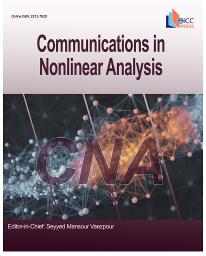

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Research Article

Benefaction of Rothe's Time Discretization Method to Solve Diffusion Equation Involving Riemann-Liouville Fractional Integral and Delay Integral Function

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Abstract:

This paper presents a diffusion equation involving α^{th} order Riemann-Liouville (R-L) fractional integral along with integral forcing function for delay and some constant coefficients. We apply $L2$ scheme to discretize the fractional integral function and Rothe's method is opted to establish the existence and uniqueness of a strong solution. In addition, we provide some error estimation and continuous on initial data. Eventually, we provide an application to manifest the results.

Keywords: Method of semidiscretization, Riemann-Liouville fractional integral, Strong solution, Delay differential equation, Initial conditions

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1. Introduction

The primary objective of this article is to establish existence, uniqueness of strong solution, and approximate solution of a given diffusion equation that incorporates the R-L fractional integral and with delay integral forcing function along with nonlocal history condition, initial conditions(ICs) and boundary conditions (BCs), respectively,

$$\frac{\partial y}{\partial t} + \mathcal{I}_t^{\alpha} y + \frac{\partial^2 y}{\partial x^2} = p(t, y) + \int_0^t k(t, s, y_t) ds, \quad \forall\, t \in (0, \tau],\ \forall\, x \in \Omega, \tag{1.1}$$

$$y(t, x) = \psi(t, x) \quad \text{on} \quad [-\tau, 0) \times \Omega, \tag{1.2}$$

$$y(0, x) = Y_0(x) \ \text{ and } \ \frac{\partial y}{\partial t}(0, x) = Y_1(x) \quad \text{on} \quad \Omega, \tag{1.3}$$

$$y(t, 0) = y(t, 1) = 0 \ \text{ and } \ \frac{\partial y}{\partial t}(t, 0) = \frac{\partial y}{\partial t}(t, 1) = 0 \quad \text{on} \quad \mathbb{I}_\tau, \tag{1.4}$$

where $\mathbb{I}_\tau = [0, \tau]$, $\Omega = (0, 1)$, $y_t(s, x) = y(t + s, x)$ for $s \in [-\tau, 0]$, and $\tau \in \mathbb{R}^+$, and $p,\ k,\ \psi,\ Y_0$ and Y_1 are some suitable given functions. Fractional differential equations (FDEs) have always been an interesting topic among researchers. Numerous application of FDEs are very useful in different fields such as, rheology, viscoelasticity, fluid flow, control theory, food science, electro-magnetic theory, mathematical modeling of diffusion process and many real life problems, see [1–6]. Let us review some research papers which deal with this type of problems. In their work, Bahuguna and Jaiswal [7] studied a fractional-order partial integro-differential equation, reformulating it as an abstract problem in Banach space. They applied Rothe's technique to demonstrate the existence, uniqueness of a strong solution. In an another study, they used Rothe's method to solve generalization of Basset problem [8]. Chaoui and Hallaci [9] studied a second-order fractional diffusion equation of differential Volterra operator. They also demonstrated the existence, uniqueness of a weak solution under fractional integral condition. Dubey [10] applied the method of lines for a nonlinear functional differential equation in reflexive Banach space and demonstrated the existence and uniqueness of a strong solution under nonlocal history condition. In a very recent study Hammad et. al. [11] studied many integral equations such that 2D Volterra, Atangana-Baleanu, Riemann-Liouville fractional integral equations in double controlled metric spaces. They obtained the existence, uniqueness of the solution with the aid of Fixed point technique. Kacar et. al. [12], established the extension inequalities of Gruss inequalities for Riemann-Liouville fractional integrals. Raheem and Bahuguna [13] used method of semidisretization to give strong solution of delay differential equation and thereby establishing the existence, uniqueness of that solution. The Rothe's time-discretization technique was suggested by Rothe in 1930. Being a powerful technique, it can be applied to establish sufficient conditions for the existence and uniqueness of a solution of both linear and nonlinear problems. It can also be used in the construct of algorithms to find numerical solutions. This technique helps to solve various functional differential equations and some other physical problems, see references [14–29] and references therein. In our previous study, we investigated the following delayed Rayleigh-Love equation to analyze the existence and uniqueness of its solutions.

$$a_1\frac{\partial^2 w}{\partial t^2}-a_2\frac{\partial^2 w}{\partial x^2}-a_3\frac{\partial^4 w}{\partial x^2\partial t^2} = h\Big(t,w,\frac{\partial w}{\partial t},w_t\Big), \quad t\in(0,\tau],\ x\in\Omega.$$

Subject to the conditions given here [30].

This manuscript is structured as follows: The preliminaries and presumptions are given in Section 2. Section 3 provides the a priori estimates and necessary lemmas. Section 4 addresses the convergence results. Section 5 discusses the existence and uniqueness of a strong solution to (1.1)-(1.4), along with the continuous dependence of the solution on the given initial data and error analysis. In Section 6, we present an application to illustrate the results.

2. Preliminaries and Presumptions

Let $\mathbb{X}=L^2(\Omega)$ be Hilbert space. For each $t\in\mathbb{I}_\tau$, let $C_t=C([-\tau,t];\mathbb{X})$ be the Banach space of all continuous functions defined from $[-\tau,t]$ to $\mathbb{X}$ endowed with the supremum norm

$$\|\ell\|_t=\sup_{-\tau\le s\le t}\|\ell(s)\|.$$

For $\ell\in C_\tau$, we describe $\ell_t(s)=\{\ell(t+s):s\in[-\tau,0]\}$. We define the unknown function $y:\mathbb{I}_\tau\to\mathbb{X}$, and the map $p:\mathbb{I}_\tau\times\mathbb{X}\to\mathbb{X}$, $k:\mathbb{I}_\tau\times C_0\to\mathbb{X}$ and $\psi:[-\tau,0)\to\mathbb{X}$ by

$$\begin{aligned} y(t)(x) &= y(t,x),\\ p(t,y(t))(x) &= p(t,y(t,x)),\\ k(t,y(\cdot))(x) &= k(t,y(\cdot,x)),\\ \psi(t)(x) &= \psi(t,x), \end{aligned}$$

respectively. Assuming the linear operator A defined as below

$$\mathbb{D}_A=\Big\{y\in L^2(\Omega)\ :\ \frac{d^2y}{dx^2}\in L^2(\Omega),\ y(0)=y(1)=0\Big\},$$
$$Ay=\frac{d^2y}{dx^2},$$

where $-A$ is infinitesimal generator of $\mathbb{C}_0$ semigroups of contractions in $\mathbb{X}$ and $\mathbb{D}(.)$ is the domain of A. Now, Considering following assumptions:

(H1) $p:\mathbb{I}_\tau\times\mathbb{X}\to\mathbb{X}$ satisfies the given inequality that $\exists\, L_p>0$

$$\|p(t_1,y_1)-p(t_2,u_2)\|\le L_p(|t_1-t_2|+\|u_1-u_2\|)$$

$\forall\, t_1,t_2\in\mathbb{I}_\tau,\ \forall\, u_1,u_2\in\mathbb{X}$.

(H2) $k:\mathbb{I}_\tau\times C_0\to\mathbb{X}$ satisfies the condition

$$\|k(t_1,\psi_1)-k(t_2,\psi_2)\|\le L_k(|t_1-t_2|+\|\psi_1-\psi_2\|_0)$$

$\forall\ t_1,t_2\in(0,\tau],\ \forall\ \psi_1,\psi_2\in C_0$ with $\|\psi-\phi(0)\|_0\le r$, and L_k is nondrecreasing function of $r>0$.

(H3) $\psi:[-\tau,0)\to\mathbb{X}$ satisfies a Lipschitz condition, that is, $\exists\, L_\psi>0$ such that

$$\|\psi(t_1)-\psi(t_2)\|\le L_\psi|t_1-t_2|$$

$\forall\, t_1,t_2\in[-\tau,0)$.

