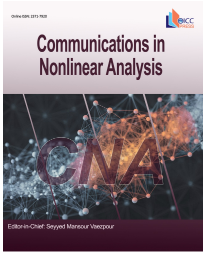

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Research Article

Fixed Point Theorems for Generalized F-Contractions of Rational Type via Iterative Set Methods in Complete Metric Spaces

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Abstract:

In this paper, we investigate a new class of F-contraction mappings defined on complete metric spaces by applying iterative set techniques. Our study focuses on introducing rational-type contractive conditions that extend and generalize several existing approaches in the literature. To ensure clarity and accessibility, we provide precise definitions, detailed step-by-step arguments, and illustrative examples that highlight the main concepts and results. The presentation is designed to make the proofs easier to follow, even for readers who are new to this area, while also offering deeper mathematical insights. Furthermore, the findings reveal potential directions for extending fixed point theory, particularly by formulating new contraction principles and exploring broader applications in mathematical analysis. This work not only contributes to the theoretical development of fixed point results but also emphasizes their relevance for future research in related fields.

Keywords: Fixed point theorems; Iterative set methods; Generalized F-contraction of rational type; Metric space; Complete metric space. MSC 2020: 47H10, 54H25

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1. Introduction

In 1922, Banach presented the fixed point theorem, which later became widely known as the Banach contraction principle [1]. This result has played a central role in mathematical analysis because of its clarity and importance. Since its introduction, many researchers have generalized the principle and extended, thereby broadening its applications. For example, in 1973, Geraghty [2] studied a modified form of the Banach contraction principle, while Ćirić [3] proposed the notion of quasi-contraction, which served as a more general framework. These contributions opened the door to a wide range of extensions that further enhanced the scope of Banach's idea (see [1; 2; 3]).

The study of metric spaces has also gained importance in various disciplines, including medicine, biology, physics, and computer science (see [4; 5]). Over time, several alternative types of metric structures have been introduced, such as semi-metric spaces, rectangular metric spaces, pseudo-metric spaces, probabilistic metric spaces, and fuzzy metric spaces. Within this setting, numerous generalizations have been explored, including cone metric spaces, quasi-metric spaces, quasi-semi-metric spaces, and D-metric spaces (see [6; 7; 8; 9; 10]). A major development occurred when Branciari introduced the generalized metric space in [11], in which the classical triangle inequality was replaced with a rectangular-type inequality. This formulation allowed Banach's contraction principle to be extended to these new spaces. Later, in 2012, Wardowski [12] introduced the concept of F-contraction, leading to a new fixed point theorem in this context. This innovation significantly broadened the scope of Banach's principle. Building on this, Secelean [13] established fixed point results for F-contractions in the setting of iterated function systems, and Piri et al. [14] proved a fixed point result for F-Suzuki contractions under less restrictive conditions for the self-map of a complete metric space, further extending Wardowski's groundbreaking work.

2. Preliminaries

Definition 2.1. *[14] Let $\mathscr{F}$ be the family of contraction mappings and $\mathcal{F} \in \mathscr{F}$. Let $\Phi : X \to X$ be a self-map on a complete metric space (X, δ). The mapping Φ is called an $\mathcal{F}$-contraction if there exists a constant $\tau > 0$, such that*

$$\forall \omega, \vartheta \in X, \quad \delta(\Phi(\omega), \Phi(\vartheta)) > 0 \Rightarrow \tau + \mathcal{F}(\delta(\Phi(\omega), \Phi(\vartheta))) \leq \mathcal{F}(\delta(\omega, \vartheta)). \tag{2.1}$$

Here, the function $\mathcal{F} : \mathbb{R}_+ \to \mathbb{R}$ is assumed to satisfy the following conditions:
(B1) $\mathcal{F}$ is strictly increasing, i.e., for any $\alpha, \beta \in \mathbb{R}_+$, $\alpha < \beta \Leftrightarrow \mathcal{F}(\alpha) < \mathcal{F}(\beta)$.
(B2) For every sequence $\{\alpha_n\} \subset \mathbb{R}^+$ with $\lim_{n\to\infty} \alpha_n = 0$, we have $\lim_{n\to\infty} \mathcal{F}(\alpha_n) = -\infty$.
(B3) There exists $\epsilon_0 > 0$, such that $\lim_{\alpha\to 0^+} \alpha^{\epsilon_0}\mathcal{F}(\alpha) = 0$.

