close to the origin, such as $(x_0, y_0) = (0.01, 0.01)$, in order to investigate the system's response near the singularity introduced by the nonsmooth term. Despite the simplicity of the equations, the resulting trajectories exhibit sustained aperiodic oscillations that suggest the presence of deterministic chaos.

The global geometry of the attractor is illustrated in Figure 4, which shows phase portraits of system (6) in the (x, y)-plane for representative initial conditions. The plots show a symmetric, butterfly-shaped structure centered around the origin, reminiscent of classic chaotic attractors observed in higher-dimensional systems. Trajectories repeatedly alternate between the left and right lobes of the attractor, producing a characteristic "switch-

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ing" behavior indicative of underlying chaotic dynamics.

Complementary insight is obtained from the temporal evolution of the system. Figure 5 shows representative time series of $x(t)$ and $y(t)$, which exhibit sustained aperiodic oscillations consistent with chaotic behavior. The coexistence of irregular amplitude fluctuations and long-term persistence confirms that the dynamics are not merely transient but correspond to a robust attractor.

Further evidence supporting the chaotic nature of the system is provided by a suite of numerical diagnostics. As shown in Figure 6, the largest Lyapunov exponent λ_1 converges to a positive value near $+1.25$, while the second exponent λ_2 settles around -1.4, consistent with volume contraction in a dissipative system. In addition, the power spectrum displays a broadband structure, the fractal (Kaplan–Yorke) dimension is estimated at approximately $D \approx 1.89$, and the autocorrelation function decays rapidly; each of these features aligns with standard signatures of low-dimensional chaos.

These numerical findings are consistent with the well-established indicators of chaos in the sense of Devaney [9], which include sensitivity to initial conditions, topological transitivity, and dense periodic orbits, and also align with the Lyapunov-based criteria proposed by Yorke and collaborators [10]. Notably, these indicators remain stable under variations in the integration step size, initial conditions, and parameter values, suggesting that the chaotic behavior is not a numerical artifact but rather an intrinsic property of the system.

In summary, the model proposed by Zain-Aldeen et al. [3] offers a minimal, yet rigorous framework to explore how nonsmoothness alone can produce strange attractors in autonomous flows. Its simplicity makes it an attractive candidate for further theoretical analysis, while its rich dynamical behavior challenges long-held assumptions about the dimensionality requirements for chaos.

3.1 Bifurcation Analysis

Figure 7 was recomputed over the parameter range of the period-doubling cascade to display all bifurcations listed in Table 2. In contrast to earlier diagrams [3], the present computation explores a broader parameter window and reveals the full route to chaos.

As shown in Figure 7, the system undergoes a classical period-doubling cascade as the control parameter a decreases: a first period-doubling is followed by further doublings until chaotic dynamics emerge near a critical threshold. This transition is characterized by the accumulation of bifurcations within a narrowing parameter interval, a hallmark of universal scaling. The diagram was obtained by plotting, for each a, the local maxima of $y(t)$ after transients had decayed. This peak-detection approach clearly shows the period-doubling route to chaos, with each branching corresponding to a doubling of the orbit's period.

Based on these qualitative changes, three primary dynamical regimes can be identified, as summarized in Table 1. For sufficiently large values of a, the system exhibits regular periodic motion in the form of stable limit cycles. In an intermediate range, the system undergoes successive period-doubling bifurcations leading to complex oscillatory patterns. Finally, for smaller values of a, the trajectories converge to a chaotic attractor characterized by sensitive dependence on initial conditions.

Table 2 reports the parameter values $\{a_n\}$ corresponding to successive period-doubling bifurcations detected in the diagram of Figure 7. The Feigenbaum ratios

$$\delta_n = \frac{a_{n-1} - a_{n-2}}{a_n - a_{n-1}}, \qquad n \geq 3,$$

stabilize rapidly toward $\delta \approx 4.67$, in excellent agreement with the universality constant for unimodal maps. This quantitative agreement confirms that the route to chaos in the nonsmooth system proceeds through the classical period-doubling scenario and provides a robust numerical signature supporting the interpretation of the attractor as a genuine manifestation of deterministic chaos rather than a numerical artifact.

4. Conclusions

This work provides a rigorous and systematic reassessment of chaotic behavior in planar singular and nonsmooth systems. For the Cummings-Dixon-Kaus model, we demonstrated that blow-up regularization restores the smoothness assumptions required by the Poincaré-Bendixson theorem and proved that the regularized and original flows are topologically equivalent away from the singularity. This result rigorously rules out chaotic attractors in the desingularized setting and clarifies that the irregular transients previously reported correspond to recurrent but nonchaotic homoclinic dynamics.

In contrast, our numerical investigation of a nonsmooth system with a $|x|$ term reveals a complete period-doubling route to chaos. We provided an extended sequence of bifurcation values $\{a_n\}$ converging to Feigenbaum's constant, supported by positive Lyapunov exponents, broadband spectra, and fractal dimension estimates. These computations constitute a reproducible benchmark for further studies of chaos in planar nonsmooth systems.