(H4) $1<\alpha<2$.

Now, we rewrite (1.1)-(1.4) as

$$\begin{aligned}&\frac{dy}{dt}+\mathcal{I}^\alpha y(t)+Ay(t)=p(t,y(t))\\ &+\int_0^t k(t,s,y_t)ds,\ y(0)=Y(0).\end{aligned} \tag{2.1}$$

Definition 2.1. *Let $\alpha,t\in\mathbb{R}$. Suppose that $t>0$ and $1<\alpha<2$. Then, the R-L fractional integral of function $g(t)$ of order α is defined as*

$$I^\alpha g(t)=\frac{1}{\Gamma(\alpha)}\int_0^t g(x)(t-x)^{\alpha-1}dx.$$

$$\frac{1}{h_n}\|y_i^n-y_{i-1}^n\|\le a+bh_n\sum_{j=1}^i\|y_j^n\|. \tag{2.2}$$

Lemma 2.2. *[31] If $-A$ is the infinitesimal generator of a $\mathbb{C}_0$-semigroup of contractions in $\mathbb{X}$, then A is m-accretive that is $\langle Av,J^*v\rangle\ge 0$ for all $v\in\mathbb{D}_A$, where J^* is the duality mapping and range of an operator $\mathbb{R}(\mathbb{I}+\gamma A)=\mathbb{X}$ for $\gamma>0$.*

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3. Discretization and a priori estimates

For the rest of this article, we will assume that the assumptions $(H1)-(H4)$ hold true. Dividing the interval $\mathbb{I}_\tau$ into the n^{th} partition $[t_{i-1}^n, t_i^n]$ of length $h_n = \frac{\tau}{n}$, where $t_i^n = ih_n$, for $i = 0, 1, 2, \ldots, n$, $n \in \mathbb{Z}^+ \cup \{0\}$. Let $y_0^n = Y_0(x)$

At $t = t_i^n$, we use $L2$ scheme to approximate the R-L fractional integral term

$$\mathcal{I}^\alpha y_i^n \approx \frac{1}{\Gamma(1+\alpha)} \sum_{j=1}^{i} (t_j^n - t_{j-1}^n)^\alpha y_j^n. \tag{3.1}$$

At time $t = t_i^n$, we replace the equation (2.1) by the following approximate equation

$$\begin{aligned} &\delta y_i^n + \frac{h_n^\alpha}{\Gamma(1+\alpha)} \sum_{j=1}^{i} y_j^n + A y_i^n \\ &= p(t_i^n, y_{i-1}^n) + \int_0^{t_i^n} k(t_i^n, s, \tilde{y}_{i-1}^n) ds \end{aligned} \tag{3.2}$$

where $y_i^n = y(t_i^n, x)$, $\delta y_i^n = \dfrac{y_i^n - y_{i-1}^n}{h_n}$ and $\tilde{y}_0^n(t) = \psi(t)\,\forall\, t \in [-\tau, 0]$ and for every $i = 2, 3, \ldots, n$, $\tilde{y}_{i-1}^n(t)$ is defined as

$$\tilde{y}_{i-1}^n(t) = \begin{cases} \psi(t_{i-1}^n + t), \quad t \in [-\tau, -t_{i-1}^n], \\ u_{k-1}^n + (t_{i-1}^n + t - t_{k-1}^n)\delta u_k^n, \\ \qquad t \in [-t_{i-k}^n, -t_{i-k-1}^n], \\ \qquad k = 1, 2, \ldots, i-1. \end{cases}$$

The existence and uniqueness of the $\{y_j^n\}$ satisfying (3.2) is a consequence of Lemma (2.2).

Lemma 3.1. *For each* $i = 1, 2, \cdots n, \ \ n \in \mathbb{Z}^+$,

$$\|y_i^n\| \leq \omega, \tag{3.3}$$

where $\omega > 0$ *is a constant which does not depend on* n *and* i.

Proof: From (3.2) for $i = 1$, we get

$$\begin{aligned} \frac{y_1^n - y_0^n}{h_n} + \frac{(h_n)^\alpha}{\Gamma(1+\alpha)} y_1^n + A y_1^n &= p(t_1^n, y_0^n) \\ &\quad + \int_0^{t_1^n} k(t_1^n, s, \tilde{y}_0^n) ds. \end{aligned}$$

Then,

$$\begin{aligned} &\langle y_1^n, y_1^n\rangle + \frac{h_n^{1+\alpha}}{\Gamma(1+\alpha)} \Big\langle y_1^n, y_1^n \Big\rangle + h_n \langle A y_1^n, y_1^n\rangle \\ = \ &\langle y_0^n, y_1^n\rangle + h_n \Big\langle p(t_1^n, y_0^n) \\ &+ \int_0^{t_1^n} k(t_1^n, s, \tilde{y}_0^n) ds, y_1^n \Big\rangle. \end{aligned}$$

By Lemma (2.2), we get

$$\begin{aligned} &\|y_1^n\| + \frac{h_n^{1+\alpha}}{\Gamma(1+\alpha)} \|y_1^n\| \\ \leq \ &\|y_0^n\| + h_n \|p(t_1^n, y_0^n)\| \\ &+ h_n \Big\| \int_0^{t_1^n} k(t_1^n, s, \tilde{y}_0^n) ds \Big\|. \end{aligned} \tag{3.4}$$

for $i \geq 2$, applying (3.1) in (3.2) and using (3.4), we get

$$\begin{aligned} &\|y_i^n\| + \frac{h_n^{1+\alpha}}{\Gamma(1+\alpha)} \|y_i^n\| \\ \leq \ &\frac{h_n^{1+\alpha}}{\Gamma(1+\alpha)} \sum_{j=1}^{i-1} \|y_j^n\| + h_n \sum_{j=1}^{i} \|p(t_j^n, y_{j-1}^n)\| \\ &+ h_n \sum_{j=1}^{i} \Big\| \int_0^{t_j^n} k(t_j^n, s, \tilde{y}_{j-1}^n) ds \Big\| + \|y_0^n\|. \end{aligned}$$

$\left(\frac{h_n^{1+\alpha}}{\Gamma(1+\alpha)} + 1\right) \geq 1.$
Therefore,

$$\begin{aligned} \|y_i^n\| \leq \ &h_n \sum_{j=1}^{i} \Big[\|p(t_j^n, y_{j-1}^n)\| \\ &+ \Big\| \int_0^{t_j^n} k(t_j^n, s, \tilde{y}_{j-1}^n) ds \Big\| \Big] \\ &+ \|y_0^n\| + \frac{h_n^{1+\alpha}}{\Gamma(1+\alpha)} \sum_{j=1}^{i} \|y_j^n\|. \end{aligned} \tag{3.5}$$

Now, consider $\|p(t_j^n, y_{j-1}^n)\|$ and $\left\| \int_0^{t_j^n} k(t_j^n, s, \tilde{y}_{j-1}^n) ds \right\|$

$$\begin{aligned} \|p(t_j^n, y_{j-1}^n)\| &\leq \|p(t_j^n, u_{j-1}^n) - p(0, y_0^n)\| \\ &\quad + \|p(0, y_0^n)\| \\ \|p(t_j^n, y_{j-1}^n)\| &\leq L_p[\tau + \|y_{j-1}^n - y_0^n\|]. \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\Big\| \int_0^{t_j^n} k(t_j^n, s, \tilde{y}_{j-1}^n) ds \Big\| \\ \leq \ &\Big\| \int_0^{t_j^n} k(t_j^n, s, \tilde{y}_{j-1}^n) ds \\ &- \int_0^{t_j^n} k(0, s, \psi(0)) ds \Big\| \\ &+ \Big\| \int_0^{t_j^n} k(0, s, \psi(0)) ds \Big\|. \\ \leq \ &\tau[L_k(\tau + r) + M_k]. \end{aligned} \tag{3.7}$$