Definition 2.2. *[15] Let $\Phi : X \to X$ be a self-map defined on a metric space (X, δ). Then Φ is called a generalized $\mathcal{F}$-contraction of rational type (A) if there exists a constant $\tau > 0$, such that*

$$\forall \omega, \vartheta \in X, \quad \delta(\Phi(\omega), \Phi(\vartheta)) > 0 \Rightarrow \tau + \mathcal{F}(\delta(\Phi(\omega), \Phi(\vartheta))) \leq \mathcal{F}(\mathcal{M}(\omega, \vartheta)), \tag{2.2}$$

where

$$\mathcal{M}(\omega, \vartheta) = \max\left\{\delta(\omega, \vartheta),\ \delta(\omega, \Phi(\omega)),\ \delta(\vartheta, \Phi(\vartheta)), \left(\frac{\delta(\omega, \Phi(\vartheta)) + \delta(\vartheta, \Phi(\omega))}{1 + \delta(\omega, \Phi(\omega)) + \delta(\vartheta, \Phi(\vartheta))}\right)\delta(\omega, \Phi(\omega))\right\}.$$

Definition 2.3. *[15] Let $\Phi : X \to X$ be a self-map defined on a metric space (X, δ). Then Φ is called a generalized an $\mathcal{F}$-contraction of rational type (B) if there exists a constant $\tau > 0$, such that*

$$\forall \omega, \vartheta \in X, \quad \delta(\Phi(\omega), \Phi(\vartheta)) > 0 \Rightarrow \tau + \mathcal{F}(\delta(\Phi(\omega), \Phi(\vartheta))) \leq \mathcal{F}(\mathcal{M}(\omega, \vartheta)), \tag{2.3}$$

where

$$\mathcal{M}(\omega, \vartheta) = \max\left\{\delta(\omega, \vartheta),\ \delta(\omega, \Phi(\omega)),\ \delta(\vartheta, \Phi(\vartheta)), \left(\frac{\delta(\omega, \Phi(\vartheta)) + \delta(\vartheta, \Phi(\omega))}{1 + \delta(\omega, \Phi(\omega)) + \delta(\vartheta, \Phi(\vartheta))}\right)\delta(\vartheta, \Phi(\vartheta))\right\}.$$

3. Main Result

In this section, we establish new rational-type conditions for a generalized $\mathcal{F}$-contraction mapping on a complete metric space by employing a novel class of iterative sets. Before presenting our main concepts, we first consider the following inequality:

$$\frac{\delta(\omega, \vartheta) + \delta(\Phi\omega, \Phi\vartheta)}{1 + \delta(\vartheta, \Phi\vartheta)} \leq 1.$$

By expressing $\delta(\Phi\omega, \Phi\vartheta)$ as $\delta(\Phi\omega, \vartheta) + \delta(\vartheta, \Phi\vartheta)$ (where ϑ lies between $\Phi\omega$ and $\Phi\vartheta$), we rewrite the inequality as:

$$\frac{\delta(\omega, \vartheta) + [\delta(\Phi\omega, \vartheta) + \delta(\vartheta, \Phi\vartheta)]}{1 + \delta(\vartheta, \Phi\vartheta)} \leq 1.$$

Since $\delta(\Phi\omega, \vartheta) = \delta(\vartheta, \Phi\omega)$ this further simplifies to:

$$\frac{\delta(\omega, \vartheta) + [\delta(\vartheta, \Phi\omega) + \delta(\vartheta, \Phi\vartheta)]}{1 + \delta(\vartheta, \Phi\vartheta)} \leq 1.$$

