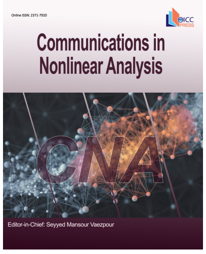

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Research Article

Analysis of Fractional Gas Dynamic Equations Using Atangana-Baleanu Derivative Operator

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Abstract:

This paper presents approximate analytical solutions for solving the nonlinear homogeneous and non-homogeneous gas dynamics equations with the Atangana-Baleanu fractional derivative operator. The validity of the solutions is verified by providing existence and uniqueness criteria for the solution using the employed technique. To illustrate the ability and efficiency of the method, absolute errors, tables, and graphical presentations of numerical solutions for specific cases are obtained.

Keywords: Atangana-Baleanu operator; Fractional gas dynamic equations; Iterative Laplace transform method

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1. Introduction

Gas dynamic equations are mathematical representations defined by the physical laws of energy conservation, mass conservation, momentum conservation, etc. Nonlinear fractional-order gas dynamics equations are applied in shock fronts, unusual fractions, and connection discontinuities. Gas dynamics is a field of study in fluid dynamics that examines gas motion and its effects on physical structures based on the concepts of fluid mechanics and fluid dynamics. In 1981, Steger and Warming [1] noted that the conservation-law form of the inviscid gas dynamic equation possesses a remarkable property by virtue of which the nonlinear flux vectors are homogeneous functions. The science emerges from the study of gas flows involving choked flows in nozzles and pipes, gas fuel streams in rocket engines, aerodynamic heating on atmospheric reentry vehicles, and shock waves around aircraft [2].

During the last decade, fractional calculus has found applications in numerous seemingly diverse fields of science and engineering. Fractional differential equations are increasingly used to model problems in fluid mechanics, acoustics, biology, electromagnetism, diffusion, signal processing, and many other physical processes [3–6].

Major importance is given to fractional calculus due to its significant role in various fields compared to integral calculus for forming various day-to-day models and studying these models. To find better approximations and convergent numerical solutions of nonlinear partial and fractional partial differential equations, numerous methods have been established in recent years by several scholars and scientists. There are many recent approaches that solve problems analytically or numerically, such as the finite difference method, differential transform method, homotopy analysis method, homotopy analysis transform method, and several iterative approaches [7–12]. Recently, Biazar and Eslami [13] and Das and Kumar [14] have applied methods to obtain solutions of the homogeneous and non-homogeneous time-fractional gas dynamics equation. Rasulov et al. [15] used the finite difference method to find solutions of nonlinear equations of gas dynamic problems for discontinuous functions. In [16], Prakash and Kumar investigated a numerical method for time-fractional gas dynamic equations. Solutions of fractional-order gas dynamics equations by effective techniques were obtained in [17] by Iqbal et al. Mohamed [18] used a new iterative method for fractional gas dynamics and coupled Burger's equations.

These fractional differential equations involve several fractional differential operators such as the Riemann-Liouville operator [19], Caputo operator [20], Caputo-Fabrizio operator [21, 22], etc. However, these operators possess a power law kernel, exponential kernel, and have singularity. Therefore, they have some limitations in modeling physical problems. To overcome this difficulty, Atangana and Baleanu recently proposed a reliable operator having a nonlocal and nonsingular kernel in the form of a Mittag-Leffler function known as the Atangana-Baleanu operator [23].

The goal of this study is to apply the iterative Laplace transform to find approximate solutions of time-fractional homogeneous and non-homogeneous gas dynamic equations using the Atangana-Baleanu operator. These equations are given below as follows:

$$\frac{{}^{ABC}\partial^\beta u}{\partial t^\beta} + u\frac{\partial u}{\partial x} - u(1-u)\log\lambda = 0, \quad \lambda > 0,\ 0 < \beta \leq 1 \tag{1}$$

with initial condition $u(x,0) = \lambda^{-x}$, and

$$\frac{{}^{ABC}\partial^\beta u}{\partial t^\beta} + u\frac{\partial u}{\partial x} + (1+t)^2u^2 = x^2, \quad 0 < \beta \leq 1 \tag{2}$$

with initial condition $u(x,0) = x$.

The paper is organized in the following order. In Section 2, basic results of fractional calculus are presented. The working rule for the iterative Laplace transform method is given in Section 3. Section 4 presents existence and uniqueness criteria for solutions of time-fractional homogeneous and non-homogeneous gas dynamic equations. In Section 5, numerical simulations, tables, and plots for the obtained solutions are provided. Finally, we give our conclusions in Section 6.