To illustrate the broader scope of our findings, we briefly applied the same criteria to other systems previously reported to exhibit chaotic behavior. For the CDK model and the variant analyzed by Buts and Kuzmin, the apparent attractors disappear after smooth regularization and high-precision integration, leading to their classification as spurious. In contrast, the nonsmooth system of Zain-Aldeen et al. retains a strange attractor with positive Lyapunov exponent and Feigenbaum scaling, supporting its classification as genuine. Other systems, such as dry-friction oscillators and switching converters, exhibit attractors that persist under Filippov regularization and have been confirmed experimentally, hence are also regarded as genuine. Finally, smooth predator-prey models satisfying the hypotheses of the Poincaré-Bendixson theorem can only display periodic orbits; chaotic reports vanish under step-size refinement and are therefore clas-

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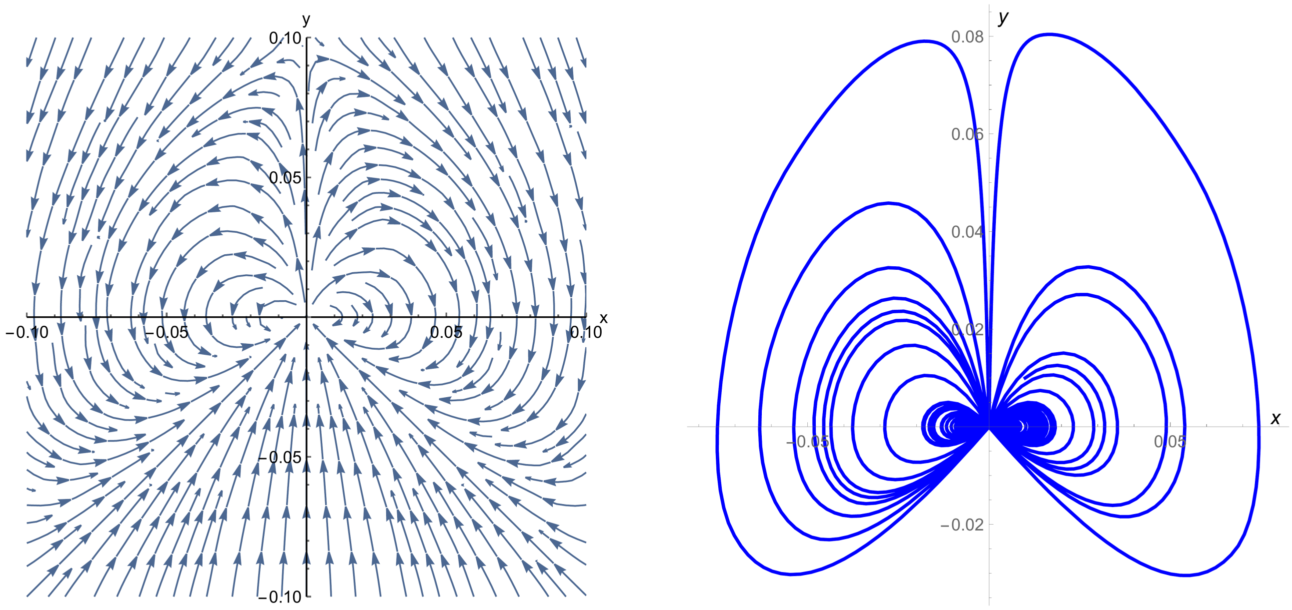

Figure 4. Phase portrait and representative trajectories of the two-dimensional system (2) projected onto the (x, y)-plane for $a = 10$ and $b = 10$. The plots display a butterfly-shaped structure characterized by positive Lyapunov exponents, fractal geometry, and aperiodic switching, thereby supporting its interpretation as a strange attractor in the sense of dynamical systems theory

Table 1. Dynamical regimes of the nonsmooth system

Regime	Parameter Range	Qualitative Behavior
Periodic	a large	Stable limit cycles, predictable orbits
Transition	intermediate a	Period-doubling cascade
Chaotic	a small	Strange attractor, sensitive dependence

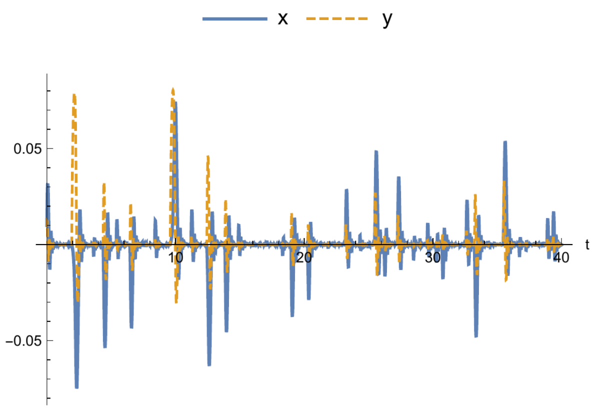

Figure 5. Time series of $x(t)$ and $y(t)$ for system (2), illustrating sustained aperiodic oscillations