Rearranging terms, we obtain:

$$\frac{[\delta(\omega, \vartheta) + \delta(\vartheta, \Phi\omega)] + \delta(\vartheta, \Phi\vartheta)}{1 + \delta(\vartheta, \Phi\vartheta)} \leq 1.$$

Using the property $\delta(\omega, \vartheta) + \delta(\vartheta, \Phi\omega) = \delta(\omega, \Phi\omega)$ the inequality becomes:

$$\frac{\delta(\omega, \Phi\omega) + \delta(\vartheta, \Phi\vartheta)}{1 + \delta(\vartheta, \Phi\vartheta)} \leq 1.$$

Multiplying both sides by $1 + \delta(\vartheta, \Phi\vartheta)$ yields:

$$\delta(\omega, \Phi\omega) + \delta(\vartheta, \Phi\vartheta) \leq 1 + \delta(\vartheta, \Phi\vartheta).$$

Subtracting $\delta(\vartheta, \Phi\vartheta)$ from both sides, we arrive at:

$$\delta(\omega, \Phi\omega) \leq 1.$$

Similarly, if we consider $\Phi\omega$ lying between ω and ϑ in $\delta(\omega, \vartheta)$, we obtain the same result:

$$\delta(\omega, \Phi\omega) \leq 1.$$

From this, we conclude that we can replace $\dfrac{\delta(\omega, \vartheta) + \delta(\Phi\omega, \Phi\vartheta)}{1 + \delta(\vartheta, \Phi\vartheta)}$ by $\delta(\omega, \Phi\omega)$.
In a similar way, we can replace $\dfrac{\delta(\omega, \vartheta) + \delta(\Phi\omega, \Phi\vartheta)}{1 + \delta(\omega, \Phi\omega)}$ by $\delta(\vartheta, \Phi\vartheta)$

Definition 3.1. *Let a self-map $\Phi : X \to X$ be defined on a metric space (X, δ), then Φ is called a generalized $\mathcal{F}$-contraction of rational type (I) if $\exists\, \tau > 0$, such that*

$$\forall \omega, \vartheta \in X,\ \delta(\Phi(\omega), \Phi(\vartheta)) > 0 \Rightarrow \tau + \mathcal{F}(\delta(\Phi(\omega), \Phi(\vartheta))) \leq \mathcal{F}(\mathcal{M}(\omega, \vartheta)), \tag{3.1}$$

where

$$\mathcal{M}(\omega, \vartheta) = \max\left\{\delta(\omega, \vartheta),\ \delta(\omega, \Phi(\omega)),\ \delta(\vartheta, \Phi(\vartheta)), \frac{\delta(\omega, \vartheta) + \delta(\Phi(\omega), \Phi(\vartheta))}{1 + \delta(\omega, \Phi(\omega))}\right\}.$$

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Theorem 3.2. *Let (X, δ) be a complete metric space. Suppose that $\Phi : X \to X$ is a generalized $\mathcal{F}$-contraction of rational type (I). If either $\mathcal{F}$ or Φ is continuous, then Φ has a fixed point.*

Proof. Let $\omega_0 \in X$ be arbitrary and define the sequence $\{\omega_n\}$ by $\omega_{n+1} = \Phi(\omega_n)$, $n \in \mathbb{N}$. If there exists n such that $\delta(\omega_n, \Phi(\omega_n)) = 0$, then ω_n is a fixed point and we are done.

