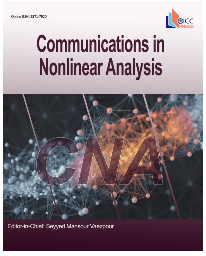

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Research Article

Integral Bounds for Generalized Polar Derivatives via Bernstein and Turán Inequalities

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Abstract:

Let $P(z) = \sum_{\mu=0}^{n} a_v z^v$ be a polynomial degree n, and β be any complex number. Then $nP(z) + (\beta - z)P'(z)$ is the polar derivative of $P(z)$, usually denoted by $D_\beta P(z)$. The derivative $D_\beta P(z)$ is the polynomial of degree at most $n-1$. The ordinary derivative follows from polar derivative. In this work, we establish integral mean inequalities of Turán and Erdős–Lax type for the Generalized polar derivative of a polynomial, incorporating specific coefficients of the polynomial. Our findings refine several earlier results in the literature, and notably, one of our theorems extends and sharpens the well-known inequality by Khangembam Babina Devi and Barchand Chanam as a particular case. These improvements take into account both the location of the zeros and the structure of the coefficients of the polynomial. To illustrate the effectiveness and enhanced sharpness of our results, we also include numerical simulations and graphical comparisons with existing inequalities.

Keywords: Bernstein-type inequalities; Turán-type inequalities; Polar derivative; polynomials; Generalized Polar Derivative; MSC 2020: 30A10, 30C10, 30C15, 60E05, 65D15

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1. Introduction

Mathematical models often arise from problems in science and engineering, with polynomial functions playing a key role across many areas of mathematics. Chebyshev pioneered the study of polynomials that best approximate continuous functions, leading to important Bernstein-type inequalities. These inequalities provide bounds on polynomial derivatives and are fundamental in approximation theory. The earliest notable result in this field dates back to the work of the Russian chemist Mendeleev [1], who studied the problem of maximizing $|P'(x)|$ for quadratic polynomials $p(x)$ with real coefficients satisfying $|P(x)| \leq 1$ on the interval $[-1, 1]$. Mendeleev showed that under these conditions, $|P'(x)| \leq 4$. Later, Markov [2] extended this result to algebraic polynomials of degree n on the real line, proving a general bound for $|P'(x)|$.

$$\max_{-1\leq x\leq 1} |P'(x)| \leq n^2 \max_{-1\leq x\leq 1} |P(x)|$$

About two decades later, Bernstein [3] sought an analogue of Markov's theorem for the unit disk in the complex plane, rather than the interval $[-1, 1]$, to establish an inverse theorem in approximation theory (see Borwein and Erdélyi [4, p. 241]). His work aimed to quantify how well a polynomial of a given degree can approximate a continuous function based on its derivatives and Lipschitz constants. This led to the well-known Bernstein inequality, which asserts that if $t \in \tau_n$ (the set of real trigonometric polynomials of degree at most n), then for $K = [0, 2\pi)$, the inequality holds: $|t'(x)| \leq n\|t\|_\infty$, for all $x \in K$.

$$\max_{\theta\in K} |t^{(m)}(\theta)| \leq n^m \max_{\theta\in K} |t^{(m)}(\theta)| \tag{1.1}$$

The above inequality also holds for all $t \in \tau_n^c$, the set of complex trigonometric polynomials of degree at most n. As a special case, this leads to the algebraic polynomial version of Bernstein's inequality on the unit disk.

Theorem 1.1. *If $P(z)$ is a polynomial of degree n, then*

$$\max_{|z|=1}|P'(z)| \leq n \max_{|z|=1}|P(z)| \tag{1.2}$$

Equality holds in (1.2) if $P(z)$ has all its zeros at the origin.

Approximating continuous functions with polynomials is important both theoretically and practically. A key method uses Bernstein's trigonometric inequality, leading to a result (Theorem 1.2) [[4], p. 241, Part (a) of E.18], which approximates m times differentiable real-valued function on a half-closed interval $[0, 2\pi]$ by trigonometric polynomials. For the sake of convenience of the readers, we state the above result more precisely.

Let Lip_α, $\alpha \in (0,1]$, denote the family of all real-valued functions g defined on K satisfying

$$Sup\left\{\frac{|g(x)-g(y)|}{|x-y|} : x \neq y \in K\right\} < \infty.$$

If $C(K)$ denotes the set of all continuous functions on K , then for $f \in C(K)$, let

$$E_n(f) := inf\left\{\sup_{\theta\in K}|t-f| : t \in \tau_n\right\}$$

Theorem 1.2 (Direct theorem)**.** *Suppose f is m times differentiable on K and $f^{(m)} \in \mathrm{Lip}_\alpha$ for some $\alpha \in (0,1]$. Then, there is a constant C depending only on f so that*

$$E_n(f) \leq Cn^{-(m+\alpha)},\ n = 1, 2, \cdots .$$

On the other hand, the converse (inverse) of Theorem 1.2 is essentially of interest and is stated below.

Theorem 1.3 (Inverse theorem)**.** *Suppose m is a non-negative integer, $\alpha \in (0,1)$, and $f \in C(K)$. Suppose there is a constant $C > 0$ depending only on f such that*

$$E_n(f) \leq Cn^{-(m+\alpha)},\ n = 1, 2, \cdots .$$

Then f is m times continuously differentiable on K and $f^{(m)} \in \mathrm{Lip}_\alpha$.

