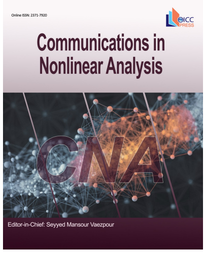

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Research Article

Applications of Wardowski-Type Contraction Mappings in Complex-Valued Branciari b-Metric Spaces to Fractional Differential Equations

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Abstract:

This research establishes some existence results for Wardowski-type contraction mappings defined on complex-valued Branciari b-metric spaces. Our analysis generalizes existing literature by improving upon classical results related to contraction mappings in similar metric structures. We provide a concrete example to validate our findings and extend the application of these results to the domain of fractional differential equations.

Keywords: Fixed point; Wardowski-type; Contraction; Complex-valued; Branciari b-metric spaces; Application; Fractional differential equations; MSC 2020: 47H10, 54H25

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1. Introduction

The Banach contraction principle [1], as one of the most famous fixed point theorems in metric spaces, has wide applications and has been extended in many papers from different perspectives (see, for instance, [2–16]). The class of F-contractions, defined by Wardowski [17], provides a robust generalization of the well-known Banach contraction principle. It is worth noting that many mappings that are not necessarily Banach contractions belong to the class of F-contractions; therefore, the concept of an F-contraction constitutes a genuine generalization of the Banach contraction principle. Subsequently, several authors have generalized Wardowski's result from different aspects (see, for instance, [18–20] and the references therein).

A notable deviation from classical metric structures was initiated by Branciari [10], who replaced the triangle inequality with a weaker rectangular condition, thereby defining Branciari metric spaces (BMS). Often termed 'generalized metric spaces,' these structures encompass cases where standard metric axioms fail to hold. Consequently, the analysis of fixed points within this setting has gained significant traction, as identifying the conditions for existence and uniqueness of fixed points in a BMS is a non-trivial process that demands particular clinical insight.

Several researchers have subsequently enriched the literature on Branciari metric spaces (BMS) by exploring diverse classes of generalized contractions. For instance, contractive conditions involving auxiliary functions and (ψ,φ)-type mappings have been extensively analyzed in [19, 21, 22]. Similarly, extensions of α-implicit and Geraghty-type contractions were investigated by Karapınar [23] and Asadi et al. [8], respectively. Furthermore, Aydi et al. [24] and Azam et al. [25] contributed to the field by establishing results for generalized (α-ψ) and Kannan-type contractions. Other significant developments include common fixed point theorems [26, 27], refinements of the Banach contraction principle [28, 29], and fundamental results such as Caristi's theorem [30] within these spaces.

The evolution of fixed point theory further progressed with the introduction of complex-valued (CV) metric spaces by Azam et al. [9], who established common fixed point results for mappings under rational contractive conditions. This framework sparked a wave of generalizations across various dimensions of CV metric spaces (refer to [31–37] and associated literature). Parallel to these developments, Ege [38] pioneered the study of complex-valued rectangular b-metric spaces (CV Branciari b-metric spaces), providing a unique fixed point result that serves as an analogue to the Banach contraction principle in this refined setting. Moreover, the methodology was enriched by Samet et al. [39] through the introduction of α-admissible and α-ψ-contractive mappings, offering a robust foundation for existence theorems in complete metric structures.

The primary objective of this research is to establish Wardowski-type contraction mappings within the framework of complex-valued (CV) Branciari b-metric spaces and to investigate the existence of their fixed points. Our findings extend and refine several established results in CV Branciari b-metric spaces, specifically enhancing the work of Ege [38]. To demonstrate the practical significance of these theoretical developments, we present an application to the field of fractional differential equations, providing sufficient criteria to ensure the existence of solutions for certain fractional systems.

2. Preliminaries

In this section, we present some notations and definitions which will be used in this paper in the sequel.