Assume $\delta(\omega_n, \Phi(\omega_n)) > 0$ for all n. Then, by the definition of generalized $\mathcal{F}$-contraction, there exists $\tau > 0$, such that

$$\begin{aligned}\tau + \mathcal{F}(\delta(\Phi(\omega_{n-1}), \Phi(\omega_n)))\\ \leq \mathcal{F}(\mathcal{M}(\omega_{n-1}, \omega_n)),\end{aligned}$$

where

$$\begin{aligned}\mathcal{M}(\omega_{n-1}, \omega_n) = \max\Big\{&\delta(\omega_{n-1}, \omega_n),\\ &\delta(\omega_{n-1}, \Phi(\omega_{n-1})),\ \delta(\omega_n, \Phi(\omega_n)),\\ &\frac{\delta(\omega_{n-1}, \omega_n) + \delta(\Phi(\omega_{n-1}), \Phi(\omega_n))}{1 + \delta(\omega_{n-1}, \Phi(\omega_{n-1}))}\Big\}.\end{aligned}$$

A simple calculation shows that

$$\mathcal{M}(\omega_{n-1}, \omega_n) = \max\{\delta(\omega_{n-1}, \omega_n),\ \delta(\omega_n, \omega_{n+1})\}.$$

From this we obtain the recursive inequality

$$\mathcal{F}(\delta(\omega_n, \omega_{n+1})) \leq \mathcal{F}(\delta(\omega_{n-1}, \omega_n)) - \tau,$$

and consequently

$$\mathcal{F}(\delta(\omega_n, \omega_{n+1})) \leq \mathcal{F}(\delta(\omega_0, \omega_1)) - n\tau.$$

Hence,

$$\begin{aligned}&\lim_{n\to\infty} \mathcal{F}(\delta(\omega_n, \omega_{n+1})) = -\infty,\\ &\text{and thus } \lim_{n\to\infty} \delta(\omega_n, \omega_{n+1}) = 0.\end{aligned}$$

From condition (B3) there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} n(\delta(\omega_n, \omega_{n+1}))^k = 0.$$

Therefore, for sufficiently large n,

$$\delta(\omega_n, \omega_{n+1}) \leq \frac{1}{n^{1/k}}.$$

By the triangle inequality, for $m > n$ we have

$$\begin{aligned}\delta(\omega_n, \omega_m) &\leq \sum_{i=n}^{m-1} \delta(\omega_i, \omega_{i+1})\\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}.\end{aligned}$$

Since $k \in (0, 1)$, we have $1/k > 1$ and hence the series converges. Thus $\{\omega_n\}$ is a Cauchy sequence, and completeness of X ensures that $\omega_n \to \omega^* \in X$.

If Φ is continuous, then

$$\begin{aligned}\Phi(\omega^*) &= \lim_{n\to\infty} \Phi(\omega_n) = \lim_{n\to\infty} \omega_{n+1}\\ &= \omega^*,\end{aligned}$$

so ω^* is a fixed point.

If instead $\mathcal{F}$ is continuous, suppose for contradiction that $\Phi(\omega^*) \neq \omega^*$. By applying the contractive condition to the subsequence $\{\omega_{n_k}\}$ and passing to the limit, one arrives at

$$\begin{aligned}\tau + \mathcal{F}(\delta(\omega^*, \Phi(\omega^*)))\\ \leq \mathcal{F}(\delta(\omega^*, \Phi(\omega^*))),\end{aligned}$$

a contradiction. Hence $\Phi(\omega^*) = \omega^*$, and Φ has a fixed point. ⊡

Example 3.3. *Let $\omega = [0, 1]$ and (X, δ) be a complete metric space, such that $\delta(\omega, \vartheta) = \max\{\omega, \vartheta\}$, $\forall \omega, \vartheta \in X$ a self-map $\Phi : X \to X$, is define by:*

$$\Phi(\omega) = \begin{cases} \dfrac{\omega}{2}, & \text{if } \omega \in [0, 1),\\ 0, & \text{if } \omega = 1.\end{cases}$$