From the above discussion, it is worth noting that Bernstein- and Markov-type inequalities play a significant role in approximation theory. Direct and inverse theorems of approximation and related matters may be found in many books on approximation theory, including Cheney [5], Lorentz [6], and DeVore and Lorentz [7]. Inequality (1.2) provides insight into the rate of change of a polynomial of degree at most n, making it significant in both pure mathematics—particularly approximation theory—and applied fields like physics. Several extensions of this inequality exist, involving generalizations of intervals, norms, and function classes. A notable example includes replacing the sup-norm with an expression based on the integral mean.
Let $P(z)$ be a polynomial of degree n over the set of complex numbers and $q(z) = z^n\overline{\left(\frac{1}{\bar{z}}\right)}$. We define the integral mean of $P(z)$ on the unit circle $|z| = 1$ by

$$||p||_r = \left\{\int_0^{2\pi}|P(e^{i\theta})|d\theta\right\}^{\frac{1}{r}},\ 0 < r < \infty. \tag{1.3}$$

By allowing $r \to \infty$ in equation (1.3) and applying a well-known result from analysis (refer to [8; 9] we obtain...

$$\lim_{r\to\infty}\left\{\frac{1}{2\pi}\int_0^{2\pi}|P(e^{i\theta})|d\theta\right\}^{\frac{1}{r}} = \max_{|z|=1}|P(z)|, \tag{1.4}$$

we can suitably denote,

$$||P(z)||_\infty = \max_{|z|=1}|P(z)|. \tag{1.5}$$

Similarly, we can define

$$||P(z)||_0 = e^{\left\{\frac{1}{2\pi}\int_0^{2\pi}|P(e^{i\theta})|d\theta\right\}}$$

We can also show that $\lim_{r\to\infty^+}\|p\|_r = \|p\|_0$. Interestingly, by taking the limit as $r \to 0^+$, the results established for $r > 0$ remain valid even when $r = 0$. Moreover, inequality (1.2) can be derived by letting $r \to \infty$ in the corresponding generalized inequality.

$$\|p'\|_r \leq n\|p\|_r, \quad r > 0. \tag{1.6}$$

Inequality (1.6) was proved by Zygmund [10] for $r \geq 1$ and by Arestov [11] for $0 < r < 1$.

If we consider only polynomials that have no zeros inside the unit disk $|z| < 1$, then inequalities (1.2) and (1.6) can be correspondingly sharpened.

$$\|p'\|_\infty \leq \frac{n}{2}\|p\|_\infty, \quad r > 0. \tag{1.7}$$

$$\|p'\|_r \leq \frac{n}{\|1+z\|_r}\|p\|_r, \quad r > 0. \tag{1.8}$$

Inequality (1.7) was originally conjectured by Erdős and later confirmed by Lax [12], while inequality (8) was established by de Bruijn [13] for $r \geq 1$, and by Rahman and Schmeisser [14] for $0 < r < 1$. Additionally, in 1939, Turán (see [15]) derived a lower bound for the maximum of $|P'(z)|$ on the unit circle, showing that if $P(z)$ is a polynomial of degree n with all its zeros in $|z| \leq 1$, then

$$\|p'\|_\infty \geq \frac{n}{2}\|p\|_\infty, \quad r > 0. \tag{1.9}$$

Extending inequality (1.9), Govil [16] showed that if $P(z)$ is a polynomial of degree n with all its zeros within $|z| \leq k$, where $k \geq 1$, then

$$\|p'\|_\infty \geq \frac{n}{1+k^n}\|p\|_\infty. \tag{1.10}$$

As a generalization of inequality (1.7), Malik [17] established that if $P(z)$is a polynomial of degree n with no zeros inside the disk $|z| < k$, where $k \geq 1$, then

$$\|p'\|_\infty \leq \frac{n}{1+k^n}\|p\|_\infty. \tag{1.11}$$

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Under the same conditions on the polynomial $P(z)$, Govil and Rahman [18] extended inequality (1.11) to the L^r-norm by proving that

$$\|p'\|_\infty \geq \frac{n}{\|k+z\|_r}\|p\|_r,\ r \geq 1 \tag{1.12}$$

Gardner and Weems [19], and independently Rather [20], confirmed that inequality (1.12) holds for $0 < r < 1$. Several extensions have been proposed by Chan and Malik [21], Dewan and Bidkham [22], Chanam and Dewan [23], and Dewan and Mir [24].

For polynomials with no zeros inside the disk $|z| < k$, where $k \leq 1$, determining an exact upper bound for the maximum of $|P'(z)|$ on $|z| = 1$ is generally challenging. It was long assumed that if $P(z)$ has no zeros in $|z| < k$, then a natural extension of inequality (1.7) would hold as

$$\|P'\|_\infty \leq \frac{n}{1+k^n}\|P\|_\infty,$$

Nevertheless, certain special cases have been explored, with partial extensions of (1.7) established. Notably, in 1980, Govil [25] offered a generalization of (1.7) under additional conditions.

Theorem 1.4. *If $P(z)$ is a polynomial of degree n having no zeros in $|z| < k,\ k \leq 1$, then*

$$\|P'\|_\infty \leq \frac{n}{1+k^n}\|P\|_\infty, \tag{1.13}$$

provided $|P'(z)|$ and $|Q'(z)|$ attain their maxima at the same point on the circle $|z| = 1$, where throughout the paper $Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$.

In 1984, Malik [26] was the first to extend Turán's inequality (1.9) to an integral mean form, showing that for a degree n polynomial $P(z)$ with all zeros inside $|z| \leq 1$ and for $r > 0$,

$$\|1+z\|_r\|P'\|_\infty \geq n\|P\|_r. \tag{1.14}$$

This result is sharp, with equality when $p(z) = (z+1)^n$. Later, in 1988, Aziz [11] derived the integral mean extension of inequality (1.10) and established a related theorem.

Theorem 1.5. *If P(z) is a polynomial of degree n having all its zeros in $|z| \leq k, k \geq 1$, then for $r \geq 1$,*

$$\|1+k^nz\|_r\|P'\|_\infty \geq n\|P\|_r \tag{1.15}$$

In 2021, Devi et al. [27] established a more general result that includes inequality (1.13) as a special case.

Theorem 1.6. *If $P(z)$ is a polynomial of degree n having no zeros in $|z| < k,\ k \leq 1$, then for every $r > 0$,*

$$k^n n\|P\|_r \leq \|k^n+z\|_r\, n\|P\|_\infty - \|P'\|_\infty \tag{1.16}$$

provided $|P(z)|$ and $|Q(z)|$ attain their maxima at the same point on the circle $|z| = 1$.

Devi et al. (1.16) further refined Theorem 1.6 by proving a sharper result.

Theorem 1.7. *If $P(z)$ is a polynomial of degree n having no zeros in $|z| < k, k \leq 1$, then for every real or complex number α with $|\alpha| < 1$ and for every $r > 0$,*

$$k^n n\left\|z^n\overline{P(z)} + \frac{m}{k^n}\alpha\right\|_r \leq \|k^n+z\|_r \times \left\{n\|P\|_\infty - \|P'\|_\infty\right\}, \tag{1.17}$$

provided $|P(z)|$ and $|Q(z)|$ attain their maxima at the same point on the circle $|z| = 1$, where throughout the paper $m = \max_{|z|=k}$.

