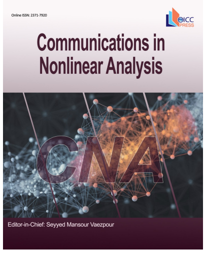

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Research Article

Common Fixed Point Theorem of Brancieri's Integral Type in Quasi b-Metric Space

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Abstract:

Let $P(z) = \sum_{j=0}^{n} a_j z^j \in \mathcal{P}_n$, with $P(z)$ not vanishing in $|z| < 1$. In this paper, we investigate a common fixed point theorem for a pair of continuous self-mappings satisfying integral-type contractions in the setting of a dislocated quasi b-metric space. The established theorem extends and generalizes several well-known results in the literature. Moreover, the constructed result can be extensively applied in the fields of integral equations and ordinary differential equations. An example is provided to support the established result.

Keywords: Complete dislocated quasi b-metric space; Cauchy sequence; Self-mapping; Contraction; Common fixed point

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1. Introduction

In 1906, Fréchet introduced the notion of metric space, which is one of the cornerstones of not only mathematics but also several quantitative sciences. Due to its importance and application potential, this notion has been extended, improved, and generalized in many different ways.

In the field of metric fixed point theory, the first significant result was proved by Banach in complete metric spaces, which may be stated as follows: "Every contraction mapping on a complete metric space $\mathcal{P}$ has a unique fixed point."

The notion of b-metric space was introduced by Czerwik [1], in connection with some problems concerning the convergence of non-measurable functions with respect to their measure. Fixed point theorems regarding b-metric spaces were obtained in [2]. Hitzler [3] introduced the concept of dislocated metric (d-metric) space, in which the distance between similar points need not necessarily be zero. The concept of dislocated metric space is widely applicable in engineering and logic programming semantics. In 2005, Zeyada et al. [4] further generalized the concept of dislocated metric space and introduced the concept of dislocated quasi-metric (dq-metric) space.

Recently, Rahman and Sarwar [5] further generalized the concept of b-metric space and dislocated quasi-metric space and initiated the notion of dislocated quasi b-metric (dq b-metric) space. Rahman and Sarwar [5] opened a new avenue for researchers to work in the field of metric fixed point theory. Fixed point theorems regarding dislocated quasi b-metric spaces were obtained in [6].

In 2002, Branciari [7] introduced the concept of integral-type contractions in metric spaces, which is very important for solving integral equations. This idea was extended by researchers in different types of distance spaces. Fixed point theorems of integral-type contractions were proved in [8, 9].

In this work, we have proved a common fixed point theorem of integral type for a new type of rational contraction condition in the setting of dislocated quasi b-metric spaces, which improves, extends, and generalizes similar fixed point results in dislocated quasi b-metric spaces. An example is provided to validate our theorem.

2. Preliminaries

We need the following definitions, which may be found in [5].

Definition 2.1. *Let $\mathcal{P}$ be a non-empty set and $k \geq 1$ be a real number. Then a mapping $d : \mathcal{P} \times \mathcal{P} \to [0, \infty)$ is called a dislocated quasi b-metric if for all $\mathrm{x}, \mathrm{y}, z \in \mathcal{P}$:*

(d_1) $d(\mathrm{x}, \mathrm{y}) = d(\mathrm{y}, \mathrm{x}) = 0$ implies $\mathrm{x} = \mathrm{y}$;

(d_2) $d(\mathrm{x}, \mathrm{y}) \leq k[d(\mathrm{x}, z) + d(z, \mathrm{y})]$.

The pair $(\mathcal{P}, d)$ is called a dislocated quasi b-metric space, or shortly a dq b-metric space.

Remark 2.2. *In the definition of dislocated quasi b-metric space, if $k = 1$, then it becomes a (usual) dislocated quasi-metric space. Therefore, every dislocated quasi-metric space is a dislocated quasi b-metric space, and every b-metric space is a dislocated quasi b-metric space with the same coefficient k and zero self-distance. However, the converse is not true, as is clear from the following example.*

Example 2.3. *Let $\mathcal{P} = \mathbb{R}$ and suppose*

$$d(\mathrm{x}, \mathrm{y}) = |2\mathrm{x} - \mathrm{y}|^2 + |2\mathrm{x} + \mathrm{y}|^2.$$

Then $(\mathcal{P}, d)$ is a dislocated quasi b-metric space with the coefficient $k = 2$. However, it is neither a dislocated quasi-metric space nor a b-metric space.