Define a function $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(\alpha) = \ln(\alpha)$, $\forall\, \alpha \in \mathbb{R}^+$. Now, we prove that the condition in (3.1) is satisfy:
$\delta(\Phi(\omega), \Phi(\vartheta)) > 0,\ \forall\, \omega, \vartheta \in X \Rightarrow \tau + F(\delta(\Phi(\omega), \delta(\vartheta))) \leq F(\mathcal{M}(\omega, \vartheta))$,
where

$$\begin{aligned}\mathcal{M}(\omega, \vartheta) = \max\Big\{&\delta(\omega, \vartheta),\ \delta(\omega, \Phi(\omega)),\ \delta(\vartheta, \Phi(\vartheta)),\\ &\frac{\delta(\omega, \vartheta) + \delta(\Phi(\omega), \Phi(\vartheta))}{1 + \delta(\omega, \Phi(\omega))}\Big\}\\ = \max&\{\delta(\omega, \vartheta), \delta(\omega, \Phi(\omega)), \delta(\vartheta, \Phi(\vartheta))\}\\ = \delta&(\omega, \vartheta).\end{aligned}$$

Now,

$$\begin{aligned}\tau + F(\delta(\Phi(\omega), \Phi(\vartheta))) &= \tau + F\left(\delta\left(\frac{\omega}{2}, \frac{\vartheta}{2}\right)\right)\\ &= \tau + F\left(\max\left\{\frac{\omega}{2}, \frac{\vartheta}{2}\right\}\right).\end{aligned} \tag{3.2}$$

We choosing $\tau \leq \ln(2)$, there are two cases:

Case (1): if $\omega < \vartheta$ then by (3.2)

$$\begin{aligned}\tau + F\left(\frac{\vartheta}{2}\right) &\leq \ln(2) + \ln\left(\frac{\vartheta}{2}\right)\\ &= \ln\left(2 \cdot \frac{\vartheta}{2}\right)\\ &= \ln(\delta(\omega, \vartheta))\\ &= F(\delta(\omega, \vartheta)).\end{aligned}$$

Case (2): if $\omega > \vartheta$ then, by (3.2)

$$\tau + F\left(\frac{\omega}{2}\right) \leq \ln(2) + \ln\left(\frac{\omega}{2}\right)$$

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$$\begin{aligned}&= \ln\left(2 \cdot \frac{\omega}{2}\right)\\ &= \ln(\delta(\omega.\vartheta))\\ &= F(\delta(\omega,\vartheta))\end{aligned}$$

From two cases, we conclude that

$$\tau + F(\delta(\Phi(\omega),\Phi(\vartheta))) \leq F(\delta(\omega,\vartheta)).$$

Definition 3.4. *Let a self-map $\Phi : X \to X$ be defined on a metric space (X,δ), then Φ is called a generalized $\mathcal{F}$-contraction of rational type (II) if $\exists\, \tau > 0$, such that*

$$\begin{aligned}&\forall\, \omega,\vartheta \in X,\ \delta(\Phi(\omega),\Phi(\vartheta)) > 0\\ &\quad \Rightarrow \tau + \mathcal{F}(\delta(\Phi(\omega),\Phi(\vartheta))) \leq \mathcal{F}(\mathcal{M}(\omega,\vartheta)),\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}\mathcal{M}(\omega,\vartheta) = \max\Bigg\{&\delta(\omega,\vartheta),\ \delta(\omega,\Phi(\omega)),\ \delta(\vartheta,\Phi(\vartheta)),\\ &\frac{\delta(\omega,\vartheta)+\delta(\Phi(\omega),\Phi(\vartheta))}{1+\delta(\vartheta,\Phi(\vartheta))}\Bigg\}.\end{aligned} \tag{3.4}$$

Theorem 3.5. *Let (X,δ) be a complete metric space and let $\Phi : X \to X$ be a generalized $\mathcal{F}$-contraction of rational type (I). Assume conditions* (B2) *and* (B3) *hold. Then Φ has a unique fixed point in X.*