Definition 2.1. *[39] Let Ω be a nonempty set, $T:\Omega\to\Omega$ be a self-mapping and $\xi:\Omega\times\Omega\to[0,\infty)$ be a function. It is said that T is ξ-admissible whenever for any $\ell,\upsilon\in\Omega$, $\xi(\ell,\upsilon)\geq 1$ implies $\xi(T\ell,T\upsilon)\geq 1$.*

Also, we say that ξ is triangular whenever for any $\ell,\upsilon,z\in\Omega$, $\xi(\ell,\upsilon)\geq 1$ and $\xi(\upsilon,z)\geq 1$ implies $\xi(\ell,z)\geq 1$.

Definition 2.2. *([39]) Let (Ω,d) be a metric space and $\xi:\Omega\times\Omega\to[0,\infty)$ be a function. It is said that Ω is ξ-regular if for any sequence $\{\ell_n\}$ in Ω that $\lim_{n\to\infty}d(\ell_n,\ell)=0$ and $\xi(\ell_n,\ell_{n+1})\geq 1$ for all $n\in\mathbb{N}$, then $\xi(\ell_n,\ell)\geq 1$ for all $n\in\mathbb{N}$.*

Definition 2.3. *[9] Assume $\mathbb{C}$ represents the set of complex numbers. For any pair $\mathcal{Z}_1,\mathcal{Z}_2\in\mathbb{C}$, a partial ordering $\preceq$ is established such that $\mathcal{Z}_1\preceq\mathcal{Z}_2$ holds if and only if $\mathcal{R}e(\mathcal{Z}_1)\leq\mathcal{R}e(\mathcal{Z}_2)$ and $\mathcal{I}m(\mathcal{Z}_1)\leq\mathcal{I}m(\mathcal{Z}_2)$. Consequently, the relation $\mathcal{Z}_1\preceq\mathcal{Z}_2$ is satisfied provided that one of the following criteria is met:*

(i) $\mathcal{R}e(\mathcal{Z}_1)=\mathcal{R}e(\mathcal{Z}_2)$, $\mathcal{I}m(\mathcal{Z}_1)<\mathcal{I}m(\mathcal{Z}_2)$,

(ii) $\mathcal{R}e(\mathcal{Z}_1)<\mathcal{R}e(\mathcal{Z}_2)$, $\mathcal{I}m(\mathcal{Z}_1)=\mathcal{I}m(\mathcal{Z}_2)$,

(iii) $\mathcal{R}e(\mathcal{Z}_1)<\mathcal{R}e(\mathcal{Z}_2)$, $\mathcal{I}m(\mathcal{Z}_1)<\mathcal{I}m(\mathcal{Z}_2)$,

(iv) $\mathcal{R}e(\mathcal{Z}_1)=\mathcal{R}e(\mathcal{Z}_2)$, $\mathcal{I}m(\mathcal{Z}_1)=\mathcal{I}m(\mathcal{Z}_2)$.

Specifically, the notation $\mathcal{Z}_1\precnsim\mathcal{Z}_2$ is employed whenever any of the aforementioned conditions (i), (ii) and (iii) is satisfied. Furthermore, we denote $\mathcal{Z}_1\prec\mathcal{Z}_2$ exclusively when the third condition (iii) holds.

Definition 2.4. *[9] Let Ω is a nonempty set. Let a mapping $\rho_c:\Omega\times\Omega\to\mathbb{C}$ satisfies the following conditions:*

(i) $0\preceq\rho_c(\ell,\upsilon)$ *and* $\rho_c(\ell,\upsilon)=0$ *if and only if* $\ell=\upsilon$;

(ii) $\rho_c(\ell,\upsilon)=\rho_c(\upsilon,\ell)$;

(iii) $\rho_c(\ell,\upsilon)\preceq\rho_c(\ell,\xi)+\rho_c(\xi,\upsilon)$, *for all* $\ell,\xi,\upsilon\in\Omega$,

Under these conditions, ρ_c defines a complex-valued (CV) metric on Ω, and the corresponding pair (Ω,ρ_c) is referred to as a complex-valued (CV) metric space.