Remark 2.4. *Like the dislocated quasi-metric space, in a dislocated quasi b-metric space, the distance between similar points need not necessarily be zero, as is clear from the above example.*

Definition 2.5. *A sequence $\{\mathrm{x}_n\}$ is called dq b-convergent in $(\mathcal{P}, d)$ if for $n \in \mathbb{N}$, we have $\lim_{n\to\infty} d(\mathrm{x}_n, \mathrm{x}) = 0$. Then x is called the dq b-limit of the sequence $\{\mathrm{x}_n\}$.*

Definition 2.6. *A sequence $\{\mathrm{x}_n\}$ in a dq b-metric space $(\mathcal{P}, d)$ is called a Cauchy sequence if for $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for $m, n \geq n_0$, we have $d(\mathrm{x}_m, \mathrm{x}_n) < \epsilon$, or equivalently, $\lim_{m,n\to\infty} d(\mathrm{x}_m, \mathrm{x}_n) = 0$.*

Definition 2.7. *A dq b-metric space $(\mathcal{P}, d)$ is said to be complete if every Cauchy sequence in $\mathcal{P}$ converges to a point of $\mathcal{P}$.*

Definition 2.8. *Let $(\mathcal{P}, d_1)$ and (Y, d_2) be two dq b-metric spaces. A mapping $\mathcal{T} : \mathcal{P} \to \mathrm{Y}$ is said to be continuous if for each sequence $\{\mathrm{x}_n\}$ that is dq b-convergent to x_0 in $\mathcal{P}$, the sequence $\{\mathcal{T}\mathrm{x}_n\}$ is dq b-convergent to $\mathcal{T}\mathrm{x}_0$ in Y.*

The following well-known results can be seen in [5].

Lemma 2.9. *The limit of a convergent sequence in a dislocated quasi b-metric space is unique.*

Lemma 2.10. *Let $(\mathcal{P}, d)$ be a dislocated quasi b-metric space and $\{\mathrm{x}_n\}$ be a sequence in the dq-b-metric space such that*

$$d(\mathrm{x}_n, \mathrm{x}_{n+1}) \leq \alpha \cdot d(\mathrm{x}_{n-1}, \mathrm{x}_n)$$

for $n = 1, 2, 3, \ldots$ and $0 \leq \alpha \cdot k < 1$, $\alpha \in [0, 1)$, where k is defined in the dq b-metric space. Then $\{\mathrm{x}_n\}$ is a Cauchy sequence in $\mathcal{P}$.

Theorem 2.11. *Let $(\mathcal{P}, d)$ be a complete dislocated quasi b-metric space. Let $\mathcal{T} : \mathcal{P} \to \mathcal{P}$ be a continuous contraction with $\alpha \in [0, 1)$ and $0 \leq k\alpha < 1$, where $k \geq 1$. Then $\mathcal{T}$ has a unique fixed point in $\mathcal{P}$.*

Recently, Rahman and Sarwar [10] proved the following result in the context of dislocated quasi-metric spaces.

Theorem 2.12. *Let $(\mathcal{P}, d)$ be a complete dq-metric space and $\mathcal{T} : \mathcal{P} \to \mathcal{P}$ be a continuous self-mapping satisfying the condition*

$$\begin{aligned} d(\mathcal{T}\mathrm{x}, \mathcal{T}\mathrm{y}) \leq \alpha \cdot d(\mathrm{x}, \mathrm{y}) + \beta \cdot \frac{d(\mathrm{x}, \mathcal{T}\mathrm{y}) d(\mathrm{y}, \mathcal{T}\mathrm{y})}{d(\mathrm{x}, \mathrm{y}) d(\mathrm{y}, \mathcal{T}\mathrm{y})} \\ + \gamma \cdot \frac{d(\mathrm{x}, S\mathrm{x}) d(\mathrm{y}, \mathcal{T}\mathrm{y})}{1 + d(\mathrm{x}, \mathrm{y})} \\ + \mu \cdot \frac{d(\mathrm{x}, S\mathrm{x}) d(\mathrm{x}, \mathcal{T}\mathrm{y})}{1 + d(\mathrm{x}, \mathrm{y})} \end{aligned}$$

for all $\mathrm{x}, \mathrm{y} \in \mathcal{P}$, where $\alpha, \beta, \gamma, \mu > 0$ with $\alpha + \beta + \gamma + 2\mu < 1$. Then $\mathcal{T}$ has a unique fixed point in $\mathcal{P}$.