Proof. Choose $\omega_0 \in X$ and define a sequence $\{\omega_n\}$ by $\omega_{n+1} = \Phi(\omega_n),\ n \geq 0$.
If $\omega_n = \Phi(\omega_n)$ for some n, then ω_n is a fixed point. Otherwise, assume $\omega_n \neq \Phi(\omega_n)$ for all n. For $n \geq 1$, set

$$\begin{aligned}\mathcal{M}(\omega_{n-1},\omega_n) = \max\Bigg\{&\delta(\omega_{n-1},\omega_n),\ \delta(\omega_n,\omega_{n+1}),\\ &\delta(\omega_{n-1},\omega_{n+1}),\\ &\frac{\delta(\omega_{n-1},\omega_n)+\delta(\omega_n,\omega_{n+1})}{1+\delta(\omega_{n-1},\omega_n)}\Bigg\}.\end{aligned}$$

Since the last two terms are bounded above by $\max\{\delta(\omega_{n-1},\omega_n),\ \delta(\omega_n,\omega_{n+1})\}$, we deduce

$$\mathcal{M}(\omega_{n-1},\omega_n) = \max\{\delta(\omega_{n-1},\omega_n),\ \delta(\omega_n,\omega_{n+1})\}.$$

From the generalized $\mathcal{F}$-contractive condition, there exists $\tau > 0$, such that

$$\begin{aligned}&\tau + \mathcal{F}(\delta(\omega_n,\omega_{n+1}))\\ &\quad \leq \mathcal{F}(\mathcal{M}(\omega_{n-1},\omega_n)).\end{aligned}$$

If $\mathcal{M}(\omega_{n-1},\omega_n) = \delta(\omega_n,\omega_{n+1})$, then $\tau \leq 0$, a contradiction. Hence

$$\mathcal{M}(\omega_{n-1},\omega_n) = \delta(\omega_{n-1},\omega_n),$$

and therefore

$$\begin{aligned}&\tau + \mathcal{F}(\delta(\omega_n,\omega_{n+1}))\\ &\quad \leq \mathcal{F}(\delta(\omega_{n-1},\omega_n)).\end{aligned}$$

It follows that the sequence $\{\delta(\omega_n,\omega_{n+1})\}$ is decreasing and converges to 0. Moreover, for $m > n$,

$$\delta(\omega_n,\omega_m) \leq \sum_{i=n}^{m-1} \delta(\omega_i,\omega_{i+1}).$$

By condition (B3), we have

$$\delta(\omega_n,\omega_{n+1}) \leq \varphi^n \delta(\omega_0,\omega_1), \qquad \varphi \in (0,1).$$

Hence

$$\begin{aligned}\delta(\omega_n,\omega_m) &\leq \sum_{i=n}^{m-1} \varphi^i\, \delta(\omega_0,\omega_1)\\ &\to 0 \quad \text{as } n,m \to \infty,\end{aligned}$$

which shows that $\{\omega_n\}$ is Cauchy. Completeness of X ensures that $\omega_n \to \vartheta \in X$. If Φ is continuous, then

$$\begin{aligned}\vartheta &= \lim_{n\to\infty} \omega_{n+1} = \lim_{n\to\infty} \Phi(\omega_n)\\ &= \Phi(\vartheta),\end{aligned}$$

so ϑ is a fixed point. If instead $\mathcal{F}$ is continuous, applying the contractive condition to (ω_{n_k},ϑ) for a subsequence and letting $k \to \infty$ yields

$$\begin{aligned}&\tau + \mathcal{F}(\delta(\vartheta,\Phi(\vartheta)))\\ &\quad \leq \mathcal{F}(\delta(\vartheta,\Phi(\vartheta))),\end{aligned}$$

which is impossible since $\tau > 0$. Hence $\Phi(\vartheta) = \vartheta$. For uniqueness, suppose ϑ_1, ϑ_2 are two distinct fixed points, then

$$\begin{aligned}&\tau + \mathcal{F}(\delta(\vartheta_1,\vartheta_2))\\ &\quad \leq \mathcal{F}(\delta(\vartheta_1,\vartheta_2)),\end{aligned}$$

which is impossible. Thus ϑ is the unique fixed point of Φ. $\boxdot$

Example 3.6. *Let $X = \{1\} \cup [0, \frac{7}{10}]$ and $\delta(\omega,\vartheta) = |\omega - \vartheta|$, $\forall\, \omega,\vartheta \in X$, let (X,δ) be a complete metric space and a self-map $\Phi : X \to X$ be define as:*

$$\Phi(\omega) = \begin{cases} \dfrac{\omega}{2}, & \omega \in \left[0, \dfrac{7}{10}\right],\\[2ex] \dfrac{1}{4}, & \omega = 1.\end{cases}$$