In this article, we use the symbol $|.|$ for the absolute value of a number and the symbol $\Big|.\Big|$ for the magnitude (length) of a complex number.

Example 2.5. *Let $\Omega=\Omega_1\cup\Omega_2$, where*

$$\Omega_1=\{\mathcal{Z}\in\mathbb{C}:\mathcal{R}e(\mathcal{Z})\geq 0 \text{ and } \mathcal{I}m(\mathcal{Z})=0\}$$

and

$$\Omega_2=\{\mathcal{Z}\in\mathbb{C}:\mathcal{R}e(\mathcal{Z})=0 \text{ and } \mathcal{I}m(\mathcal{Z})\geq 0\}.$$

Define $\rho_c:\Omega\times\Omega\to\mathbb{C}$ as follows:

$$\rho_c(\mathcal{Z}_1,\mathcal{Z}_2)=\begin{cases}\frac{1}{2}|\delta_1-\delta_2|(1+i), & \mathcal{Z}_1,\mathcal{Z}_2\in\Omega_1,\\ \frac{1}{2}|\xi_1-\xi_2|(1+i), & \mathcal{Z}_1,\mathcal{Z}_2\in\Omega_2,\\ \frac{1}{2}(\delta_1+\xi_2)(1+i), & \mathcal{Z}_1\in\Omega_1,\\ & \mathcal{Z}_2\in\Omega_2\end{cases}$$

and $\rho_c(\mathcal{Z}_1,\mathcal{Z}_2)=\rho_c(\mathcal{Z}_2,\mathcal{Z}_1)$, where $\mathcal{Z}_1=\delta_1+i\xi_1$ and $\mathcal{Z}_2=\delta_2+i\xi_2$. Then, (Ω,ρ_c) is a CV metric space.

Example 2.6. *Let $\Omega=\Omega_1\cup\Omega_2$, where Ω_1 and Ω_2 are the same ones introduced in the previous example. Define $\rho_c:\Omega\times\Omega\to\mathbb{C}$ as follows:*

$$\rho_c(\mathcal{Z}_1,\mathcal{Z}_2)=\rho_c(\mathcal{Z}_2,\mathcal{Z}_1)=\begin{cases}|\delta_1-\delta_2|, & \mathcal{Z}_1,\mathcal{Z}_2\in\Omega_1,\\ |\xi_1-\xi_2|i, & \mathcal{Z}_1,\mathcal{Z}_2\in\Omega_2,\\ \delta_1+\xi_2 i, & \mathcal{Z}_1\in\Omega_1,\mathcal{Z}_2\in\Omega_2,\end{cases}$$

where $\mathcal{Z}_1=\delta_1+\xi_1 i$ and $\mathcal{Z}_2=\delta_2+\xi_2 i$. Then (Ω,ρ_c) is a CV metric space.

Example 2.7. *Let $\Omega=\mathbb{C}$. Define $\rho_c:\Omega\times\Omega\to\mathbb{C}$ as follows:*

$$\rho_c(\mathcal{Z}_1,\mathcal{Z}_2)=a|\delta_1-\delta_2|+b|\xi_1-\xi_2|i,$$

where $\mathcal{Z}_1=\delta_1+\xi_1 i$ and $\mathcal{Z}_2=\delta_2+\xi_2 i$ and a,b are two positive real constants. Then (Ω,ρ_c) is a CV metric space.

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Example 2.8. *[40] Let $\Omega = [0,1]$ and $\ell, \upsilon \in \Omega$. Define $\rho_c : \Omega \times \Omega \to \mathbb{C}$ by*

$$\rho_c(\ell, \upsilon) = \begin{cases} 0, & \ell = \upsilon, \\ \frac{i}{2}, & \ell \neq \upsilon. \end{cases}$$

Then ρ_c is a CV metric and hence (Ω, ρ_c) is a CV metric space.