Theorem 2.13. *[11] Let $(\mathcal{P}, d)$ be a complete metric space and $\mathcal{T} : \mathcal{P} \to \mathcal{P}$ be a continuous self-mapping satisfying the condition*

$$\begin{aligned} d(\mathcal{T}\mathrm{x}, \mathcal{T}\mathrm{y}) &\leq \alpha \cdot d(\mathrm{x}, \mathrm{y}) \\ &\quad + \beta \cdot \frac{d(\mathrm{y}, \mathcal{T}\mathrm{y})[1 + d(\mathrm{x}, \mathcal{T}\mathrm{x})]}{1 + d(\mathrm{x}, \mathrm{y})} \end{aligned}$$

for all $\mathrm{x}, \mathrm{y} \in \mathcal{P}$, where $\alpha, \beta > 0$ with $\alpha + \beta < 1$. Then $\mathcal{T}$ has a unique fixed point in $\mathcal{P}$.

Remark 2.14. *It is obvious that the following statement holds for real numbers a, b, and c: If $a < b$ and $c > 0$, then $ac < bc$.*

Branciari [7] proved the following theorem in complete metric spaces.

Theorem 2.15. *Let $(\mathcal{P}, d)$ be a complete metric space for $\alpha \in (0, 1)$. Let $\mathcal{T} : \mathcal{P} \to \mathcal{P}$ be a mapping such that for all $\mathrm{x}, \mathrm{y} \in \mathcal{P}$,*

$$\int_0^{d(\mathcal{T}\mathrm{x}, \mathcal{T}\mathrm{y})} \rho(t) dt \leq \alpha \cdot \int_0^{d(\mathrm{x}, \mathrm{y})} \rho(t) dt,$$

where $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue integrable mapping that is summable on each compact subset of $\mathbb{R}^+$, non-negative, and such that for any $s > 0$, $\int_0^s \rho(t) dt > 0$. Then $\mathcal{T}$ has a unique fixed point in $\mathcal{P}$.

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The following important lemma can be found in [9].

Lemma 2.16. *Let $(\mathcal{P}, d)$ be a dq b-metric space and $\{x_n\}$ be a sequence in the dq b-metric space such that*

$$\int_0^{d(x_n, x_{n+1})} \rho(t)dt \leq \alpha \int_0^{d(x_{n-1}, x_n)} \rho(t)dt \qquad (2.1)$$

for $n = 1, 2, 3, \dots$ and $0 \leq \alpha k < 1$, where $\alpha \in [0, 1)$ and k is defined in the dq b-metric space. With $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ being a Lebesgue integrable mapping that is summable on each compact subset of $\mathbb{R}^+$, non-negative, and such that for any $s > 0$, $\int_0^s \rho(t)dt > 0$, then $\{x_n\}$ is a Cauchy sequence in $\mathcal{P}$.

Theorem 2.17. *[5] Let $(\mathcal{P}, d)$ be a complete dislocated quasi b-metric space for $\alpha \in (0, 1)$ and $\alpha k < 1$. Let $\mathcal{T} : \mathcal{P} \to \mathcal{P}$ be a mapping such that for all $x, y \in \mathcal{P}$,*

$$\int_0^{d(\mathcal{T}x, \mathcal{T}y)} \rho(t)dt \leq \alpha \cdot \int_0^{d(x,y)} \rho(t)dt,$$

where $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue integrable mapping that is summable on each compact subset of $\mathbb{R}^+$, non-negative, and such that for any $s > 0$, $\int_0^s \rho(t)dt > 0$. Then $\mathcal{T}$ has a unique fixed point in $\mathcal{P}$.

Rahman [6] obtained the following result in the setting of dislocated quasi b-metric spaces.

Theorem 2.18. *Let $(\mathcal{P}, d)$ be a complete dislocated quasi b-metric space. Let $\mathcal{T} : \mathcal{P} \to \mathcal{P}$ be a continuous contraction for $k \geq 1$ satisfying the condition*

$$d(\mathcal{T}x, \mathcal{T}y) \leq \phi(d(x, y))$$

for all $x, y \in \mathcal{P}$, where ϕ is a comparison function. Then $\mathcal{T}$ has a unique fixed point in $\mathcal{P}$.