Before proceeding to some other results, let us introduce the concept of the polar derivative involved.

Definition 1.8 (Polar Derivative). *For a polynomial $P(z)$ of degree n, the polar derivative with respect to a point β is defined as*

$$D_\beta P(z) = nP(z) + (\beta - z)P'(z), \tag{1.18}$$

as introduced in [28] and [29, Chap. 6]. This derivative, which is a polynomial of degree at most $n-1$, extends the concept of the ordinary derivative. In fact, as $\beta \to \infty$,

$$\lim_{\beta\to\infty} D_\beta P(z) = P'(z),$$

uniformly for $|z| \leq R$, where $R > 0$.

Definition 1.9 (Generalized Derivative). *Given a polynomial $P(z) = c(z-z_1)(z-z_2)\cdots(z-z_n)$ of degree n and an n-tuple $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ of non-negative real numbers (not all zero), Sz-Nagy [30] introduced a generalized derivative of $P(z)$ defined by:*

$$P^\gamma(z) = P(z)\sum_{j=1}^{n}\frac{\gamma_j}{z-z_j} = c\sum_{j=1}^{n}\gamma_j P_j(z), \tag{1.19}$$

where $P_j(z) = c\prod_{\substack{i=1\\ i\neq j}}^{n}(z-z_i)$ for $1 \leq j \leq n$.
Note that the ordinary derivative $A'(z)$ of $P(z)$ can be obtained from $P^\gamma(z)$ by setting $\gamma_j = 1$ for $j = 1, 2, \ldots, n$, that is,

$$P^\gamma(z) = P'(z),\ \text{for } \gamma = (\underbrace{1, 1, \ldots, 1}_{n\text{-times}}).$$

Definition 1.10 (Generalized polar Derivative of a polynomial). *Let $P(z)$ be a polynomial of degree n and $P^\gamma(z)$ be its generalized derivative. Then the generalized polar derivative of $P(z)$ with respect to the complex number β is defined as :*

$$D^\gamma_\beta P(z) = \Lambda P(z) + (\beta - z)P^\gamma(z) \tag{1.20}$$

It generalizes the derivative $P^\gamma(z)$ in the sense that,

$$\lim_{\beta\to\infty}\frac{D^\gamma_\beta P(z)}{\beta} = P^\gamma(z),$$

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uniformly with respect to z for $|z| \le R,\ R > 0$. Where $\Lambda = \sum_{\mu=1}^{n} \gamma_j$ and $0 \neq (\gamma = \gamma_1, \gamma_2, \cdots, \gamma_n)$ is n-tuple of non-negative real numbers. For a unit n-tuple $\gamma = (1, 1, \cdots, 1)$, equation (1.20) reduced to polar derivative as

$$\begin{aligned} D_{\beta}^{1,1,\cdots,1} P(z) &= nP(z) + (\beta - z)P'(z) \\ &= D_{\beta}P(z), \end{aligned}$$

which is the polar derivative of $P(z)$.

Extensive studies on the polar derivative of polynomials can be found in the works of Milovanović et al. [31], Marden [28], and Rahman and Schmeisser [32], where techniques from geometric function theory are used to derive polynomial inequalities.

In 1998, Aziz and Rather [33] extended inequality (1.10) to the polar derivative, proving that if $P(z)$ is a polynomial of degree n with all its zeros in $|z| \le k$ (where $k \ge 1$, then the result holds for any complex number β satisfying $|\beta| \ge k$.

$$\|D_{\beta}P(z)\|_{\infty} \ge n \left(\frac{|\beta| - k}{1 + k^n} \right) \|P(z)\|_{\infty} \tag{1.21}$$

Recently, Mir and Breaz [34] provided the polar derivative counterpart of inequality (13). They showed that if $P(z)$ is a polynomial of degree n with no zeros inside $|z| < k$, where $k \le 1$, then the result holds for every complex number β with $|\beta| \ge 1$.

$$\|D_{\beta}P(z)\|_{\infty} \le n \left(\frac{|\beta| - k}{1 + k^n} \right) \|P(z)\|_{\infty} \tag{1.22}$$

provided $|P(z)|$ and $|Q(z)|$ attain their maxima at the same point on the circle $|z| = 1$.

They [34] also demonstrated that if $P(z)$ is a polynomial of degree n with no zeros in $|z| < k$, where $k \le 1$, then the result holds for every complex number β satisfying $|\beta| \ge 1$.

$$\begin{aligned} \|D_{\beta}P(z)\|_{\infty} \le \left(\frac{n}{1 + k^n} \right) \Big\{ &(|\beta| + k^n) \|P\|_{\infty} \\ &- (|\beta| - 1)\, m \Big\}, \end{aligned} \tag{1.23}$$

provided $|P(z)|$ and $|Q(z)|$ attain their maxima at the same point on the circle $|z| = 1$.

Efforts to derive an integral version of inequality (19) by Aziz and Rather [33] culminated in 2017 when Rather and Bhat [35] successfully established such an integral form. More recently, in 2023, Singha and Chanam [36] extended a related result of Kumar [37] in this area. In this paper, our primary objective is to develop an integral refinement of inequality (19) from Aziz and Rather [33], which also strengthens the classical inequality (1.9) by Túran [15] for the ordinary derivative. As a consequence, we obtain improved versions of inequalities recently established by Mir and Breaz [34] for polar derivatives. The paper is organized as follows: Section 2 presents the main results and their significant implications; Section 3 contains auxiliary lemmas essential for the proofs; and Section 4 is dedicated to proving the main theorems.

2. Main results

This section focuses on presenting our primary findings in the form of theorems and corollaries. The significance of the initial theorem lies in its ability to generate numerous important consequences related to previously established polynomial inequalities. We first derive the following result, which extends Theorems generalized polar derivative.