Substitute (3.6) and (3.7) in (4.13), we obtain

$$\begin{aligned} \|y_i^n\| \ \leq \ &\tau \Big[\tau L_p + \tau\{L_k(\tau + r) + \omega_k\} + \|Y_0\| \Big] \\ &+ \frac{h_n^{1+\alpha}}{\Gamma(1+\alpha)} \sum_{j=1}^{i} \|y_j^n\|. \end{aligned}$$

Thus, we get

$$\|y_i^n\| \ \leq \ a + b h_n \sum_{j=1}^{i} \|y_j^n\|. \tag{3.8}$$

where

$$\begin{aligned} a &= \tau \Big[\tau L_p + \tau\{L_k(\tau + r) + \omega_k\} + \|Y_0\| \Big], \\ b &= \frac{h_n^\alpha}{\Gamma(1+\alpha)}. \end{aligned}$$

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Applying discrete Gronwall's inequality, we have

$$\|y_i^n\| \leq \frac{a}{1-bh_n} \exp\left(\frac{b(i-1)h_n}{1-bh_n}\right). \tag{3.9}$$

Since the sequence $\frac{a}{1-bh_n} \exp\left(\frac{b\tau}{1-bh_n}\right)$ is convergent, $\|y_i^n\|$ is bounded $\forall\, n \in \mathbb{Z}^+$ and $\forall\, i = 1, 2, \ldots, n$. It has been demonstrated.

Lemma 3.2. *For each* $i = 1, 2, \cdots n,\ \ n \in \mathbb{Z}^+$,

$$\frac{1}{h_n}\|y_i^n - y_{i-1}^n\| \leq \omega, \tag{3.10}$$

where $\omega > 0$ *is a constant which does not depend on* n *and* i.

Proof: For $i \geq 2$,, by subtracting the equation (3.2) at $i-1$ from the equation (3.2) at i, we obtain

$$\begin{aligned}
&\frac{y_i^n - y_{i-1}^n}{h_n} + \frac{(h_n)^\alpha}{\Gamma(1+\alpha)}\sum_{j=1}^{i} y_j^n - \frac{(h_n)^\alpha}{\Gamma(1+\alpha)}\sum_{j=1}^{i-1} y_j^n \\
&+A(y_i^n - y_{i-1}^n) \\
= \quad &\frac{y_{i-1}^n - y_{i-2}^n}{h_n} + [p(t_i^n, y_{i-1}^n) - p(t_{i-1}^n, y_{i-2}^n)] \\
&+\Big[\int_0^{t_i^n} k(t_i^n, s, \tilde{y}_{i-1}^n)ds \\
&-\int_0^{t_{i-1}^n} k(t_{i-1}^n, s, \tilde{y}_{i-2}^n)ds\Big]
\end{aligned} \tag{3.11}$$

Then,

$$\begin{aligned}
&\Big\langle \frac{y_i^n - y_{i-1}^n}{h_n}, y_i^n - y_{i-1}^n\Big\rangle + \frac{h_n^{1+\alpha}}{\Gamma(1+\alpha)}\Big\langle \frac{y_i^n - y_{i-1}^n}{h_n}, \\
&y_i^n - y_{i-1}^n\Big\rangle + \langle A(y_i^n - y_{i-1}^n), y_i^n - y_{i-1}^n\rangle \\
= \quad &\Big\langle \frac{y_{i-1}^n - y_{i-2}^n}{h_n}, y_i^n - y_{i-1}^n\Big\rangle \\
&+\langle p(t_i^n, y_{i-1}^n) - p(t_{i-1}^n, y_{i-2}^n), y_i^n - y_{i-1}^n\rangle \\
&+\Big\langle \int_0^{t_i^n} k(t_i^n, s, \tilde{y}_{i-1}^n)ds \\
&-\int_0^{t_{i-1}^n} k(t_{i-1}^n, s, \tilde{y}_{i-2}^n)ds, y_i^n - y_{i-1}^n\Big\rangle \\
&+\frac{h_n^\alpha}{\Gamma(1+\alpha)}\Big\langle \sum_{j=1}^{i-2} y_j^n - \sum_{j=1}^{i-1} y_j^n, y_i^n - y_{i-1}^n\Big\rangle.
\end{aligned}$$

By mean of Lemma (2.2), we get

$$\begin{aligned}
&\left(1 + \frac{h_n^{1+\alpha}}{\Gamma(1+\alpha)}\right)\frac{1}{h_n}\|y_i^n - y_{i-1}^n\| \\
\leq \quad &\frac{1}{h_n}\|y_{i-1}^n - y_{i-2}^n\| + \|p(t_i^n, y_{i-1}^n) - p(t_{i-1}^n, y_{i-2}^n)\| \\
&+\Big\| \int_0^{t_i^n} k(t_i^n, s, \tilde{y}_{i-1}^n)ds \\
&-\int_0^{t_{i-1}^n} k(t_{i-1}^n, s, \tilde{y}_{i-2}^n)ds\Big\| \\
&+\frac{h_n^\alpha}{\Gamma(1+\alpha)}\|y_{i-1}^n\|.
\end{aligned}$$

Therefore, $\left(1 + \frac{h_n^{1+\alpha}}{\Gamma(1+\alpha)}\right) \geq 1$.

Then

$$\begin{aligned}
&\frac{1}{h_n}\|y_i^n - y_{i-1}^n\| \\
\leq \quad &\frac{1}{h_n}\|y_{i-1}^n - y_{i-2}^n\| + \frac{h_n^\alpha}{\Gamma(1+\alpha)}\|y_{i-1}^n\| \\
&+\|p(t_i^n, y_{i-1}^n) - p(t_{i-1}^n, y_{i-2}^n)\| \\
&+\Big\| \int_0^{t_i^n} k(t_i^n, s, \tilde{y}_{i-1}^n)ds \\
&-\int_0^{t_{i-1}^n} k(t_{i-1}^n, s, \tilde{y}_{i-2}^n)ds\Big\|.
\end{aligned}$$

Therefore, for $i = 2, 3 \ldots, n$, we get

$$\begin{aligned}
&\frac{1}{h_n}\|y_i^n - y_{i-1}^n\| \\
\leq \quad &\frac{1}{h_n}\|y_1^n - y_0^n\| + \sum_{j=2}^{i}\Big[\|p(t_j^n, y_{j-1}^n) - p(t_{j-1}^n, y_{j-2}^n)\| \\
&+\Big\| \int_0^{t_j^n} k(t_j^n, s, \tilde{y}_{j-1}^n)ds \\
&-\int_0^{t_{j-1}^n} k(t_{j-1}^n, s, \tilde{y}_{j-2}^n)ds\Big\|\Big] \\
&+\frac{h_n^\alpha}{\Gamma(1+\alpha)}\sum_{j=1}^{i}\|y_j^n\|.
\end{aligned} \tag{3.12}$$

By putting $i = 1$ in (3.2), we obtain

$$\begin{aligned}
&\frac{y_1^n - y_0^n}{h_n} + \frac{(h_n)^\alpha}{\Gamma(1+\alpha)}y_1^n + Ay_1^n \\
= \quad &p(t_1^n, y_0^n) + \int_0^{t_1^n} k(t_1^n, s, \psi(0))ds
\end{aligned}$$

By using Lemma (2.2), we obtain

$$\begin{aligned}
&\left(1 + \frac{h_n^{1+\alpha}}{\Gamma(1+\alpha)}\right)\frac{1}{h_n}\|y_1^n - y_0^n\| \\
\leq \quad &\|p(t_1^n, y_0^n)\| + \Big\|\int_0^{t_1^n} k(t_1^n, s, \psi(0))ds\Big\| \\
&+\frac{h_n^\alpha}{\Gamma(1+\alpha)}\|y_0^n\| + \|Ay_0^n\|
\end{aligned}$$