In [9], the authors proved the following theorem for integral-type contractions in dislocated quasi b-metric spaces.

Theorem 2.19. *Let $(\mathcal{P}, d)$ be a complete dislocated quasi b-metric space, for $a, b, c, e, f \geq 0$ with $\frac{a+b}{1-(c+e+f)} < \frac{1}{k}$, where $k \geq 1$, and let $\mathcal{T} : \mathcal{P} \to \mathcal{P}$ be a continuous self-mapping such that for all $x, y \in \mathcal{P}$, the following condition is satisfied:*

$$\begin{aligned}\int_0^{d(\mathcal{T}x, \mathcal{T}y)} \rho(t)dt &\leq a \cdot \int_0^{d(x,y)} \rho(t)dt \\ &+ b \cdot \int_0^{d(x, \mathcal{T}x)} \rho(t)dt \\ &+ c \cdot \int_0^{d(y, \mathcal{T}y)} \rho(t)dt \\ &+ e \cdot \int_0^{\frac{d(y, \mathcal{T}y)[1+d(x, \mathcal{T}x)]}{1+d(x,y)}} \rho(t)dt \\ &+ f \cdot \int_0^{\frac{d(x, \mathcal{T}y)d(y, \mathcal{T}y)}{k[d(x,y)+d(y, \mathcal{T}y)]}} \rho(t)dt,\end{aligned}$$

where $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue integrable mapping that is summable on each compact subset of $\mathbb{R}^+$, non-negative, and such that for any $s > 0$, $\int_0^s \rho(t)dt > 0$. Then $\mathcal{T}$ has a unique fixed point.

3. Main Result

Theorem 3.1. *Let $(\mathcal{P}, d)$ be a complete dq b-metric space with coefficient $k \geq 1$, and let $S, \mathcal{T}$ be continuous self-mappings $S, \mathcal{T} : \mathcal{P} \to \mathcal{P}$ satisfying the condition*

$$\begin{aligned}\int_0^{d(Sx, \mathcal{T}y)} \rho(t)dt &\leq \alpha \cdot \int_0^{d(x,y)} \rho(t)dt \\ &+ \beta \cdot \int_0^{\frac{d(y, \mathcal{T}y)[1+d(x, Sx)]}{1+d(x,y)}} \rho(t)dt \\ &+ \gamma \cdot \int_0^{\frac{d(x, Sx)d(y, \mathcal{T}y)}{1+d(x,y)}} \rho(t)dt \\ &+ \mu \cdot \int_0^{\frac{d(x, Sx)d(x, \mathcal{T}y)}{1+d(x,y)}} \rho(t)dt\end{aligned}$$

for all $x, y \in \mathcal{P}$, where $\alpha, \beta, \gamma, \mu \geq 0$ with $\frac{\alpha+\mu}{1-(\beta+\gamma+\mu)} < \frac{1}{k}$, and where $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue integrable mapping that is summable on each compact subset of $\mathbb{R}^+$, non-negative, and such that for any $s > 0$, $\int_0^s \rho(t)dt > 0$. Then S and $\mathcal{T}$ have a unique

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common fixed point in $\mathcal{P}$.

Proof. Let x_0 be arbitrary in $\mathcal{P}$. We define a sequence $\{x_n\}$ for $n = 0, 1, 2, \ldots$ using the rule

$$x_0, x_1 = Sx_0, x_3 = Sx_2, \ldots, x_{2n+1} = Sx_{2n}$$

and

$$x_2 = \mathcal{T}x_1, x_4 = \mathcal{T}x_3, \ldots, x_{2n} = \mathcal{T}x_{2n-1}.$$

Now we show that $\{x_n\}$ is a Cauchy sequence in $\mathcal{P}$. For this, consider

$$\int_0^{d(x_{2n+1},x_{2n+2})} \rho(t)dt = \int_0^{d(Sx_{2n},\mathcal{T}x_{2n+1})} \rho(t)dt.$$