Theorem 2.1. *If $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree $n \ge 2$ having all its zeros in $|z| \le k,\ k \ge 1$ then for every complex number α and β with $|\alpha| < 1$, $|\beta| \ge k$ and for each $r > 0$.*

$$\begin{aligned} \Lambda\big(|\beta| - k\big)\|P(z)\|_r \ &\le \|1 + k^n z\|_r \\ &\times \Big\{ \Big[\|D_{\beta}^{\gamma}P(z) + \alpha m n\|_{\infty} \\ &\quad - \left(1 - \frac{1}{k^2}\right) |na_0 + \beta a_1 + \alpha m n| \Big] \Big\},\ \text{if } n > 2 \end{aligned} \tag{2.1}$$

which is equivalent to

$$\begin{aligned} \Lambda\big(|\beta| - k\big)\|P(z)\|_r \ &\le \|1 + k^n z\|_r \\ &\times \Big\{ \Big[\|D_{\beta}^{\gamma}P(z) + \alpha m n\|_{\infty} \\ &\quad - \left(1 - \frac{1}{k}\right) |na_0 + \beta a_1 + \alpha m n| \Big] \Big\},\ \text{if } n = 2 \end{aligned} \tag{2.2}$$

Remark 2.2. *For a polynomial of degree 1, simply $P(z) = a_0 + a_1 z$, and hence, we can easily evaluierte $m = \min_{|z|=k} |P(z)| = k|a_1| - |a_0|$ and $\max_{|z|=1} |D_{\beta}^{\gamma}P(z) + \alpha m| = |a_0 + \beta a_1 + \alpha(k|a_1| - |a_0|)|$. So, we need not find their estimates as their exact values are readily obtainable.*

Remark 2.3. *If $\alpha = 0$, inequalities (2.1) and (2.2) of Theorem 2.1 amount to a remarkable result, which following the results*

$$\begin{aligned} \Lambda\big(|\beta| - k\big)\|P(z)\|_r \ &\le \|1 + k^n z\|_r \\ &\times \Big\{ \Big[\|D_{\beta}^{\gamma}P(z)\|_{\infty} \\ &\quad - \left(1 - \frac{1}{k^2}\right) |na_0 + \beta a_1| \Big] \Big\},\ \text{if } n > 2 \end{aligned} \tag{2.3}$$

which is equivalent to

$$\begin{aligned} \Lambda\big(|\beta| - k\big)\|P(z)\|_r \ &\le \|1 + k^n z\|_r \\ &\times \Big\{ \Big[\|D_{\beta}^{\gamma}P(z)\|_{\infty} \\ &\quad - \left(1 - \frac{1}{k}\right) |na_0 + \beta a_1| \Big] \Big\},\ \text{if } n = 2 \end{aligned} \tag{2.4}$$

Remark 2.4. *Dividing both inequalities (2.1) and (2.2)of Theorem 2.1 by $|\beta|$ and allowing $|\beta| \to \infty$ yield*

$$\Lambda\|P(z) + \alpha m\|_r \ \le \|1 + k^n z\|_r$$

$$\times\left[\|P^{\gamma}(z)\|_{\infty}-|a_1|\left(1-\frac{1}{k^2}\right)\right],\ \text{if } n>2 \quad (2.5)$$

and

$$\Lambda\|P(z)+\alpha m\|_r \le \|1+k^n z\|_r \times\left[\|P^{\gamma}(z)\|_{\infty}-|a_1|\left(1-\frac{1}{k}\right)\right],\ \text{if } n=2 \quad (2.6)$$

Remark 2.5. *Putting $\alpha=0$ in inequalities (2.6) and (2.7), the following result follows for polynomials of degree $n\ge 2$.*

$$\Lambda\|P(z)\|_r \le \|1+k^n z\|_r \times\left[\|P^{\gamma}(z)\|_{\infty}-|a_1|\left(1-\frac{1}{k^2}\right)\right],\ \text{if } n>2 \quad (2.7)$$

and

$$\Lambda\|P(z)\|_r \le \|1+k^n z\|_r \times\left[\|P^{\gamma}(z)\|_{\infty}-|a_1|\left(1-\frac{1}{k}\right)\right],\ \text{if } n=2 \quad (2.8)$$

Corollary 2.6. *If $P(z)=\sum_{v=0}^{n}a_v z^v$ is a polynomial of degree $n\ge 2$ having all its zeros in $|z|\le k$, where $k\ge 1$, then for every $r>0$,*

$$\Lambda\|P(z)\|_r \le \|1+k^n z\|_r \times\left[\|P^{\gamma}(z)\|_{\infty}-|a_1|\left(1-\frac{1}{k^2}\right)\right], \quad (2.9)$$

and

$$\Lambda\|P(z)\|_r \le \|1+k^n z\|_r \times\left[\|P^{\gamma}(z)\|_{\infty}-|a_1|\left(1-\frac{1}{k}\right)\right], \quad (2.10)$$

Theorem 2.7. *If $P(z)=\sum_{v=0}^{n}a_v z^v$ is a polynomial of degree $n\ge 2$ having no zeros in $|z|<k$, where $k\le 1$, then for every $r>0$ and for every complex number β with $|\beta|\ge 1$,*

$$k^n\Lambda\left(|\beta|-1\right)\|P(z)\|_r \le \|k^n+z\|_r \times\left[\left(\Lambda|\beta|\|P(z)\|_{\infty}-\|D_{\beta}^{\gamma}P(z)\|\right)-\left(|\beta|-1\right)\left(1-k^2\right)|a_n-1|\right],\ \text{if } n>2 \quad (2.11)$$

and

$$k^n\Lambda\left(|\beta|-1\right)\|P(z)\|_r \le \|k^n+z\|_r \times\left[\left(\Lambda|\beta|\|P(z)\|_{\infty}-\|D_{\beta}^{\gamma}P(z)\|\right)-\left(|\beta|-1\right)\left(1-k\right)|a_n-1|\right],\ \text{if } n=2 \quad (2.12)$$

Theorem 2.8. *If $P(z)=\sum_{v=0}^{n}a_v z^v$ is a polynomial of degree $n\ge 2$ having no zeros in $|z|<k$, $k\le 1$ then for every complex number α and β with $|\alpha|<1$, $|\beta|\ge k$ and for each $r<0$,*

$$k^n\Lambda\left(|\beta|-1\right)\left\|z^nP(e^{i\theta})+\alpha\frac{m}{k^n}\right\|_r \le \|k^n+z\|_r \times\Big\{\Big[\Big(\Lambda|\beta|\|P(z)\|_{\infty}-\|D_{\beta}^{\gamma}P \ldots -\left(|\beta|-1\right)\left(1-k^2\right)|a_n \ldots \quad \text{if } n>2 \quad (2.13)$$

and

$$k^n\Lambda\left(|\beta|-1\right)\left\|z^nP(e^{i\theta})+\alpha\frac{m}{k^n}\right\|_r \le \|k^n+z\|_r \times\Big\{\Big[\Big(\Lambda|\beta|\|p(z)\|_{\infty}-\|D_{\beta}^{\gamma}P(z \ldots -\left(|\beta|-1\right)\left(1-k\right)|a_n-1 \ldots \quad \text{if } n=2 \quad (2.14)$$

3. Lemmas

We require the following lemmas to establish our theorems.