Since, $\left(1 + \frac{h_n^{1+\alpha}}{\Gamma(1+\alpha)}\right) \geq 1$

Hence,

$$\begin{aligned}
&\frac{1}{h_n}\|y_1^n - y_0^n\| \\
\leq \quad &\|p(t_1^n, y_0^n)\| + \Big\|\int_0^{t_1^n} k(t_1^n, s, \psi(0))ds\Big\| \\
&+\frac{h_n^\alpha}{\Gamma(1+\alpha)}\|y_0^n\| + \|Ay_0^n\| \\
\leq \quad &\tau(L_p + \tau L_k + L_k r) \\
&+\left(L_p + \frac{\tau^\alpha}{\Gamma(1+\alpha)}\right)\|Y_0\| + \|AY_0\|.
\end{aligned} \tag{3.13}$$

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By (3.12) and (3.13), we have

$$\begin{aligned}
&\frac{1}{h_n}\|y_i^n - y_{i-1}^n\| \\
\leq\ & \tau(L_p + \tau L_k + L_k r) + \Big(L_p + \frac{\tau^\alpha}{\Gamma(1+\alpha)}\Big)\|Y_0\| \\
&+\|AY_0\| + \sum_{j=2}^{i}\Big[\|p(t_j^n, y_{j-1}^n) - p(t_{j-1}^n, y_{j-2}^n)\| \\
&+\Big\|\int_0^{t_j^n} k(t_j^n, s, \tilde{y}_{j-1}^n)ds \\
&-\int_0^{t_{j-1}^n} k(t_{j-1}^n, s, \tilde{y}_{j-2}^n)ds\Big\|\Big] \\
&+\frac{h_n^\alpha}{\Gamma(1+\alpha)}\sum_{j=1}^{i}\|y_j^n\|. \qquad (3.14)
\end{aligned}$$

Using Lemma 3.1, we have

$$\begin{aligned}
&\frac{1}{h_n}\|y_i^n - y_{i-1}^n\| \\
\leq\ & \tau(L_p + \tau L_k + L_k r) + \Big(L_p + \frac{\tau^\alpha}{\Gamma(1+\alpha)}\Big)\|Y_0\| \\
&+\|AY_0\| + \frac{\tau^\alpha \omega}{\Gamma(1+\alpha)} + \sum_{j=2}^{i}\Big[\|p(t_j^n, y_{j-1}^n) \\
&-p(t_{j-1}^n, y_{j-2}^n)\| + \Big\|\int_0^{t_j^n} k(t_j^n, s, \tilde{y}_{j-1}^n)ds \\
&-\int_0^{t_{j-1}^n} k(t_{j-1}^n, s, \tilde{y}_{j-2}^n)ds\Big\|\Big]. \qquad (3.15)
\end{aligned}$$

Now consider, $\|p(t_j^n, y_{j-1}^n) - p(t_{j-1}^n, y_{j-2}^n)\|$ and $\Big\|\int_0^{t_j^n} k(t_j^n, s, \tilde{y}_{j-1}^n)ds - \int_0^{t_{j-1}^n} k(t_{j-1}^n, s, \tilde{y}_{j-2}^n)ds\Big\|$.

$$\begin{aligned}
&\|p(t_j^n, y_{j-1}^n) - p(t_{j-1}^n, y_{j-2}^n)\| \\
\leq\ & L_p[|t_j^n - t_{j-1}^n| + \|y_{j-1}^n - y_{j-2}^n\|] \\
\leq\ & L_p h_n[1 + \frac{1}{h_n}\|y_{j-1}^n - y_{j-2}^n\|. \qquad (3.16)
\end{aligned}$$

$$\begin{aligned}
&\Big\|\int_0^{t_j^n} k(t_j^n, s, \tilde{y}_{j-1}^n)ds \\
&-\int_0^{t_{j-1}^n} k(t_{j-1}^n, s, \tilde{y}_{j-2}^n)ds\Big\| \\
\leq\ & 2TL_k(h_n + r) + h_n(rL_k + \omega_k). \qquad (3.17)
\end{aligned}$$

By (3.15), (3.16) and (3.17), we obtain

$$\begin{aligned}
&\frac{1}{h_n}\|y_i^n - y_{i-1}^n\| \\
\leq\ & \tau(L_p + \tau L_k + L_k r) + \Big(L_p + \frac{\tau^\alpha}{\Gamma(1+\alpha)}\Big)\|Y_0\| \\
&+\|AY_0\| + \frac{\tau^\alpha \omega}{\Gamma(1+\alpha)} \\
&+\sum_{j=2}^{i}\Big[L_p h_n[1 + \frac{1}{h_n}\|y_{j-1}^n - y_{j-2}^n\|] \\
&+2\tau L_k(h_n + r) + h_n(rL_k + \omega_k)\Big] \\
\leq\ & \Big[3\tau(L_p + \tau L_k + L_k r + \tau L_k r + \omega_k) \\
&+\Big(L_p + \frac{\tau^\alpha}{\Gamma(1+\alpha)}\Big)\|Y_0\| + \|AY_0\| + \frac{\tau^\alpha \omega}{\Gamma(1+\alpha)}\Big] \\
&+L_p h_n \sum_{j=1}^{i}\frac{1}{h_n}\|y_{j-1}^n - y_{j-2}^n\|.
\end{aligned}$$

Thus, we have

$$\frac{1}{h_n}\|y_i^n - y_{i-1}^n\| \leq a + bh_n\sum_{j=1}^{i}(\frac{1}{h_n}\|y_j^n - y_{j-1}^n\|), \qquad (3.18)$$

where

$$\begin{aligned}
a\ =\ & 3\tau(L_p + \tau L_k + L_k r + \tau L_k r + \omega_k) \\
&+\Big(L_p + \frac{\tau^\alpha}{\Gamma(1+\alpha)}\Big)\|Y_0\| + \|AY_0\| + \frac{\tau^\alpha \omega}{\Gamma(1+\alpha)},
\end{aligned}$$

and $b = L_p$ By Gronwall's inequality, we obtain $\frac{1}{h_n}\|y_i^n - y_{i-1}^n\| \leq \frac{a}{1-bh_n}\exp\Big(\frac{b(i-1)h_n}{1-bh_n}\Big)$. Due to the converges nature of sequence $\frac{a}{1-L_p h_n}\exp\Big(\frac{L_p\tau}{1-L_p h_n}\Big)$, $\frac{1}{h_n}\|y_i^n - y_{i-1}^n\|$ is bounded $\forall$ $n \in \mathbb{Z}^+$ and for all $i = 1, 2, \ldots, n$. It has been demonstrated.

4. Convergence

Defining the sequences $\{\bar{Y}^n(t)\}$ and $\{\bar{X}^n(t)\}$ by

$$\bar{Y}^n(t) = \begin{cases} y_0^n; t = 0, \\ y_{i-1}^n + \frac{1}{h_n}(t - t_{i-1}^n)(y_i^n - y_{i-1}^n) \\ t \in (t_{i-1}^n, t_i^n], \end{cases} \qquad (4.1)$$

$$\bar{X}^n(t) = \begin{cases} y_0^n, & \text{at} \quad t = 0, \\ y_i^n, & \text{if} \quad t \in (t_{i-i}^n, t_i^n], \end{cases} \qquad (4.2)$$

Defining a sequence of step function $\bar{\mathcal{I}}^\alpha \bar{X}^n(t)$

$$\bar{\mathcal{I}}^\alpha \bar{X}^n(t) = \begin{cases} 0, & \text{at} \quad t = 0, \\ \frac{h_n^\alpha}{\Gamma(1+\alpha)}\sum_{j=1}^{i} y_j^n, & \text{if} \quad t \in (t_{i-i}^n, t_i^n], \end{cases} \qquad (4.3)$$

Remark 4.1. *It is clear from Lemmas 3.1 and 3.2 that the function $\bar{Y}^n(t)$ is uniformly Lipschitz continuous on*