Now using the given condition in the theorem and the definition of the sequence defined above, we have

$$\begin{aligned}\int_0^{d(x_{2n+1},x_{2n+2})} \rho(t)dt &\le \alpha \cdot \int_0^{d(x_{2n},x_{2n+1})} \rho(t)dt \\ &+ \beta \cdot \int_0^{\frac{d(x_{2n+1},\mathcal{T}x_{2n+1})[1+d(x_{2n},Sx_{2n})]}{1+d(x_{2n},x_{2n+1})}} \rho(t)dt \\ &+ \gamma \cdot \int_0^{\frac{d(x_{2n},Sx_{2n})d(x_{2n+1},\mathcal{T}x_{2n+1})}{1+d(x_{2n},x_{2n+1})}} \rho(t)dt \\ &+ \mu \cdot \int_0^{\frac{d(x_{2n},Sx_{2n})d(x_{2n},\mathcal{T}x_{2n+1})}{1+d(x_{2n},x_{2n+1})}} \rho(t)dt.\end{aligned}$$

This simplifies to

$$\begin{aligned}\int_0^{d(x_{2n+1},x_{2n+2})} \rho(t)dt &\le \alpha \cdot \int_0^{d(x_{2n},x_{2n+1})} \rho(t)dt \\ &+ \beta \cdot \int_0^{\frac{d(x_{2n+1},x_{2n+2})[1+d(x_{2n},x_{2n+1})]}{1+d(x_{2n},x_{2n+1})}} \rho(t)dt \\ &+ \gamma \cdot \int_0^{\frac{d(x_{2n},x_{2n+1})d(x_{2n+1},x_{2n+2})}{1+d(x_{2n},x_{2n+1})}} \rho(t)dt \\ &+ \mu \cdot \int_0^{\frac{d(x_{2n},x_{2n+1})d(x_{2n},x_{2n+2})}{1+d(x_{2n},x_{2n+1})}} \rho(t)dt.\end{aligned}$$

Now using the triangular inequality and Remark 2.14, we have

$$\begin{aligned}\int_0^{d(x_{2n+1},x_{2n+2})} \rho(t)dt &\le \alpha \cdot \int_0^{d(x_{2n},x_{2n+1})} \rho(t)dt \\ &+ \beta \cdot \int_0^{d(x_{2n+1},x_{2n+2})} \rho(t)dt \\ &+ \gamma \cdot \int_0^{d(x_{2n+1},x_{2n+2})} \rho(t)dt \\ &+ \mu \cdot \int_0^{k[d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})]} \rho(t)dt.\end{aligned}$$

Simplification yields

$$\int_0^{d(x_{2n+1},x_{2n+2})} \rho(t)dt \le \left(\frac{\alpha + \mu k}{1 - (\beta + \gamma + \mu k)}\right) \int_0^{d(x_{2n},x_{2n+1})} \rho(t)dt.$$

Let $h = \frac{\alpha+\mu k}{1-(\beta+\gamma+\mu k)}$, so the above equation takes the form

$$\int_0^{d(x_{2n+1},x_{2n+2})} \rho(t)dt \le h \cdot \int_0^{d(x_{2n},x_{2n+1})} \rho(t)dt.$$

Similarly, we can obtain

$$\int_0^{d(x_{2n},x_{2n+1})} \rho(t)dt \le h \cdot \int_0^{d(x_{2n-1},x_{2n})} \rho(t)dt.$$

Continuing the same procedure, one can get

$$\int_0^{d(x_n,x_{n+1})} \rho(t)dt \le h \cdot \int_0^{d(x_{n-1},x_n)} \rho(t)dt.$$

Since $h < \frac{1}{k}$, by using Lemma 2.16, we get that $\{x_n\}$ is a Cauchy sequence in the complete dislocated quasi b-metric space $\mathcal{P}$. Consequently, there exists $u \in \mathcal{P}$ such that $\lim_{n\to\infty} x_n = u$. Also, the subsequences $\{x_{2n+1}\}$ and $\{x_{2n+2}\}$ will also converge to $u \in \mathcal{P}$.

Using the continuity of S and $\mathcal{T}$, we have $\mathcal{T} \cdot \lim_{n\to\infty} x_{2n+1} = \mathcal{T}u \Rightarrow \mathcal{T}u = u$. Similarly, we can show that $Su = u$. Therefore, u is the common fixed point of S and $\mathcal{T}$.