Lemma 3.1 (see in [38]). *Let $P(z)$ is a polynomial of degree n, then for $R\ge 1$*

$$\max_{|z|=R}|P(z)| \le R^n \max_{|z|=1}|P(z)| \quad (3.1)$$

Lemma 3.2 (see in [39]). *Let $P(z)$ is a polynomial of degree n, then for $R>1$,*

$$\max_{|z|=R}|P(z)| \le R^n\max_{|z|=1}|P(z)|-(R^n-R^{n-2})|P(0)| \ for\ n\ge 2 \quad (3.2)$$

and

$$\max_{|z|=R}|P(z)| \le R\max_{|z|=1}|P(z)|-(R-1)|P(0)| \ for\ n=1. \quad (3.3)$$

Lemma 3.3. *Let $P(z)$ be a polynomial of degree n with no zeros inside the disk $|z|<1$. Then, for any $R\ge 1$ and any positive $r>0$*

$$\int_0^{2\pi}|P(Re^{i\theta})|^r d\theta \ge (C_r)^r\int_0^{2\pi}|P(e^{i\theta})|^r d\theta, \quad (3.4)$$

where

$$C_r=\frac{\left(\int_0^{2\pi}|1+R^ne^{i\theta}|^r\,d\theta\right)^{\frac{1}{r}}}{\left(\int_0^{2\pi}|1+e^{i\theta}|\,d\theta\right)^{\frac{1}{r}}}.$$

Lemma 3.4 (see in [40]). *If $P(z)=a\prod_{j=1}^{n}(z-z_j)$ is a polynomial of degree n and $Q(z)=z^n\overline{P\left(\frac{1}{\bar z}\right)}=\bar a\prod_{j=1}^{n}(1-z\bar z_j)$, and $\gamma=(\gamma_1,\gamma_2,\ldots,\gamma_n)$ is an n-tuple of non-negative real numbers with $\gamma\ne 0$, then*

$$z^{n-1}\overline{P^{\gamma}\left(\frac{1}{\bar z}\right)}=\Lambda Q(z)-zQ^{\gamma}(z), \quad (3.5)$$

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and

$$z^{n-1}\overline{Q^{\gamma}\left(\frac{1}{\bar{z}}\right)} = \Lambda P(z) - zP^{\gamma}(z), \tag{3.6}$$

where $P^{\gamma}(z)$ *is defined as in the equation 1.19 and* $Q^{\gamma}(z) = Q(z)\sum_{j=1}^{n}\frac{\gamma_j\overline{z_j}}{z\overline{z_j}-1}$.

Lemma 3.5 (see in [40]). *If all the zeros of polynomial* $P(z)$ *lie in* $|z| \le k, k \ge 1$ *and* $0 \ne \gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n)$, $\gamma_j \ge 0$, $\forall\, 1 \le j \le n$ then for $|z| = 1$,

$$|Q^{\gamma}(z)| \le k|P^{\gamma}(z)|, \tag{3.7}$$

where

$$z^{n-1}\overline{Q^{\gamma}(\frac{1}{\bar{z}})} = P(z) - zP^{\gamma}(z).$$

Lemma 3.6 (see in [41]). *If* $p(z)$ *is a polynomial of degree* n*, then*

$$|P^{\gamma}(z)| + |Q^{\gamma}(z)| \ge \Lambda \max_{|z|=1}|P(z)| \text{ for } |z| = 1 \tag{3.8}$$

4. Proof of main results

Proof of Theorem 2.1. Since $P(z) = \sum_{v=0}^{n} a_v z_v$ has all its zeros $|z| \le k$, $k \ge 1$, then for every complex number α and β with $|\alpha| \le 1$, By Rouche's theorem, the polynomial $R(z) = P(z) + \alpha m$ has all its zeros in $|z| \le k$, $k \ge 1$. Hence, the polynomial $E(z) = R(kz)$ has all its zeros in $|z| \le 1$, and hence the polynomial. $F(z) = z^n E\left(\frac{1}{\bar{z}}\right)$ has all its zeros in $|z| \ge 1$. If z_ν, $\nu = 1, 2, 3, \ldots, n$ are the zeros of $F(z)$, then obviously $|z_\nu| \ge 1$, $1 \le \nu \le n$, and

$$\frac{zF^{\gamma}(z)}{F(z)} = \sum_{v=1}^{n}\frac{z\gamma_j}{z - z_v} \tag{4.1}$$

so that for points $e^{i\theta}, 0 \le \theta < 2\pi$, for which $F(e^{i\theta}) \ne 0$, we have

$$\begin{aligned}\text{Re}\left(\frac{e^{i\theta}F^{\gamma}(e^{i\theta})}{F(e^{i\theta})}\right) &= \sum_{v=1}^{n}\text{Re}\left(\frac{e^{i\theta}\gamma_j}{e^{i\theta} - z_v}\right)\\ &\le \Lambda\frac{n}{2}\end{aligned} \tag{4.2}$$

which gives

$$\left|\frac{e^{i\theta}F^{\gamma}(e^{i\theta})}{\Lambda nF(e^{i\theta})}\right| \le \left|1 - \frac{e^{i\theta}F^{\gamma}(e^{i\theta})}{\Lambda nF(e^{i\theta})}\right| \tag{4.3}$$

for points $e^{i\theta}, 0 \le \theta < 2\pi$, for which $F(e^{i\theta}) \ne 0$. Inequality above (4.3) is equivalent to

$$\left|e^{i\theta}F^{\gamma}(e^{i\theta})\right| \le \left|\Lambda nF(e^{i\theta}) - e^{i\theta}F^{\gamma}(e^{i\theta})\right| \tag{4.4}$$