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the interval $(0,\tau]$ *and that* $\bar{Y}^n(t)-\bar{X}^n(t)\to 0$ *in* $\mathbb{X}$ *as* $n\to\infty$ *on* $(0,\tau]$*. Additionally, the sequences* $\bar{Y}^n(t)$ *and* $\bar{X}^n(t)$ *are uniformly bounded in* $\mathbb{X}$*.*

Lemma 4.2. $\|\bar{\mathcal{I}}^\alpha\bar{X}^n(t)-\mathcal{I}^\alpha\bar{X}^n(t)\|\to 0$ *as* $n\to\infty$ *uniformly on* $(0,\tau]$*.*

Proof: For a given t, there exists a i such that $t\in(t_{i-1},t_i]$.

$$\begin{aligned}
&\|\bar{\mathcal{I}}^\alpha\bar{X}^n(t)-\mathcal{I}^\alpha\bar{X}^n(t)\|\\
=\;&\frac{1}{\Gamma(\alpha)}\Big\|\sum_{j=1}^{i}(t_j^n-t_{j-1}^n)^\alpha y_j^n\\
&-\sum_{j=1}^{i}(t-t_{j-1}^n)^\alpha y_j^n\Big\|\\
\le\;&\frac{1}{\Gamma(\alpha)}\left[\Big\|\sum_{j=1}^{i}(h_n)^\alpha y_j^n\Big\|+\Big\|\sum_{j=1}^{i}(h_n)^\alpha y_j^n\Big\|\right]\\
\le\;&\frac{2}{\Gamma(\alpha)}h_n^{\alpha-1}\tau\omega\to 0\quad\text{as}\quad n\to\infty.
\end{aligned}$$

Hence proved. For $t\in(t_{i-1}^n,t_i^n]$, $1\le i\le n$, define $\bar{P}^n(t)$ as

$$\bar{P}^n(t)=p(t_j^n,y_{j-1}^n)+\int_0^{t_j^n}k(t_j,s,\tilde{y}_{j-1}^n)ds,\tag{4.4}$$

then (3.2) can be rewrite as

$$\begin{aligned}&\tfrac{d^-\bar{Y}^n}{dt}(t)+\bar{\mathcal{I}}^\alpha\bar{X}^n(t)+A\bar{X}^n(t)\\&=\bar{P}^n(t),\quad t\in(0,\tau],\end{aligned}\tag{4.5}$$

where $\dfrac{d^-}{dt}$ denotes the left hand derivative in $(0,\tau]$. For $t\in(0,\tau]$, we get

$$\begin{aligned}&\bar{Y}^n(t)+\int_0^t\Big(\bar{\mathcal{I}}^\alpha\bar{X}^n(t)+A\bar{X}^n(t)\Big)ds\\&=Y_0+\int_0^t\bar{P}^n(s)ds.\end{aligned}\tag{4.6}$$

Lemma 4.3. *There exists* $y\in C(\mathbb{I}_\tau;L^2(\Omega))$ *such that* $\bar{Y}^n(t)\to y(t)$ *uniformly on* $\mathbb{I}_\tau$*.* $y(t)$ *is differentiable almost everywhere on* $\mathbb{I}_\tau$ *as well.*

Proof: Consider $t\in(t_{i-1}^n,t_i^n]$ and $t\in(t_{k-1}^l,t_k^l]$, $t_i^n\le t_k^l$, $1\le i\le n$, $1\le k\le l$. Then, from (4.5), we have

$$\begin{aligned}&\Big\langle\frac{d\bar{Y}^n}{dt}-\frac{d\bar{Y}^l}{dt},\bar{X}^n(t)-\bar{X}^l(t)\Big\rangle\\&+\langle A(\bar{X}^n(t)-\bar{X}^l(t)),\bar{X}^n(t)-\bar{X}^l(t)\rangle\\&+\Big\langle\bar{\mathcal{I}}^\alpha\bar{X}^n(t)-\bar{\mathcal{I}}^\alpha\bar{X}^l(t),\bar{X}^n(t)-\bar{X}^l(t)\Big\rangle\\&=\langle\bar{P}^n(t)-\bar{P}^l(t),\bar{X}^n(t)-\bar{X}^l(t)\rangle.\end{aligned}\tag{4.7}$$

Now, consider

$$\begin{aligned}&\Big\langle\frac{d}{dt}(\bar{Y}^n-\bar{Y}^l),\bar{X}^n(t)-\bar{X}^l(t)\Big\rangle_B\\&=\Big\langle\frac{d}{dt}(\bar{Y}^n-\bar{Y}^l),\bar{X}^n(t)-\bar{Y}^n(t)\\&+\bar{Y}^l(t)-\bar{X}^l(t)\Big\rangle\\&+\frac{1}{2}\frac{d}{dt}\|\bar{Y}^n(t)-\bar{Y}^l(t)\|^2,\end{aligned}\tag{4.8}$$

and

$$\begin{aligned}&\Big\langle\bar{\mathcal{I}}^\alpha\bar{X}^n(t)-\bar{\mathcal{I}}^\alpha\bar{X}^l(t),\bar{X}^n(t)-\bar{X}^l(t)\Big\rangle\\
&=\Big\langle\frac{h_n^\alpha}{\Gamma(1+\alpha)}\sum_{j=1}^{i}\bar{X}^n(t)\\
&-\frac{h_l^\alpha}{\Gamma(1+\alpha)}\sum_{m=1}^{k}\bar{X}^l(t),\bar{X}^n(t)-\bar{X}^l(t)\Big\rangle\\
&=\frac{ih_n^\alpha}{\Gamma(1+\alpha)}\langle\bar{X}^n(t)-\bar{X}^l(t),\bar{X}^n(t)-\bar{X}^l(t)\rangle\\
&-\Big|\frac{kh_l^\alpha}{\Gamma(1+\alpha)}-\frac{ih_n^\alpha}{\Gamma(1+\alpha)}\Big|\langle\bar{X}^l(t),\bar{X}^l(t)-\bar{X}^n(t)\rangle,\\
&=\frac{ih_n^\alpha}{\Gamma(1+\alpha)}\|\bar{X}^n(t)-\bar{X}^l(t)\|^2\\
&-\Big|\frac{kh_l^\alpha-ih_n^\alpha}{\Gamma(1+\alpha)}\Big|\langle\bar{X}^l(t),\bar{X}^l(t)-\bar{X}^n(t)\rangle.\end{aligned}\tag{4.9}$$

Substituting (4.8) and (4.9) into (4.7), we obtain

$$\begin{aligned}&\frac{d}{dt}\|\bar{Y}^n-\bar{Y}^l\|^2+\frac{ih_n^\alpha}{\Gamma(1+\alpha)}\|\bar{X}^n(t)-\bar{X}^m(t)\|^2\\
&+\langle 2A(\bar{X}^n(t)-\bar{X}^m(t)),\bar{X}^n(t)-\bar{X}^m(t)\rangle\\
&=2\Big\langle\frac{d}{dt}(\bar{Y}^n-\bar{Y}^l),\bar{Y}^n(t)-\bar{X}^n(t)\\
&+\bar{X}^l(t)-\bar{Y}^l(t)\Big\rangle\\
&+\Big|\frac{kh_l^\alpha-ih_n^\alpha}{\Gamma(1+\alpha)}\Big|\langle\bar{X}^l(t),\bar{X}^l(t)-\bar{X}^n(t)\rangle\\
&+2\langle\bar{P}^n(t)-\bar{P}^l(t),\bar{X}^n(t)-\bar{X}^l(t)\rangle.\end{aligned}$$

Using Lemma (2.2), we get

$$\begin{aligned}&\frac{d}{dt}\|\bar{Y}^n-\bar{Y}^l\|^2\\
&\le 2\Big\langle\frac{d}{dt}(\bar{Y}^n-\bar{Y}^l),\bar{Y}^n(t)-\bar{X}^n(t)\\
&+\bar{X}^l(t)-\bar{Y}^l(t)\Big\rangle\\
&+\Big|\frac{kh_l^\alpha-jh_n^\alpha}{\Gamma(1+\alpha)}\Big|\langle\bar{X}^l(t),\bar{X}^n(t)-\bar{X}^l(t)\rangle\\
&+2\langle\bar{P}^n(t)-\bar{P}^l(t),\bar{X}^n(t)-\bar{X}^l(t)\rangle.\end{aligned}\tag{4.10}$$