We choosing $\mathcal{F}(\alpha) = \ln(\alpha)$ and $\tau = \ln(7)$. Since Φ is not continuous, so that Φ is not F-contraction. When, $\omega = \frac{1}{4},\ \vartheta = 1 \Rightarrow \delta\left(\Phi\left(\frac{1}{4}\right),\Phi(1)\right) = \left|\frac{1}{8} - \frac{1}{4}\right| = \frac{1}{8} > 0$

$$\begin{aligned}\mathcal{M}(\tfrac{1}{4},1) = \max\Bigg\{&\delta\left(\frac{1}{4},1\right),\ \delta\left(\frac{1}{4},\Phi\left(\frac{1}{4}\right)\right),\\ &\delta(1,\Phi(1)),\\ &\frac{\delta\left(\frac{1}{4},1\right)+\delta\left(\Phi\left(\frac{1}{4}\right),\Phi(1)\right)}{1+\delta(1,\Phi(1))}\Bigg\}\end{aligned}$$

$$= \max\left\{\delta\left(\frac{1}{4},1\right), \delta\left(\frac{1}{4},\frac{1}{8}\right), \delta\left(1,\frac{1}{4}\right), \frac{\delta\left(\frac{1}{4},1\right)+\delta\left(\frac{1}{8},\frac{1}{4}\right)}{1+\delta\left(1,\frac{1}{4}\right)}\right\}$$

$$= \max\left\{\frac{3}{4},\frac{1}{8},\frac{3}{4},\frac{\frac{3}{4}+\frac{1}{8}}{1+\frac{3}{4}}\right\}$$

$$= \max\left\{\frac{3}{4},\frac{1}{8},\frac{3}{4},\frac{\frac{7}{8}}{\frac{7}{4}}\right\}$$

$$= \max\left\{\frac{3}{4},\frac{1}{8},\frac{3}{4},\frac{1}{2}\right\} = \frac{3}{4}.$$

Now,

$$\begin{aligned}\tau + \mathcal{F}\left(\delta\left(\Phi\left(\frac{1}{4}\right),\Phi(1)\right)\right) &= \ln(7) + Ln\left(\frac{1}{8}\right)\\ &= \ln\left(\frac{7}{8}\right)\\ &> \ln\left(\frac{3}{4}\right)\\ &= \mathcal{F}\left(\mathcal{M}\left(\frac{1}{4},1\right)\right),\end{aligned}$$

so that Φ is not F-contraction. But if $\tau = \ln(k)$, $k \in [1,6]$, then

$$\tau + \mathcal{F}\left(\delta\left(\Phi\left(\frac{1}{4}\right),\Phi(1)\right)\right) \leq \mathcal{F}\left(\mathcal{M}\left(\frac{1}{4},1\right)\right),$$

and Φ satisfy F-contraction.

4. Conclusion

In this study, we explored a new class of generalized F-contraction mappings within complete metric spaces by employing iterative set techniques and rational-type conditions. The results establish fixed point theorems that not only extend and unify several existing frameworks but also provide a clearer and more flexible foundation for the study of contractive-type mappings. By combining rigorous analysis with illustrative examples, this work enhances understanding of the underlying concepts and offers promising directions for further research in fixed point theory and its potential applications.

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Authors contributions
V. C. Borkar and Saeed A. A. Al-Salehi jointly contributed to the conception and design of the study. Both authors carried out the mathematical analysis, developed the contractive conditions and proofs, and constructed the illustrative examples. Both authors participated in drafting and revising the manuscript, and approved the final version for publication.

Availability of data and materials
The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflict of interests
The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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