for points $e^{i\theta}, 0 \le \theta < 2\pi$, for which $F(e^{i\theta}) \ne 0$. Inequality above 6 trivially also holds for the points $e^{i\theta}$, $0 \le \theta < 2\pi$ for which $e^{i\theta} = 0$. Hence, it follows that for $|z| = 1$

$$|F^{\gamma}(z)| \le |\Lambda nF(z) - zF^{\gamma}(z)|. \tag{4.5}$$

Since $E(z)$ has all its zeros in $|z| \le 1$, by Gauss-Lucas theorem $E^{\gamma}(z)$ has all its in $|z| \le 1$, and hence the polynomial from the Lemma

$$z^{n-1}\overline{E^{\gamma}(\frac{1}{\bar{z}})} \le \Lambda F(z) - zF^{\gamma}(z) \tag{4.6}$$

has all its zeros in $|z| \ge 1$. From the above equation (4.6), it follows that the function

$$w(z) = \frac{zF^{\gamma}(z)}{\Lambda F(z) - zF^{\gamma}(z)} \tag{4.7}$$

is analytic in $|z| \le 1$, with $|w(z)| \le 1$ for for $|z| \le 1$ and $W(0) = 0$, hence the function $1 + W(z)$ is subordinate to the function $1 + z$ for $|z| \le 1$. Hence, by a $well - known$ property of subordination [27], we have for every $r > 0$,

$$\int_0^{2\pi}\left|1 + w(e^{i\theta})\right|^r d\theta \le \int_0^{2\pi}\left|1 + e^{i\theta}\right|^r d\theta. \tag{4.8}$$

Now

$$1 + w(z) = \frac{\Lambda F(z)}{\Lambda F(z) - zF^{\gamma}(z)} \tag{4.9}$$

For $|z| = 1$, we have from equation (4.6)

$$|E^{\gamma}(z) \le |z^{n-1}\overline{E^{\gamma}(\frac{1}{\bar{z}})}| = |\Lambda F(z) - zF^{\gamma}(z)| \tag{4.10}$$

For $|z| = 1$, relation (4.9) gives, on using equation (4.10),

$$\begin{aligned}\Lambda|F(z)| &\le |1 + w(z)||\Lambda F(z) - zF^{\gamma}(z)|\\ &= |1 + w(z)|\max_{|z|=1}|E^{\gamma}(z)|\\ &= k|1 + w(z)|\max_{|z|=k}|E^{\gamma}(z)|\end{aligned} \tag{4.11}$$

combining inequality (4.8) and inequality (4.10), we have for every $r > 0$

$$\Lambda^r\int_0^{2\pi}\left|F(e^{i\theta})\right|^r d\theta \le k^r\left(\int_0^{2\pi}\left|1 + e^{i\theta}\right|^r d\theta.\right)\left(\max_{|z|=k}|R'(z)\right)^r \tag{4.12}$$

Using inequality (3.4) of Lemma 3.3 to $F(z)$, we get for each $k \ge 1$ and every $r > 0$

$$\int_0^{2\pi}\left|F(ke^{i\theta})\right|^r d\theta \le (C_r)^r\left(\int_0^{2\pi}\left|F(e^{i\theta})\right|^r d\theta\right) \tag{4.13}$$

where $C_r = \frac{\left(\int_0^{2\pi}|1+k^n e^{i\theta}|^r d\theta\right)^{\frac{1}{r}}}{\left(\int_0^{2\pi}|1+e^{i\theta}|^r d\theta\right)^{\frac{1}{r}}}$ since

$$\begin{aligned}|F(z)| &\le z^n\overline{E\left(\frac{1}{\bar{z}}\right)}\\ &= z^n\overline{R\left(\frac{k}{\bar{z}}\right)}\\ &= z^n\left(\overline{R\left(\frac{k}{\bar{z}}\right)} + \bar{\alpha}m\right),\end{aligned}$$

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we have for $0 \le \theta < 2\pi$ and

$$|F(ke^{i\theta})| = \left|k^n e^{i\theta}\left\{\overline{P(e^{i\theta}) + \overline{\alpha} m}\right\}\right| \tag{4.14}$$

From (4.12), (4.13) and (4.14), it follows that for every $r > 0$

$$\Lambda^n k^{nr} \int_0^{2\pi} \left|P(e^{i\theta}) + \alpha m\right|^r d\theta \le \Lambda^r (C_r)^r \int_0^{2\pi} |F(e^{i\theta})|^r d\theta$$
$$= k^r \int_0^{2\pi} |1 + k^n e^{i\theta}|^r d\theta \left(\max_{|z|=k} |R'(z)|\right)^r \tag{4.15}$$

Applying Lemma 3.4 to $E(z)$, we have for $|z| = 1$

$$|F^\gamma(z)| \le |E^\gamma(z)|. \tag{4.16}$$

Now, using inequality (1.20), we have for $\left|\frac{\beta}{k}\right| \ge 1$ and $|z| = 1$

$$\begin{aligned}\left|D^\gamma_{\frac{\beta}{k}}\right| &= \left|\Lambda E(z) + \left(\frac{\beta}{k}\right) - zE^\gamma(z)\right| \\ &\ge \left|\frac{\beta}{k}\right| |E^\gamma(z)| - |\Lambda E(z) - E^\gamma(z)| \\ &\ge \left|\frac{\beta}{k}\right| |E^\gamma(z)| - |F^\gamma(z)| \\ &\ge \left(\left|\frac{\beta}{k}\right| - 1\right) E^\gamma(z)\end{aligned} \tag{4.17}$$

Because $(F^\gamma(z) = \Lambda E(z) - zE^\gamma \; for \; |z| = 1)$. Replacing $E(z)$ by $R(kz)$ in (4.17), we get

$$\left|\Lambda R(kz) + \left(\frac{\beta}{k} kR'(kz)\right)\right| \ge (|\beta| - k)\, R'(kz)$$

which is equivalent to

$$|\Lambda R(kz) + (\beta - kz)\, R'(kz)| \ge (|\beta| - k)\, |R'(kz)|,$$

so that we obtain

$$\max_{|z|=1} |D^\gamma_\beta R(kz)| \ge (|\beta| - k)\, |R'(kz)|,$$

which is equivalent to

$$\max_{|z|=k} |D^\gamma_\beta R(z)| \ge (|\beta| - k)\, |R'(z)|, \tag{4.18}$$

Applying Inequality above (4.18) to inequality (4.15)

$$\Lambda^r k^{nr} (|\beta| - k)^r \int_0^{2\pi} |P(e^{i\theta}) + \alpha m|^r d\theta \le k^r \int_0^{2\pi} |1 + k^n e^{i\theta}|^r d\theta \times \left(\max_{|z|=k} |D^\gamma_\beta R(z)|\right)^r.$$