2. Preliminaries

Definition 1. *The left and right Atangana-Baleanu fractional derivatives for a given function of order β in Caputo sense are defined as*

$$ {}^{ABC}_{a}D^\beta_t f(t) = \frac{B(\beta)}{1-\beta}\int_a^t \frac{df(\tau)}{d\tau} \times E_\beta\left[-\frac{\beta}{1-\beta}(t-\tau)^\beta\right] d\tau \tag{3}$$

(left derivative)

$$ {}^{ABC}_{a}D^\beta_t f(t) = -\frac{B(\beta)}{1-\beta}\int_t^b \frac{df(\tau)}{d\tau} \times E_\beta\left[-\frac{\beta}{1-\beta}(t-\tau)^\beta\right] d\tau \tag{4}$$

(right derivative) where $B(\beta) = (1-\beta) + \frac{\beta}{\Gamma(\beta)}$ is a normalization function and $E_\beta(\cdot)$ is the Mittag-Leffler function.

Definition 2. *The Atangana-Baleanu fractional integral of order β is defined as*

$$ {}^{AB}_{a}I^\beta_t f(t) = \frac{1-\beta}{B(\beta)}f(t) + \frac{\beta}{B(\beta)\Gamma(\beta)}\int_a^t (t-\tau)^{\beta-1}f(\tau)d\tau \tag{5}$$

Definition 3. *The Laplace transform of the Atangana-Baleanu fractional derivative is given by*

$$\mathcal{L}\left\{{}^{ABC}_{0}D^\beta_t f(t)\right\} = \frac{B(\beta)}{1-\beta}\frac{s^\beta\mathcal{L}\{f(t)\} - s^{\beta-1}f(0)}{s^\beta + \frac{\beta}{1-\beta}} \tag{6}$$

where $0 < \beta \leq 1$.

3. Iterative Laplace Transform Method

The iterative Laplace transform method (ILTM) is a combination of the Laplace transform and the iterative method proposed by Daftardar-Gejji and Jafari [24]. Consider the general fractional differential equation

$$ {}^{ABC}_{0}D^\beta_t u(x,t) = L[u(x,t)] + N[u(x,t)] + g(x,t) \tag{7}$$

with initial condition $u(x,0) = h(x)$, where L is a linear operator, N is a nonlinear operator, and $g(x,t)$ is a source term.

Applying the Laplace transform to both sides and using the initial condition, we obtain

$$\mathcal{L}\{u(x,t)\} = \frac{s^{\beta-1}h(x)}{s^\beta + \frac{\beta}{1-\beta}} + \frac{1-\beta}{B(\beta)s^\beta + \beta}\mathcal{L}\{L[u] + N[u] + g\} \tag{8}$$

The iterative scheme is then defined as

$$u_0(x,t) = \mathcal{L}^{-1}\left[\frac{s^{\beta-1}h(x)}{s^\beta + \frac{\beta}{1-\beta}}\right] \tag{9}$$

$$u_{n+1}(x,t) = \mathcal{L}^{-1}\left[\frac{1-\beta}{B(\beta)s^\beta + \beta} \times \mathcal{L}\{L[u_n] + N[u_n] + g\}\right] \tag{10}$$

for $n \geq 0$.

The approximate solution is given by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{11}$$

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4. Existence and Uniqueness

Theorem 4. *Let $X = C([0,a] \times [0,T])$ be the Banach space of all continuous functions on $[0,a] \times [0,T]$ with the norm*

$$\|u\| = \max_{(x,t)\in[0,a]\times[0,T]} |u(x,t)| \tag{12}$$

If the nonlinear operator N satisfies the Lipschitz condition

$$\|N[u] - N[v]\| \leq K\|u - v\| \tag{13}$$

for all $u, v \in X$, where K is the Lipschitz constant, and if

$$\frac{K(1-\beta)T^\beta}{B(\beta)\Gamma(\beta+1)} < 1 \tag{14}$$

then the fractional gas dynamic equation has a unique solution in X.

Proof. The proof follows from the Banach fixed-point theorem applied to the operator defined by the iterative scheme. □

5. Numerical Results and Discussion

5.1 Homogeneous Gas Dynamic Equation

Consider the homogeneous fractional gas dynamic equation (1) with initial condition $u(x,0) = \lambda^{-x}$ where $\lambda = e$. Applying the ILTM, we obtain the following iterative scheme. The zeroth-order approximation is

$$u_0(x,t) = e^{-x} \tag{15}$$

For subsequent iterations, we apply the scheme systematically. After four iterations, the approximate solution for $\beta = 1$ converges to the exact solution

$$u(x,t) = e^{-(x+t)} \tag{16}$$

Table 1 presents the comparison of absolute errors between the exact solution and the four-term approximations obtained by ILTM for equation (1).