For $t\in(t_{i-1}^n,t_i^n]$ and $t\in(t_{k-1}^l,t_k^l]$, $1\le i\le n$,

$1 \leq k \leq l$, $t_i^n < t_k^l$, we consider

$$\begin{aligned}
&\|\bar{P}^n(t) - \bar{P}^l(t)\| \\
&= \Big\| p(t_i^n, y_i^n) + \int_0^{t_i^n} k(t_i^n, s, \tilde{y}_i^n)ds - p(t_k^l, y_k^l) \\
&- \int_0^{t_k^l} k(t_k^l, s, \tilde{y}_k^l)ds \Big\| \\
&\leq \|p(t_i^n, y_i^n) - p(t_k^l, y_k^l)\| + \Big\| \int_0^{t_i^n} k(t_i^n, s, \tilde{y}_i^n)ds \\
&- \int_0^{t_k^l} k(t_k^l, s, \tilde{y}_k^l)ds \Big\| \\
&\leq L_p \Big[|t_i^n - t_k^l| + \|y_{i-1}^n - y_{k-1}^l\| \Big] \\
&+ \tau L_k \Big[|t_i^n - t_l^l| + \|\tilde{y}_{i-1}^n - \tilde{y}_{k-1}^l\|_0 \Big] \\
&+ (t_k^l - t_i^n) L_k [|t_j^n| + \|\tilde{y}_{i-1}^n - \psi(0)\|_0] \\
&+ (t_k^l - t_i^n)\omega_k \\
&\leq L_p[(1+\omega)(h_n + h_l) + \|\bar{Y}^n(t) - \bar{Y}^l(t)\|] \\
&+ (h_n + h_l)[\tau L_k(1 + L_\psi) \\
&+ L_k(h_n + r) + \omega_k].
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
&\Big\langle \frac{d}{dt}(\bar{Y}^n - \bar{Y}^l), \bar{Y}^n(t) - \bar{X}^n(t) \\
&+ \bar{X}^m(t) - \bar{Y}^l(t) \Big\rangle \\
&\leq \left(\left\| \frac{d\bar{Y}^n}{dt} \right\| + \left\| \frac{d\bar{Y}^l}{dt} \right\| \right) \Big(\|\bar{Y}^n(t) - \bar{X}^n(t)\|_B \\
&+ \|\bar{X}^l(t) - \bar{Y}^l(t)\| \Big) \\
&\leq 2\omega(h_n + h_l),
\end{aligned} \tag{4.12}$$

$$\begin{aligned}
&\left| \frac{(kh_l^\alpha - jh_n^\alpha)}{\Gamma(1+\alpha)} \right| \langle \bar{Z}^l(t), \bar{X}^l(t) - \bar{X}^n(t) \rangle \\
&\leq \left| \frac{kh_l^\alpha - ih_n^\alpha}{\Gamma(1+\alpha)} \right| 2C^2,
\end{aligned} \tag{4.13}$$

and

$$\|\bar{X}^n(t) - \bar{X}^l(t)\| \leq \omega(h_n + h_l) + \|\bar{Y}^n(t) - \bar{Y}^l(t)\|. \tag{4.14}$$

Therefore, by (4.10), we obtain

$$\begin{aligned}
&\frac{d}{dt}\|\bar{Y}^n - \bar{Y}^l\|^2 \leq 2\omega(h_n + h_l)(h_n + h_l + 2) \\
&\times [L_p(1+\omega) + \tau L_k(1 + L_\psi + L(h_n + r) + \omega_k] \\
&+ \left| \frac{kh_l^\alpha - ih_n^\alpha}{\Gamma(1+\alpha)} \right| 2\omega^2 + 4\omega(h_n + h_l)[L_p + 1] \\
&+ 2L_p\|\bar{Y}^n(t) - \bar{Y}^l(t)\|^2.
\end{aligned}$$

$$\begin{aligned}
\|\bar{Y}^n(t) - \bar{Y}^l(t)\|^2 &\leq \nu_{nl} \\
&+ L_p \int_0^t \|\bar{Y}^n(s) - \bar{Y}^l(s)\|^2 ds,
\end{aligned}$$

where

$$\begin{aligned}
\nu_{nl} &= 2\omega\tau(h_n + h_l)(h_n + h_l + 2)[L_p(1+\omega) \\
&\quad + \tau L_k(1 + L_\psi + L(h_n + r) + \omega_k] \\
&\quad + 4\omega\tau(h_n + h_l)[L_p + 1] \\
&\quad + \left| \frac{kh_l^\alpha - ih_n^\alpha}{\Gamma(1+\alpha)} \right| 2\tau\omega^2,
\end{aligned}$$

and $\nu_{nl} \to 0$ as $n, l \to 0$. Using Gronwall's inequality, we have

$$\|\bar{Y}^n(t) - \bar{Y}^l(t)\|^2 \leq \nu_{nl} e^{L_p t} \leq \nu_{nl} e^{L_p \tau} \quad \text{for all } t \in \mathbb{I}_\tau. \tag{4.15}$$

Thus, $\{\bar{Y}^n(t)\}$ is cauchy sequences in $C(\mathbb{I}_\tau; L^2(\Omega))$. Therefore, we get $y \in C(\mathbb{I}_\tau; L^2(\Omega))$ such that $\bar{Y}^n(t) \to y(t)$ uniformly on $\mathbb{I}_\tau$. Since $\|\bar{Y}^n(t) - \bar{Y}^n(s)\| \leq |t - s|$. Hence $y(t)$ is Lipschitz continuous functions on $\mathbb{I}_\tau$. Thus, $\frac{dy}{dt} \in L^\infty(\mathbb{I}_\tau; L^2(\Omega))$. Hence proved.

Lemma 4.4. *There exists a subsequence $\{\bar{X}^{n_k}\}$ of $\{\bar{X}^n\}$ such that $\mathcal{I}^\alpha \bar{X}^{n_k} \rightharpoonup \mathcal{I}^\alpha y$ in $L^2(\mathbb{I}_\tau, \mathbb{X})$, as $n \to \infty$.*

Proof: By above lemma $\mathcal{I}^\alpha \bar{X}^n(t)$ is uniformly bounded in $L^2(\mathbb{I}_\tau, \mathbb{X})$, there exist a subsequence $\mathcal{I}^\alpha \bar{X}^{n_k}(t)$ and a function $\varphi \in L^2(\mathbb{I}_\tau, \mathbb{X})$ such that

$$\mathcal{I}^\alpha \bar{X}^{n_k} \rightharpoonup \varphi \quad \text{in} \quad L^2(\mathbb{I}_\tau, \mathbb{X}).$$

Define $\chi^n : \mathbb{I}_\tau \to \mathbb{X}$ by

$$\chi^n(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \bar{X}^n(s)(t-s)^{\alpha-1} ds.$$

For $t \in (0, t_1^n)$, $\bar{X}^n(t) = y_1^n$. Hence

$$\begin{aligned}
\chi^n(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t y_1^n (t-s)^{\alpha-1} ds \\
&= \frac{y_1^n}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\
&= \frac{y_1^n}{\Gamma(1+\alpha)} t^\alpha.
\end{aligned}$$

Since y_1^n is bounded, $\chi^n(0^+) = 0$ for every n. For $g^* \in \mathbb{X}^*$, we can observe

$$\langle \chi^{n_k}(t), g^* \rangle = \int_0^t \langle \mathcal{I}^\alpha \bar{X}^{n_k}, g^* \rangle ds.$$

Since $\bar{X}^{n_k} \to y$ in $C(\mathbb{I}_\tau, \mathbb{X})$, $\chi^{n_k}(t) \to \frac{1}{\Gamma(\alpha)} \int_0^t y(s)(t-s)^{\alpha-1} ds$ as $k \to \infty$. Letting $k \to \infty$, we have

$$\left\langle \frac{1}{\Gamma(\alpha)} \int_0^t y(s)(t-s)^{\alpha-1} ds, g^* \right\rangle = \int_0^t \langle \varphi(s), g^* \rangle.$$

Hence

$$\varphi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t y(s)(t-s)^{\alpha-1} ds.$$

Hence proved.