Since $D^\gamma_\beta R(z) = D^\gamma_\beta P(z) + \alpha m n$ is a polynomial of degree $n{-}1 \ge 1$, then using inequality (3.2) of Lemma 3.2, we have

$$\begin{aligned}\max_{|z|=1} |D^\gamma_\beta R(z)| &\le \Big\{k^{n-1} \max_{|z|=1} |D^\gamma_\beta R(z) + \alpha m n| \\ &\quad - \left(k^{n-1} - k^{n-3}\right) |D^\gamma_\beta P(0) + \alpha m n|\Big\} \\ &= k^{n-1}\Big\{\max_{|z|=1} |D^\gamma_\beta R(z) + \alpha m n| \\ &\quad - \left(1 - \frac{1}{k^2}\right) |na_0 + \beta a_1 + \alpha m n|\Big\}\end{aligned} \tag{4.19}$$

Applying inequality (4.19) to inequality (4.15)

$$\Lambda^r k^{nr} (|\beta| - k)^r \int_0^{2\pi} |P(z) + \alpha m|^r d\theta \le k^r \int_0^{2\pi} |1 + k^n z|^r d\theta \times \Big\{k^{n-1}\Big[\max_{|z|=1} |D^\gamma_\beta P(z) + \alpha \ldots - \left(1 - \frac{1}{k^2}\right) |na_0 + \beta \ldots \tag{4.20}$$

which is equivalent to

$$\Lambda(|\beta| - k)\left(\int_0^{2\pi} |P(z) + \alpha m|^r d\theta\right)^{\frac{1}{r}} \le \left(\int_0^{2\pi} |1 + k^n z|^r d\theta\right)^{\frac{1}{r}} \times \Big\{\max_{|z|=1} |D^\gamma_\beta P(z) + \alpha m n| - \left(1 - \frac{1}{k^2}\right) |na_0 + \beta a \ldots \tag{4.21}$$

$\boxdot$

Proof of Theorem 2.8. If we let $P(z) = \sum_{v=0}^n a_v z_v$ be a polynomial of degree $n \ge 2$ having no zero in $|z| < k$, $k \le 1$. then, $Q(z) = z^n \overline{P\left(\frac{1}{\overline{z}}\right)}$ has all its zeros in $|z| \le \frac{1}{k}$, $\frac{1}{k} \ge 1$. If

$$\begin{aligned}m' = \min_{|z|=\frac{1}{k}} |q(z)| &= \frac{1}{k^n} \min_{|z|=k} |P(z)| \\ &= \frac{m}{k^n},\end{aligned} \tag{4.22}$$

or every real or complex number α with $|\alpha| < 1$, by Rouche's theorem, the polynomial

$$\begin{aligned}Q(z) &= q(z) + \alpha m' \\ &= q(z) + \alpha \frac{m}{k^n}\end{aligned} \tag{4.23}$$

where $m = \min_{|z|=k} |P(z)|$ and all zeros of $P(z)$ lie in $|z| \le k_1$, $k_1 \ge 1$.

Applying Corollary (2.6) to the polynomial $Q(z)$, we have for every $r > 0$

$$\Lambda\left(\int_0^{2\pi} \left|Q(e^{i\theta})\right|^r d\theta\right)^{1/r} \le \left(\int_0^{2\pi} \left|1 + \frac{e^{i\theta}}{k^n}\right|^r d\theta\right)^{1/r} \times \left[\max_{|z|=1} |Q^\gamma(z)| - \left(1 - \frac{1}{(1/k)^2}\right) |a_n - \ldots\right. \tag{4.24}$$

and

$$\Lambda\left(\int_0^{2\pi} \left|Q(e^{i\theta})\right|^r d\theta\right)^{1/r} \le \left(\int_0^{2\pi} \left|1 + \frac{e^{i\theta}}{k^n}\right|^r d\theta\right)^{1/r}$$

$$\times \left[\max_{|z|=1}|Q^{\gamma}(z)| - \left(1 - \frac{1}{(1/k)}\right)|a_n - 1|\right], \text{ if } n = 2, \tag{4.25}$$

which is equivalent to

$$k^n\Lambda\left(\int_0^{2\pi}\left|Q(e^{i\theta}) + \alpha\frac{m}{k^n}\right|^r d\theta\right)^{1/r} \leq \left(\int_0^{2\pi}\left|k^n + e^{i\theta}\right|^r d\theta\right)^{1/r} \times \left[\max_{|z|=1}|Q^{\gamma}(z)| - \left(1 - k^2\right)|a_n - 1|\right], \text{ if } n > 2 \tag{4.26}$$

and

$$k^n\Lambda\left(\int_0^{2\pi}\left|Q(e^{i\theta}) + \alpha\frac{m}{k^n}\right|^r d\theta\right)^{1/r} \leq \left(\int_0^{2\pi}\left|k^n + e^{i\theta}\right|^r d\theta\right)^{1/r} \times \left[\max_{|z|=1}|Q^{\gamma}(z)| - (1 - k)|a_n - 1|\right], \text{ if } n = 2 \tag{4.27}$$

by Lemma 3.6, we have for $|z| = 1$

$$|P^{\gamma}(z)| + |Q^{\gamma}(z)| \leq n\max_{|z|=1}|P(z)|. \tag{4.28}$$

Since $|P^{\gamma}(z)|$ and $|Q^{\gamma}(z)|$ attains their maxima at the same point on $|z| = 1$, let z_0 on $|z| = 1$ be such that $\max_{|z|=1}|Q^{\gamma}(z)| = |Q^{\gamma}(z_0)|$ then

$$\max_{|z|=1}|P^{\gamma}(z)| = |P^{\gamma}(z_0)|$$

Now, in particular 4.28 gives

$$|P^{\gamma}(z_0)| + |Q^{\gamma}(z_0)| \leq \Lambda\max_{|z|=1}|P(z)|,$$

which implies

$$\max_{|z|=1}|Q^{\gamma}(z)| \leq \Lambda\max_{|z|=1}|P(z)| - \max_{|z|=1}|P^{\gamma}(z)|. \tag{4.29}$$