Remark 2.9. *Note that there exist several examples of mappings defined on a metric space that do not satisfy special contractive conditions such as Banach-type contractions; however, by defining a CV metric on the space, we obtain a contraction on the CV metric space. For instance, see Example 7 in [9].*

Definition 2.10. *([41]) Consider a non-empty set Ω and a mapping $\wp : \Omega \times \Omega \to [0, \infty)$. The mapping $\wp$ is classified as a b-metric on Ω provided that it satisfies the following axioms for all $\ell, \eta, \xi \in \Omega$ and a fixed constant $s \geq 1$:*

(b1) $\wp(\ell, \eta) = 0 \iff \ell = \eta$;

(b2) $\wp(\ell, \eta) = \wp(\eta, \ell)$;

(b3) $\wp(\ell, \eta) \leq s[\wp(\ell, \xi) + \wp(\xi, \eta)]$.

Under these criteria, the pair $(\Omega, \wp)$ is referred to as a b-metric space.

Definition 2.11. *([10]) Let $\Omega \neq \emptyset$. A mapping $\wp : \Omega \times \Omega \to [0, \infty)$ is termed a Branciari metric (BM) on Ω provided that for all $\ell, \ell' \in \Omega$:*

$(BM1)$ *$\wp(\ell, \ell') = 0$ if and only if $\ell = \ell'$;*

$(BM2)$ *$\wp(\ell, \ell') = \wp(\ell', \ell)$;*

$(BM3)$ *$\wp(\ell, \ell') \leq \wp(\ell, \xi) + \wp(\xi, \eta) + \wp(\eta, \ell')$ holds for any distinct elements $\xi, \eta \in \Omega \setminus \{\ell, \ell'\}$ (the rectangular-type inequality).*

Consequently, the pair $(\Omega, \wp)$ is designated as a Branciari metric space (BMS).

Definition 2.12. *([38]) Let Ω be a non-empty set. A mapping $\rho_c : \Omega \times \Omega \to \mathbb{C}$ is designated as a complex-valued Branciari b-metric (CVBBM) on Ω provided that it adheres to the following axioms for all $\ell, \ell' \in \Omega$:*

$(BM1)$ *The distance $\rho_c(\ell, \ell')$ is always non-negative and equals zero precisely when the points ℓ and ℓ' coincide;*

$(BM2)$ *the symmetry property holds: $\rho_c(\ell, \ell') = \rho_c(\ell', \ell)$;*

$(BM3)$ *there exists a real constant $s \geq 1$ such that for any pair of distinct intermediate points $\xi, \eta \in \Omega \setminus \{\ell, \ell'\}$, the following inequality is fulfilled:*

$$\rho_c(\ell, \ell') \preceq s\left[\rho_c(\ell, \xi) + \rho_c(\xi, \eta) + \rho_c(\eta, \ell')\right].$$

Under these axioms, the structure (Ω, ρ_c) is defined as a complex-valued Branciari b-metric space (CVBBMS).

It is obvious that every complex-valued b-metric space is a CVBBMS, while the converse is generally false. To see this, consider the following example:

Example 2.13. *Let $A = \{\xi_n \mid \xi_n \in \mathbb{Q}, n = 1, 2, \cdots, \xi_i \neq \xi_j \text{ for } i \neq j\}$, $B = \{a, b\}$ where $a, b \notin \mathbb{Q}$. Take $\Omega = A \cup B$. Define $\rho_c : \Omega \times \Omega \to [0, \infty)$ by*

$$\rho_c(\ell, \xi) = \rho_c(\xi, \ell) = \begin{cases} 1 + 2i, & \ell, \xi \in A \ (\ell \neq \xi), \\ \frac{1+i}{n}, & \ell = \xi_n \in A, \ \xi \in B, \\ 2 + i, & \ell, \xi \in B \ (\ell \neq \xi) \end{cases}$$