Table 1. Absolute error for equation (1) with $\lambda = e$

t	x	$\beta = 1$	$\beta = 0.8$	$\beta = 0.6$
0.2	0	8.21×10^{-9}	1.35×10^{-3}	9.21×10^{-3}
0.4	0.5	2.15×10^{-7}	4.82×10^{-3}	2.76×10^{-2}
0.6	1	1.93×10^{-6}	1.12×10^{-2}	5.93×10^{-2}
0.8	1.5	8.44×10^{-6}	2.08×10^{-2}	1.06×10^{-1}
1	2	2.54×10^{-5}	3.43×10^{-2}	1.73×10^{-1}

Figure 1 shows the approximate solutions of equation (1) for different values of $\beta = 0.95, 0.75, 0.55$.

5.2 Non-homogeneous Gas Dynamic Equation

Consider the non-homogeneous fractional gas dynamic equation (2) with initial condition $u(x,0) = x$. Applying the ILTM, the zeroth-order approximation is

$$u_0(x,t) = x \tag{17}$$

After systematic iterations, the approximate solution for $\beta = 1$ converges to the exact solution

$$u(x,t) = \frac{x}{1+t} \tag{18}$$

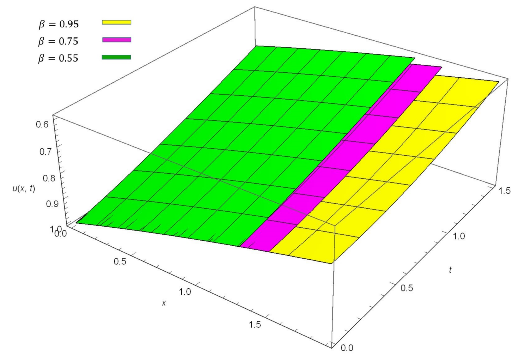

Figure 1. Approximate solution of Eq. (1) for $\beta = 0.95, 0.75, 0.55$

Table 2 presents the comparison of absolute errors between the exact solution and the four-term approximations obtained by ILTM for equation (2).

Table 2. Absolute error for equation (2)

t	x	$\beta = 1$	$\beta = 0.8$	$\beta = 0.6$
0.2	0	1.24×10^{-7}	3.52×10^{-4}	1.87×10^{-3}
0.4	1	3.89×10^{-6}	9.41×10^{-4}	4.25×10^{-3}
0.6	2	2.47×10^{-5}	1.82×10^{-3}	7.63×10^{-3}
0.8	3	9.15×10^{-5}	3.06×10^{-3}	1.21×10^{-2}
1	5	5.29×10^{-4}	6.52×10^{-3}	2.49×10^{-2}

Figure 2 shows the approximate solutions of equation (2) for different values of $\beta = 0.95, 0.75, 0.55$.

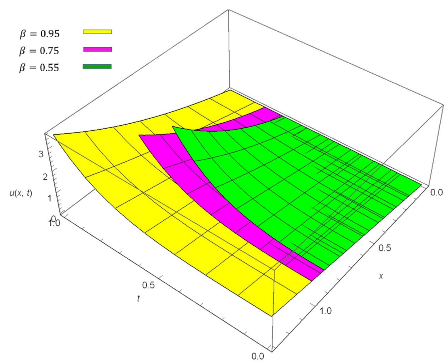

Figure 2. Approximate solution of Eq. (2) for $\beta = 0.95, 0.75, 0.55$

In Tables 1 and 2, the computational results demonstrate that this technique provides accurate numerical solutions even when lower-order approximations are used. The accuracy for the time-fractional nonlinear homogeneous and non-homogeneous gas dynamics equations is demonstrated by the absolute errors of equations (1) and (2) compared with their exact solutions.

Also, Figures 1 and 2 show surfaces for the approximate solutions of equations (1) and (2) and the exact solutions of the homogeneous and non-homogeneous fractional gas dynamics equations, respectively. From these surfaces, it can be observed that the approximate solutions and exact solutions for both equations are very close to each other.

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6. Conclusions

In this work, the iterative Laplace transform method has been successfully applied for numerical solutions of time-fractional homogeneous and non-homogeneous gas dynamic equations. The method provides more realistic series solutions that converge very rapidly in physical problems. The theoretical analysis of solutions is carried out by providing existence and uniqueness criteria for the solution using the employed technique. It is apparently seen from the comparison of absolute errors that the proposed method is a very efficient and powerful numerical method for finding approximate solutions. Also, it is important to point out that even when the problem is nonlinear, this method does not require linearization or perturbation. Therefore, ILTM is easy to implement and more convenient than other methods.

Authors contributions

All authors have participated sufficiently in the intellectual content, conception, and design of this work, as well as the writing of the manuscript. B.R. Sontakke formulated the problem and developed the methodology. S.R. Raut conducted the numerical computations and analysis. Both authors reviewed and approved the final manuscript.