Applying inequality (4.29) to (4.26) and (4.27) we have,

$$k^n\Lambda\left(\int_0^{2\pi}\left|Q(e^{i\theta}) + \alpha\frac{m}{k^n}\right|^r d\theta\right)^{1/r} \leq \left(\int_0^{2\pi}\left|k^n + e^{i\theta}\right|^r d\theta\right)^{1/r} \times \left[\left(\Lambda\max_{|z|=1}|P(z)| - \max_{|z|=1}|P^{\gamma}(z)|\right) - \left(1 - k^2\right)|a_n - 1|\right], \text{ if } n > 2 \tag{4.30}$$

and

$$k^n\Lambda\left(\int_0^{2\pi}\left|Q(e^{i\theta}) + \alpha\frac{m}{k^n}\right|^r d\theta\right)^{1/r} \leq \left(\int_0^{2\pi}\left|k^n + e^{i\theta}\right|^r d\theta\right) \times \left\{\left[\left(\Lambda\max_{|z|=1}|P(z)| - \max_{|z|=1}|P^{\gamma}_0(z)|\right) - (1 - k)|a_n - 1|\right],\right\} \text{ if } n = 2. \tag{4.31}$$

From $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$, we have

$$Q(e^{i\theta}) = e^{ni\theta}\overline{P(e^{i\theta})}. \tag{4.32}$$

On using equation (4.32) to (4.30) and (4.31)

$$k^n\Lambda\left(\int_0^{2\pi}\left|e^{ni\theta}P(e^{i\theta}) + \alpha\frac{m}{k^n}\right|^r d\theta\right)^{1/r} \leq \left(\int_0^{2\pi}\left|k^n + e^{i\theta}\right|^r d\theta\right)^{1} \times \left\{\left[\left(\Lambda\max_{|z|=1}|P(z)| - \mathrm{ma}\right.\right.\right. - \left(1 - k^2\right)|a_n - 1|\Big], \text{ if } n > 2 \tag{4.33}$$

and

$$k^n\Lambda\left(\int_0^{2\pi}\left|e^{ni\theta}P(e^{i\theta}) + \alpha\frac{m}{k^n}\right|^r d\theta\right)^{1/r} \leq \left(\int_0^{2\pi}\left|k^n + e^{i\theta}\right|^r d\theta\right)^{1} \times \left\{\left[\left(\Lambda\max_{|z|=1}|P(z)| - \mathrm{ma}\right.\right.\right. - (1 - k)|a_n - 1|\Big]\Big\} \text{ if } n = 2. \tag{4.34}$$

Using Lemma 3.3, we have for $|\beta| \geq 1$ and $|z| = 1$

$$\begin{aligned}|D^{\gamma}_{\beta}P(z)| &= |\Lambda P(z) + (\beta - z)p^{\gamma}(z)|\\ &\leq |\Lambda P(z) - zP^{\gamma}(z)| + |\beta||p^{\gamma}(z)|\\ &= |Q^{\gamma}(z)| + |\beta||P^{\gamma}(z)| \text{ where } Q^{\gamma}(z) = \Lambda P(z) - zP^{\gamma}(z\\ &= |Q^{\gamma}(z)| + |Q^{\gamma}(z)| - |Q^{\gamma}(z)| + |\beta||P^{\gamma}(z)| \text{ for } |z| = 1\\ &\leq \Lambda\max_{|z|=1}|P(z)| + (|\beta| - 1)|P^{\gamma}(z)|\\ &\leq \Lambda\max_{|z|=1}|P(z)| + (|\beta| - 1)\max_{|z|=1}|P^{\gamma}(z)|.\end{aligned} \tag{4.35}$$

Using inequality (4.35) in inequality (4.33) and (4.34)

$$k^n\Lambda(|\beta| - 1)\left(\int_0^{2\pi}\left|e^{ni\theta}P(e^{i\theta}) + \alpha\frac{m}{k^n}\right|^r d\theta\right)^{1/r} \leq \left(\int_0^{2\pi}\left|k^n + e^{i\theta}\right|^r d\theta\right)^{1/r} \cdot \left\{\left[\Lambda|\beta|\max_{|z|=1}|P(z)| - \max_{|z|=1}|D^{\gamma}_{\beta}P(z)| - (|\beta| - 1)(1 - k^2)|a_n -\right.\right. \text{ if } n > 2. \tag{4.36}$$

and

$$k^n\Lambda(|\beta| - 1)\left(\int_0^{2\pi}\left|e^{ni\theta}P(e^{i\theta}) + \alpha\frac{m}{k^n}\right|^r d\theta\right)^{1/r} \leq \left(\int_0^{2\pi}\left|k^n + e^{i\theta}\right|^r d\theta\right)^{1/r} \cdot \left\{\left[\Lambda|\beta|\max_{|z|=1}|P(z)| - \max_{|z|=1}|D^{\gamma}_{\beta}P(z)| - (|\beta| - 1)(1 - k)|a_n -\right.\right. \text{ if } n = 2. \tag{4.37}$$

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⊡

Proof of Theorem 2.7. The proof of this theorem follows on the same lines as that of Theorem 2.8 but instead of applying Corollary 2.6 to $Q(z)$ given by (4.23), we simply apply the same lemma to $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$, and we omit it. ⊡

5. Conclusion

This study focuses on deriving integral mean estimates for certain well-known Turán-type and Erdős-Lax-type inequalities related to the Generalized polar derivative of polynomials. The findings lead to a series of improved inequalities that surpass those previously established in the extensive literature on the topic.

Declarations

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Authors contributions
S.A., A.U. and R.S. contributed to the conception and design of the study. S.A. developed the main theorems and carried out the mathematical analysis. A.U. and R.S. contributed to the lemmas, verification of the results and the literature review. All authors participated in drafting and revising the manuscript, and approved the final version for publication.

Availability of data and materials
The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflict of interests
The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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