and $\rho_c(\ell, \ell) = 0$ for all $\ell \in \Omega$.
Then it is easy to show that (Ω, ρ_c) is a CVBBMS with coefficient $s = 2$. Note that (Ω, ρ_c) is not a CV b-metric space, since

$$\begin{aligned} \lim_{n\to\infty} \frac{\left|\rho_c(\xi_n, \xi_{2n})\right|}{\left|\rho_c(\xi_n, a) + \rho_c(a, \xi_{2n})\right|} &= \lim_{n\to\infty} \frac{\sqrt{5}}{\left|\frac{1+i}{n} + \frac{1+i}{2n}\right|} \\ &= \lim_{n\to\infty} \frac{\sqrt{5}}{\frac{3\sqrt{2}}{2n}} \\ &= \lim_{n\to\infty} \frac{2\sqrt{5}n}{3\sqrt{2}} = \infty. \end{aligned}$$

Thus there is no real number $s \geq 1$ such that $\rho_c(\xi_n, \xi_{2n}) \preceq s(\rho_c(\xi_n, a) + \rho_c(a, \xi_{2n}))$.

Definition 2.14. *[38] Let (Ω, ρ_c) be a CVBBMS, $\{\ell_n\}$ be a sequence in Ω and $\ell \in \Omega$.*
(a) The sequence $\{\ell_n\}$ is said to be convergent in (Ω, ρ_c) and converges to ℓ if for every $c \succ 0$ there exists $n_0 \in \mathbb{N}$ such that $\rho_c(\ell_n, \ell) \prec c$ for all $n > n_0$ and is denoted by $\lim_{n\to\infty} \rho_c(\ell_n, \ell) = 0$ or $\ell_n \to \ell$ as $n \to \infty$.

(b) A sequence $\{\ell_n\}$ within (Ω, ρ_c) is classified as a Cauchy sequence provided that $\lim_{n\to\infty} \rho_c(\ell_n, \ell_{n+p}) = 0$ holds universally for each $p \in \mathbb{N}$.

(c) The space (Ω, ρ_c) is designated as a complete CVBBMS if every Cauchy sequence generated in Ω converges to an element belonging to Ω.

Lemma 2.15. *[38] In a CVBBMS (Ω, ρ_c), a sequence $\{\ell_n\} \subset \Omega$ converges to a limit point $\ell \in \Omega$ if and only if*

$$\lim_{n\to\infty} \left|\rho_c(\ell_n, \ell)\right| = 0.$$

Lemma 2.16. *[38] In a CVBBMS (Ω, ρ_c), a sequence $\{\ell_n\} \subset \Omega$ exhibits the Cauchy property if and only if*

$$\lim_{m,n\to\infty} \left|\rho_c(\ell_n, \ell_m)\right| = 0.$$

Lemma 2.17. *Let (Ω, ρ_c) be a CVBBMS and suppose $\{\ell_n\}$ is a Cauchy sequence in Ω comprising mutually distinct terms (meaning $\ell_n \neq \ell_m$ for distinct indices $n \neq m$). Under this setting, the sequence $\{\ell_n\}$ can converge to at most one limit point.*

Proof. Let ℓ and υ be two arbitrary limits in Ω such that $\lim_{n\to\infty} \rho_c(\ell_n, \ell) = 0$ and $\lim_{n\to\infty} \rho_c(\ell_n, \upsilon) = 0$.

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Assuming there exists a subsequence $\{\ell_{n_i}\}$ that identically equals ℓ for all indices $i \in \mathbb{N}$, we readily obtain:

$$\rho_c(\ell, \upsilon) = \lim_{i\to\infty} \rho_c(\ell_{n_i}, \upsilon) = 0,$$

which yields $\ell = \upsilon$. By applying an analogous argument, if a subsequence $\{\ell_{n_i}\}$ constantly equals υ for all $i \in \mathbb{N}$, it follows that:

$$\rho_c(\ell, \upsilon) = \lim_{i\to\infty} \rho_c(\ell_{n_i}, \ell) = 0,$$

leading again to $\ell = \upsilon$.