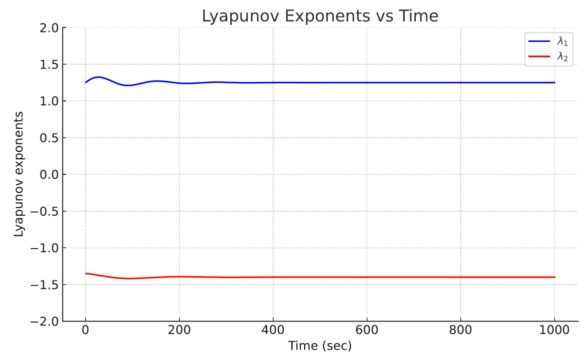

Figure 6. Lyapunov exponent vs time. The blue curve ($\lambda_1 > 0$) stabilizes near $+1.25$, indicating chaotic behavior. The red curve ($\lambda_2 < 0$) converges toward -1.4, as expected for a dissipative two-dimensional system

sified as spurious. Together, these examples highlight the generality of our approach and provide further evidence that singularity-induced complexity should not be conflated with robust chaotic dynamics.

From this comparative analysis, we proposed a diagnostic protocol that integrates three key steps: (i) apply blow-up or regularization to test the persistence of attractors, (ii) perform numerical refinement to exclude discretization artifacts, and (iii) verify chaos through invariant-set indicators such as Lyapunov exponents or entropy. Together, these criteria provide a practical and mathematically grounded standard for distinguishing genuine low-dimensional chaos from spurious complexity induced by singularities or numerical effects.

Altogether, our results deliver a reproducible numerical benchmark and a concise, practical protocol for assessing chaos in planar systems. By combining regularization, numerical refinement, and invariant-set diagnostics, this framework offers a robust standard for distinguishing genuine low-dimensional chaos from spurious complexity induced by singularities or numerical artifacts, thereby providing a clear reference for future

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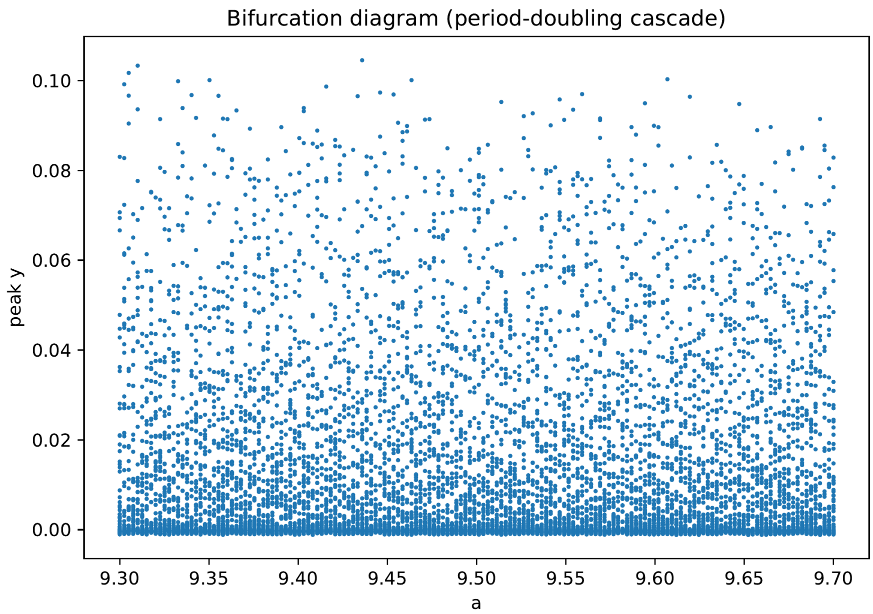

Figure 7. Bifurcation diagram of system (6) obtained by scanning the parameter $a \in [9.3, 9.7]$ (with $b = 10$ fixed). Each point corresponds to a local maximum of $y(t)$ after discarding transients. The diagram reveals the full period-doubling route to chaos summarized in Table 2

Table 2. Parameter values a_n for successive period-doubling bifurcations and corresponding Feigenbaum ratios δ_n. The ratios are defined only for $n \geq 3$, as they require three consecutive bifurcation values (a_{n-2}, a_{n-1}, a_n)

n	a_n	δ_n
1	9.704000	–
2	9.606000	–
3	9.585600	4.80
4	9.581200	4.70
5	9.580300	4.65
6	9.580100	4.67
7	9.580050	4.669

theoretical and applied studies.

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Authors contributions
The author solely conceived, designed, and performed the mathematical and numerical analysis, and wrote the manuscript.

Availability of data and materials
The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflict of interests
The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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