Uniqueness: Let u and v be two distinct common fixed points of S and $\mathcal{T}$. Using the given condition in the theorem and the triangular inequality, one can easily prove that

$$d(u,u)=d(v,v)=0. \tag{3.1}$$

Now consider

$$\int_0^{d(u,v)} \rho(t)dt = \int_0^{d(Su,\mathcal{T}v)} \rho(t)dt$$
$$\leq \alpha \cdot \int_0^{d(u,v)} \rho(t)dt$$
$$+\beta \cdot \int_0^{\frac{d(v,\mathcal{T}v)[1+d(u,Su)]}{1+d(u,v)}} \rho(t)dt$$
$$+\gamma \cdot \int_0^{\frac{d(u,Su)d(v,\mathcal{T}v)}{1+d(u,v)}} \rho(t)dt$$
$$+\mu \cdot \int_0^{\frac{d(u,Su)d(u,\mathcal{T}v)}{1+d(u,v)}} \rho(t)dt.$$

Now using (3.1) and the fact that u and v are fixed points of S and $\mathcal{T}$, we have

$$\int_0^{d(u,v)} \rho(t)dt = \int_0^{d(Su,\mathcal{T}v)} \rho(t)dt$$
$$\leq \alpha \cdot \int_0^{d(u,v)} \rho(t)dt$$
$$+\beta \cdot \int_0^{\frac{d(v,v)[1+d(u,u)]}{1+d(u,v)}} \rho(t)dt$$
$$+\gamma \cdot \int_0^{\frac{d(u,u)d(v,v)}{1+d(u,v)}} \rho(t)dt$$
$$+\mu \cdot \int_0^{\frac{d(u,u)d(u,v)}{1+d(u,v)}} \rho(t)dt.$$

This simplifies to

$$\int_0^{d(u,v)} \rho(t)dt \leq \alpha \cdot \int_0^{d(u,v)} \rho(t)dt,$$

which implies that $d(u,v)=0$, as $\alpha<1$. Similarly, we can show that $d(v,u)=0$.

Thus, $d(u,v)=d(v,u)=0$, which gives $u=v$. Hence, u is the unique common fixed point of S and $\mathcal{T}$. □

The following corollaries may be deduced from Theorem 3.1.

Corollary 3.2. *Let $(\mathcal{P},d)$ be a complete dq b-metric space with coefficient $k\geq 1$, and let $\mathcal{T}$ be a continuous self-mapping $\mathcal{T}:\mathcal{P}\to\mathcal{P}$ satisfying the condition*

$$\int_0^{d(\mathcal{T}x,\mathcal{T}y)} \rho(t)dt$$
$$\leq \alpha \cdot \int_0^{d(x,y)} \rho(t)dt$$
$$+\beta \cdot \int_0^{\frac{d(y,\mathcal{T}y)[1+d(x,\mathcal{T}x)]}{1+d(x,y)}} \rho(t)dt$$
$$+\gamma \cdot \int_0^{\frac{d(x,\mathcal{T}x)d(y,\mathcal{T}y)}{1+d(x,y)}} \rho(t)dt$$
$$+\mu \cdot \int_0^{\frac{d(x,\mathcal{T}x)d(x,\mathcal{T}y)}{1+d(x,y)}} \rho(t)dt$$

for all $x,y\in\mathcal{P}$, where $\alpha,\beta,\gamma,\mu\geq 0$ with $\frac{\alpha+\mu}{1-(\beta+\gamma+\mu)}<\frac{1}{k}$, and where $\rho:\mathbb{R}^+\to\mathbb{R}^+$ is a Lebesgue integrable mapping that is summable on each compact subset of $\mathbb{R}^+$, non-negative, and such that for any $s>0$, $\int_0^s \rho(t)dt>0$. Then $\mathcal{T}$ has a unique fixed point in $\mathcal{P}$.