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5. Main result and error estimates

This section shows that the function $y(t)$, derived in Lemma (4.3), acts as a strong solution to the system (2.1) over the interval $\mathbb{I}_\tau$. Furthermore, we examine how the solution y continuously depends on the initial condition Y_0 and the function f and error estimation.

Theorem 5.1. *Under the assumptions (H1)-(H4), the system (2.1) has a unique strong solution $y(t)$ on $\mathbb{I}_\tau$.*

Proof: Assuming the functions $y(t)$, derived in Lemma (4.3). Let $t \in (t_{i-1}^n, t_i^n],\ i = 1, 2, \ldots, n,\ n \in \mathbb{Z}^+$. Since

$$\begin{aligned} \|\bar{X}^n(t) - y(t)\| &\leq \|\bar{X}^n(t) - Y^n(t)\| \\ &+ \|\bar{Y}^n(t) - y(t)\| \\ &\leq \omega h_n + \|\bar{Y}^n(t) - y(t)\|, \end{aligned}$$

Thus, $\bar{X}^n(t) \to y(t)$ uniformly on $\mathbb{I}_\tau$. Hence, $\|\bar{X}^n(t)\|$ is bounded uniformly on $\mathbb{I}_\tau$.

For $t \in (t_{i-1}^n, t_i^n]$, we obtain

$$\begin{aligned} &\left\|\bar{P}^n(t) - p(t, y(t)) - \int_0^t k(t, s, y_t)\right\| \\ &= \left\|p(t_i^n, y_{i-1}^n) + \int_0^{t_i^n} k(t_i^n, s, \tilde{y}_{i-1}^n)ds \right. \\ &\left. -p(t, y(t)) - \int_0^t k(t, s, y_t)ds\right\| \\ &\leq L_p(|t - t_i^n| + \|y_{i-1}^n - y(t)\|) + TL_k(|t - t_i^n| \\ &+ \|\tilde{y}_{i-1}^n - y_t\|_0) \\ &+ \int_t^{t_i^n} \|k(t_i^n, s, \tilde{y}_{i-1}^n)ds\| \\ &\leq L_p[h_n(1+\omega) + \|\bar{Y}^n(t) - y(t)\|] \\ &+ h_n(2\tau L_k + \tau L_k(\omega + L_\phi) + L_k r + \omega_k). \end{aligned}$$

Thus, we determine that $\bar{P}^n(t) \to p(t, y(t)) + \int_0^t k(t, s, y_t)$ as $n \to \infty$ uniformly on $\mathbb{I}_\tau$. By Lemma (3.2) $\|\frac{dy^n}{dt}\|$ is bounded uniformly on $\mathbb{I}_\tau$. Hence, $\|A\bar{X}^n(t)\|$ is bounded uniformly on $\mathbb{I}_\tau$.
By (4.6), for every $w \in \mathbb{X}$, we get

$$\begin{aligned} &\int_0^t \langle \mathcal{I}^\alpha \bar{X}^n(s) + A\bar{X}^n(s), w\rangle ds = \langle Y_0, w\rangle \\ &-\langle \bar{Y}^n(t), w\rangle + \int_0^t \Big[p(s, y(s)) \\ &+ \int_0^s k(s, x, y_s)dx\Big] ds. \end{aligned}$$

By bounded convergence theorem, we have

$$\begin{aligned} &\int_0^t \langle \mathcal{I}^\alpha y(s) + Ay(s), w\rangle ds = \langle Y_0, w\rangle - \langle y(t), w\rangle \\ &+ \int_0^t \Big[p(s, y(s)) + \int_0^s k(s, x, u_s)dx\Big] ds. \end{aligned} \tag{5.1}$$

Since $\mathcal{I}^\alpha y(t)$ and $Ay(t)$ are Bochner integrable functions on $\mathbb{I}_\tau$, by (5.1), we obtain

$$\begin{cases} \dfrac{dy}{dt} + \mathcal{I}^\alpha y(t) + Ay(t) = p(t, y(t)) \\ + \displaystyle\int_0^t k(t, s, y_t)ds \\ \quad a.e.\ t \in \mathbb{I}_\tau, y(0) = Y_0, \end{cases} \tag{5.2}$$

Hence, the function y is a strong solution of (2.1)). Suppose that y_1, y_2 are distinct strong solutions of (2.1)). Assume that $y' = y_1 - y_2$. Then

$$\begin{aligned} &\frac{dy'}{dt} + \frac{1}{\Gamma(\alpha)} \int_0^t y'(s)(t-s)^{\alpha-1}ds(t) + Ay'(t) \\ &= \Big(p(t, y_1(t)) + \int_0^t k(t, s, y_{1t})ds\Big) \\ &-\Big(p(t, y_2(t)) + \int_0^t k(t, s, y_{2t})ds\Big) \end{aligned}$$

By Lemma (2.2), we have

$$\begin{aligned} \frac{d}{dt}\|y'(t)\|^2 &\leq 2\|y(t)\|\Big[\|p(t, y_1(t)) - p(t, y_2(t))\| \\ &+ \Big\|\int_0^t k(t, s, y_{1t})ds - \\ &\int_0^t k(t, s, y_{2t})ds\Big\|\Big] \end{aligned}$$

Therefore,

$$\frac{d}{dt}\|y'(t)\|^2 \leq [(L_p + L_k\tau)\sup_{t\in\mathbb{I}_\tau}\|y(t)\|^2]$$

Thus,

$$\|y(t)\|^2 \leq 2(L_p + L_k\tau)\int_0^t \Big(\sup_{t\in\mathbb{I}_\tau}\|y(s)\|\Big)^2 ds, \quad t \in \mathbb{I}_\tau$$

Applying Gronwall's lemma, $\|y'(t)\|^2 = 0\ \forall\ t \in \mathbb{I}_\tau$. Thus $y_1(t) = y_2(t)\ \forall\ t \in \mathbb{I}_\tau$.
We now present a result regarding the continuous dependence of the solution y on the initial condition Y_0 and the function f, where f is

$$p(t, y(t)) + \int_0^t k(t, s, y_t)ds.$$

Theorem 5.2. *Let $y,\ y_1$ be different functions of strong solutions of (2.1) corresponding to (Y_0, f) and (Z_0, f^*) satisfy the all presumptions then the following condition*

$$\begin{aligned} &\|y(t) - y_1(t)\|^2 \leq \\ &\|Y_0 - Z_0\|^2 + \int_0^\tau \|f_y(s) - f_{y_1}^*(s)\|ds \end{aligned}$$

holds for all $t \in \mathbb{I}_\tau$, where

$$f_y(t) = p(t, y(t)) + \int_0^t k(t, s, y_t)ds,$$

$$f_{y_1}^*(t) = p(t, u(t)) + \int_0^t k(t, s, u_t)ds.$$

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Proof: Consider $z(t) = y(t) - y_1(t)$. Then,

$$\frac{dz}{dt} + \frac{1}{\Gamma(\alpha)} \int_0^t z(t)(t-s)^{\alpha-1} ds$$
$$+ Az(t) = f_y(t) - f_u^*(t)$$

Using Lemma 2.2, for $t \in \mathbb{I}_\tau$, we get

$$\frac{d}{dt} \|z(t)\|^2 \le 2\|z(t)\| \|f_y(t) - f_u^*(t)\|,$$
$$\le \|z(t)\|^2 + \|f_y(t) - f_u^*(t)\|^2$$