Availability of data and materials

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflict of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] J. L. Steger and R. Warming. Flux vector splitting of the inviscid gasdynamic equations with application to finite-difference methods. *Journal of computational physics*, 40(2):263–293, 1981.

[2] T. G. Elizarova. Quasi-gas-dynamic equations. In *Quasi-Gas Dynamic Equations*, pages 37–62. Springer, Berlin, Heidelberg, 2009.

[3] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo. *Theory and Applications of Fractional Differential Equations*, volume 204 of *North-Holland Mathematics Studies*. Elsevier Science B.V., Amsterdam, The Netherlands, 2006.

[4] K. S. Miller and B. Ross. *An Introduction to the Fractional Calculus and Fractional Differential Equations*. Wiley, New York, 1993.

[5] I. Podlubny. *Fractional Differential Equations*. Academic Press, San Diego, 1999.

[6] X. J. Yang. *Local Fractional Functional Analysis and Its Applications*. Asian Academic, Hong Kong, 2011.

[7] S. Abbasbandy. The application of homotopy analysis method to nonlinear equations arising in heat transfer. *Physics Letters A*, 360:109–113, 2006.

[8] G. Adomian and R. Rach. Modified adomian polynomials. *Math. Comput. Modelling*, 24(11):39–46, 1996.

[9] J. H. He. Variational iteration method-a kind of non-linear analytical technique: some examples. *Int J Nonlinear Mech*, 34(4):699–708, 1999.

[10] J. H. He. Homotopy perturbation technique. *Comp. Meth. Appl. Mech. Eng*, 178:257–262, 1999.

[11] A. Shaikh, K. S. Nisar, V. Jadhav, S. K. Elagan, and M. Zakarya. Dynamical behaviour of hiv/aids model using fractional derivative with mittag-leffler kernel. *Alexandria Engineering Journal*, 61(4):2601–2610, 2022.

[12] B. R. Sontakke, A. S. Shelke, and A. S. Shaikh. Solution of non-linear fractional differential equations by variational iteration method and applications. *Far East Journal of Mathematical Sciences*, 110(1):113–129, 2019.

[13] J. Biazar and M. Eslami. Differential transform method for nonlinear fractional gas dynamic equation. *Int. J. Phys. Sci.*, 6:1203, 2011.

[14] S. Das and R. Kumar. Approximate analytical solutions of fractional gas dynamic equations. *Applied Mathematics and Computation*, 217(24): 9905–9915, 2011.

[15] M. Rasulov and T. Karaguler. Finite difference schemes for solving system equations of gas dynamic in a class of discontinuous functions. *Appl Math Comput*, 143:45–164, 2003.

[16] A. Prakash and M. Kumar. Numerical method for time-fractional gas dynamic equations. *Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci.*, 89:559–570, 2019.

[17] N. Iqbal, A. Akgu, R. Shah, A. Bariq, M. Mossa Al-Sawalha, and A. Ali. On solutions of fractional-order gas dynamics equation by effective techniques. *Journal of Function Spaces*, 2022:3341754, 2022. Article ID 3341754, 14 pages.

https://doi.org/10.57647/cna.2025.y5zb-4a54

[18] M. S. Al-luhaibi. New iterative method for fractional gas dynamics and coupled burger's equations. *The Scientific World Journal*, 2015:153124, 2015. Article ID 153124, 8 pages.

[19] B. R. Sontakke, A. Shaikh, and K. S. Nisar. Approximate solutions of a generalized hirota-satsuma coupled kdv and a coupled mkdv systems with time fractional derivatives. *Malaysian Journal of Mathematical Sciences*, 12(2):175–196, 2018.

[20] B. R. Sontakke and A. S. Shaikh. Properties of caputo operator and its applications to linear fractional differential equations. *Int. journal of engineering research and applications*, 5(5):22–27, 2015.

[21] A. Shaikh, A. Tassaddiq, K. S. Nisar, and D. Baleanu. Analysis of differential equations involving caputo-fabrizio fractional operator and its applications to reaction-diffusion equations. *Advances in Difference Equations*, 2019(1):1–14, 2019.

[22] B. R. Sontakke and S. R. Raut. Analysis of fractional kawahara and modified kawahara equations based on caputo-fabrizio derivative operator. *J. Math. Comput. Sci.*, 11(6):7105–7125, 2021.

[23] A. Atangana and D. Baleanu. New fractional derivatives with non-local and non-singular kernel: Theory and application to heat transfer model. *Thermal Science*, 20(2):763–769, 2016.

[24] V. Daftardar-Gejji and H. Jafari. An iterative method for solving non linear functional equations. *J. Math. Anal. Appl.*, 316:753–763, 2006.

https://doi.org/10.57647/cna.2025.y5zb-4a54