Proof. By putting $S=\mathcal{T}$ in Theorem 3.1, one can easily obtain the required result. □

Corollary 3.3. *Let $(\mathcal{P},d)$ be a complete dq b-metric space with coefficient $k\geq 1$, and let $\mathcal{T}$ be a continuous self-mapping $\mathcal{T}:\mathcal{P}\to\mathcal{P}$ satisfying the condition*

$$\int_0^{d(\mathcal{T}x,\mathcal{T}y)} \rho(t)dt$$
$$\leq \alpha \cdot \int_0^{d(x,y)} \rho(t)dt$$
$$+\beta \cdot \int_0^{\frac{d(y,\mathcal{T}y)[1+d(x,\mathcal{T}x)]}{1+d(x,y)}} \rho(t)dt$$

for all $x,y\in\mathcal{P}$, where $\alpha,\beta\geq 0$ with $\frac{\alpha}{1-\beta}<\frac{1}{k}$, and where $\rho:\mathbb{R}^+\to\mathbb{R}^+$ is a Lebesgue integrable mapping that is summable on each compact subset of $\mathbb{R}^+$, non-negative, and such that for any $s>0$, $\int_0^s \rho(t)dt>0$. Then $\mathcal{T}$ has a unique fixed point in $\mathcal{P}$.

Proof. By putting $\gamma=\mu=0$ in Corollary 3.2, one can easily obtain the required result. □

Corollary 3.4. *Let $(\mathcal{P}, d)$ be a complete dq b-metric space with coefficient $k \geq 1$, and let $\mathcal{T}$ be a continuous self-mapping $\mathcal{T} : \mathcal{P} \to \mathcal{P}$ satisfying the condition $\int_0^{d(\mathcal{T}x,\mathcal{T}y)} \rho(t)dt \leq \alpha \cdot \int_0^{d(x,y)} \rho(t)dt$ for all $x, y \in \mathcal{P}$, where $\alpha \geq 0$ with $\alpha < \frac{1}{k}$, and where $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue integrable mapping that is summable on each compact subset of $\mathbb{R}^+$, non-negative, and such that for any $s > 0$, $\int_0^s \rho(t)dt > 0$. Then $\mathcal{T}$ has a unique fixed point in $\mathcal{P}$.*

Proof. By putting $\beta = 0$ in Corollary 3.3, one can easily obtain the required result. □

Remark 3.5.
- *In Corollary 3.3, if we put $\rho(t) = t$ (identity mapping), then it generalizes Theorem 2.13 in the context of dislocated quasi b-metric spaces.*
- *Corollary 3.4 gives Theorem 2.17 of [5].*
- *Corollary 3.4 also generalizes Theorem 2.15 in the setting of dislocated quasi b-metric spaces.*
- *In Corollary 3.4, if we put $\alpha = \phi(t)$ (comparison function) and $\rho(t) = t$ (identity mapping), then Corollary 3.4 gives Theorem 2.18.*
- *By putting different constants equal to zero and $\rho(t) = t$ in Theorem 3.1, one can obtain several results from the literature in the context of dislocated quasi b-metric spaces.*

The following example is given in support of Theorem 3.1.

Example 3.6. *Let $\mathcal{P} = \mathbb{R}$ and the complete dq b-metric defined on $\mathcal{P}$ is given by $d(x, y) = |2x - y|^2 + |2x + y|^2$ with a self-mapping defined on $\mathcal{P}$ as $\mathcal{T}x = \frac{x}{2}$ and $\rho(t) = \frac{t}{2}$. Then*

$$\int_0^{d(\mathcal{T}x,\mathcal{T}y)} \rho(t)dt = \int_0^{|x-\frac{y}{2}|^2+|x+\frac{y}{2}|^2} \frac{t}{2}dt = \int_0^{\frac{1}{4}(|2x-y|^2+|2x+y|^2)} \frac{t}{2}dt.$$

Integrating with respect to t and applying limits, we have

$$\int_0^{d(\mathcal{T}x,\mathcal{T}y)} \rho(t)dt = \frac{1}{64}\left(|2x - y|^2 + |2x + y|^2\right)^2 \leq \frac{1}{4}\int_0^{d(x,y)} \rho(t)dt.$$

This satisfies all the conditions of Corollary 3.2 for $\alpha \in [\frac{1}{4}, 1)$, having $x = 0$ as the unique fixed point.

Conclusion: This work contributes a more generalized, unified, and extended result for the future literature in the setting of the newly initiated notion of dislocated quasi b-metric spaces. One can easily apply the above-proved theorem to the solution of integral equations.

Authors Contributions
All the authors have contributed equally and have read and approved the final version of the paper.
Availability of Data and Materials
The datasets used and/or analyzed during the current study are available from the corresponding author upon reasonable request.
Competing Interests
The authors declare that they have no competing interests.
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