Thus,

$$\|z(t)\|^2 \le \|z(0)\|^2 + \int_0^t \|z(s)\|^2 ds + \int_0^t \|f_y(s) - f_{y_1}^*(s)\|^2 ds$$

By applying Gronwall's inequality, we get

$$\|z(t)\|^2 \le \Big(\|z(0)\|^2 + \int_0^t \|f_y(s) - f_{y_1}^*(s)\|^2 ds\Big) e^t.$$

Lemma 5.3. *For each $i = 1, 2, \dots, n$, and $n \in \mathbb{Z}^+$, the errors $\|\bar{Y}^n(t) - y(t)\|$ and $\|y_i^n - y(t_i^n)\|$ satisfy the following bounds*
(i) $\|\bar{Y}^n(t) - y(t)\| \le \frac{Q_1}{\sqrt{n}}$ for all $t \in [t_0, \tau]$ and for some constant $Q_1 > 0$.
(ii) $\|y_i^n - y(t_i^n)\| \le \frac{Q_2}{\sqrt{n}}$, where $Q_2 > 0$ is some constant. Furthermore, $\|y_i^n - y(t)\| \le \frac{Q_2}{\sqrt{n}}$ for all $t \in (t_{i-1}^n, t_i^n]$.

Proof: Considering $l \to \infty$ (4.15) becomes,

$$\|\bar{Y}^n(t) - y(t)\| \le \frac{Q_1}{\sqrt{n}}$$

for each $t \in [0, \tau]$, for each $n \in \mathbb{Z}^+$ and for some $Q_1 > 0$. Therefore $\|\bar{Y}^n(t) - y(t)\| \le \frac{Q_1}{\sqrt{n}}$ for all $t \in [t_0, \tau]$, $\forall$ $n \in \mathbb{Z}^+$.
For $t \in (t_{i-1}^n, t_i^n],\ i = 1, 2, \dots, n,\ n \in \mathbb{Z}^+$, we get

$$\begin{aligned} \|y_{i-1}^n - y(t)\| &\le \|y_{i-1}^n - \bar{Y}^n(t)\| + \|\bar{Y}^n(t) - y(t)\| \\ &\le \omega h_n + \|\bar{Y}^n(t) - y(t)\| \\ &\le \frac{\omega\tau}{n} + \frac{Q_1}{\sqrt{n}} \le \frac{2\omega\tau + Q_1}{\sqrt{n}}. \end{aligned}$$

Consequently, by Lemma 3.2, we get

$$\begin{aligned} \|y_i^n - y(t)\| &\le \|y_i^n - y_{i-1}^n\| + \|y_{i-1}^n - y(t)\| \\ &\le \omega h_n + \|y_{i-1}^n - y(t)\| \\ &\le \frac{2\omega\tau + Q_1}{\sqrt{n}}. \end{aligned}$$

Hereof, $\|y_i^n - y(t)\| \le \frac{Q_2}{\sqrt{n}}\ \forall\ t \in [t_{i-1}^n, t_i^n],\ i = 1, 2, \dots, n,\ n \in \mathbb{Z}^+$, where $Q_2 = 2\omega\tau + Q_1$.
In particular, $\|y_i^n - y(t_i^n) \le \frac{Q_2}{\sqrt{n}}$.

6. Application

Considering partial differential equation given below

$$\frac{\partial y}{\partial t} - \frac{\partial^2 y}{\partial x^2} + \frac{15\sqrt{\pi}}{8}(t)^{\frac{5}{2}} y = \upsilon(t, y(t)) + \int_0^t \sin(t, s, y_t) ds,$$
$$t \in (0, \tau],\ x \in (\Omega), \tag{6.1}$$

the history condition

$$y(t, x) = \sigma \tan t \quad \text{on} \quad [-\tau, 0) \times (\Omega), \tag{6.2}$$

and the initial conditions

$$y(0, x) = x^2 + 1 \ \text{ and } \ \frac{\partial y}{\partial t}(0, x) = 2x \quad \text{on} \quad (\Omega), \tag{6.3}$$

where $\sigma, \upsilon, \tau \in \mathbb{R}^+$.
Here, $p(t, y) = \sigma(t, y)$, $k(t, s, \psi) = \sin(t, s, \psi)$, $\psi(t, x) = \upsilon \tan t$, $Y_0(x) = x^2 + 1$, and $Y_1(x) = 2x$. Clearly, both functions Y_0 and Y_1 belong to $L^2(\Omega)$.
Consider,

$$\begin{aligned} \|\psi(t_1) - \psi(t_2)\|^2 &= \int_0^1 \Big(\psi(t_1) - \psi(t_2)(x)\Big)^2 dx \\ &= \upsilon^2 \Big[\frac{\sin(t_1 - t_2)}{\cos t_1 \cos t_2}\Big]^2 \\ &\le \upsilon^2 |t_1 - t_2|^2. \end{aligned}$$

Therefore, we have

$$\|\psi(t_1) - \psi(t_2)\| \le \upsilon |t_1 - t_2|. \tag{6.4}$$

Consider,

$$\begin{aligned} &\|p(t_1, y_1) - p(t_2, y_2)\|^2 \\ &= \int_0^1 (p(t_1, y_1) - p(t_2, y_2)(x))^2 dx \\ &= \int_0^1 \sigma^2 (t_1 + y_1(x) - t_2 - y_2(x))^2 dx \\ &\le 2\sigma^2 \Big(\int_0^1 (t_1 - t_2)^2 dx \\ &+ \int_0^1 (y_1(x) - y_2(x))^2 dx \Big) \\ &\le 2\sigma^2 \Big(|t_1 - t_2|^2 + \|y_1 - y_2\|^2\Big) \end{aligned}$$

Thus,

$$\|p(t_1, y_1) - p(t_2, y_2)\| \le \sqrt{2}\sigma \Big(|t_1 - t_2| + \|y_1 - y_2\|\Big).$$

and

$$\begin{aligned} &\|k(t_1, s, \phi_1) - k(t_2, s, \phi_2)\|^2 \\ &= \|\sin(t_1, s, \phi_1) - \sin(t_2, s, \phi_2)\|^2 \\ &= \Big| \int_0^1 \Big[(\sin(t_1, s, \phi_1) - \sin(t_2, s, \phi_2))(x)\Big]^2 dx \Big| \\ &\le \int_0^1 \Big[(|t_1 - t_2| + |\phi_1 - \phi_2|)(x)\Big]^2 dx \\ &\le 2\Big(|t_1 - t_2|^2 + \|\phi_1 - \phi_2\|_0^2\Big) \end{aligned}$$

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Therefore,

$$\|k(t_1,s,\phi_1)-k(t_2,s,\phi_2)\| \leq \sqrt{2}\Big(|t_1-t_2|+\|\phi_1-\phi_2\|_0\Big).$$

Therefore, if $\sigma \leq \frac{1}{\sqrt{2}}$, then by Theorem 5.1, the boundary value problem (6.1)-(6.3) admits a unique strong solution.

7. Conclusion

We introduced a fractional integro-differential equation with a delay integral function (1.1), along with the corresponding history, initial, and boundary conditions (1.2)-(1.4). Utilizing the semigroup theory of bounded linear operators and an appropriate function, we first transformed the problem (1.1)-(1.4) into the system (2.1). We then applied the $L2$ scheme to discretize the Riemann-Liouville fractional integral term and used a semidiscretization method to establish certain a priori estimates. Additionally, we constructed a sequence of functions and demonstrated that this sequence converges uniformly to the unique strong solution of (2.1). We also examined the continuous dependence of the strong solution on the given initial data. The error estimates confirmed that our results closely approximate the exact solution. Finally, we presented an application to illustrate the main findings.

Authors contributions
All the authors have participated sufficiently in the intellectual content, conception and design of this work or the analysis and interpretation of the data (when applicable), as well as the writing of the manuscript.
Availability of data and materials
The data that support the findings of this study are available from the corresponding author upon reasonable request.
Conflict of interests
The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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