Consequently, we can restrict our analysis to the scenario where there exists a threshold index $N \in \mathbb{N}$ such that $\ell_n \neq \ell$ and $\ell_n \neq \upsilon$ for all $n \geq N$. Utilizing the rectangular b-metric inequality, we observe:

$$\rho_c(\ell, \upsilon) \preceq s\big[\rho_c(\ell, \ell_n) + \rho_c(\ell_n, \ell_{n+1}) + \rho_c(\ell_{n+1}, \upsilon)\big]$$

for all $n \geq N$. Taking the absolute value on both sides of this inequality yields:

$$\begin{aligned}\Big|\rho_c(\ell, \upsilon)\Big| \leq s\Big[\Big|\rho_c(\ell, \ell_n)\Big| + \Big|\rho_c(\ell_n, \ell_{n+1})\Big| \\ + \Big|\rho_c(\ell_{n+1}, \upsilon)\Big|\Big].\end{aligned}$$

Passing to the limit as $n \to \infty$ in the inequality above, we obtain $|\rho_c(\ell, \upsilon)| = 0$. Consequently, $\rho_c(\ell, \upsilon) = 0$, which guarantees that $\ell = \upsilon$. $\square$

Throughout this work, the symbols $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}^+$ are employed to represent the sets of natural, real, and positive real numbers, respectively.

Furthermore, we utilize the notations $\mathbb{C}^{\succeq 0}$, $\mathbb{C}^{\succsim 0}$, and $\mathbb{C}^{\succ 0}$ to signify the classes of complex numbers c that fulfill the respective conditions $0 \preceq c$, $0 \precsim c$, and $0 \prec c$.

In [17], Wardowski proposed a significant extension of the Banach contraction principle by introducing a novel class of mappings. Let $\mathcal{F}$ denote the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ that satisfy the following axioms:

(F1) F exhibits a strictly increasing behavior;

(F2) For any sequence $\{\varepsilon_n\}$ in $\mathbb{R}^+$, the convergence $\lim_{n\to\infty} \varepsilon_n = 0$ is equivalent to $\lim_{n\to\infty} F(\varepsilon_n) = -\infty$;

(F3) There exists a constant $k \in (0, 1)$ such that the product $\varepsilon^k F(\varepsilon)$ vanishes as $\varepsilon \to 0^+$.

Definition 2.18. *[17] Suppose (Ω, ρ) is a complete metric space. A self-mapping $T : \Omega \to \Omega$ is classified as an F-contraction if there exist a positive constant $\tau \in \mathbb{R}^+$ and a function $F \in \mathcal{F}$ such that the following implication holds for all $\ell, \upsilon \in \Omega$:*

$$\rho(T\ell, T\upsilon) > 0 \implies \tau + F(\rho(T\ell, T\upsilon)) \leq F(\rho(\ell, \upsilon)).$$

Theorem 2.19. *[17] Let (Ω, ρ) be a complete metric space. Every F-contraction mapping $T : \Omega \to \Omega$ possesses a unique fixed point $z \in \Omega$. Additionally, for an arbitrary starting point $\ell \in \Omega$, the Picard iteration sequence $\{T^n \ell\}$ is convergent to z.*

Motivated by Wardowski's idea, we want to introduce the Wardowski-type contractions in a CVBBMS.

3. Main results

Let $\mathcal{G}$ represent the family of all functions $\mathfrak{F} : \mathbb{C}^{\succsim 0} \to \mathbb{R}$ that fulfill the following requirements:

($\mathfrak{F}$1) $\mathfrak{F}$ possesses the strictly increasing property; that is, for any $c_1, c_2 \in \mathbb{C}$, the relation $c_1 \precsim c_2$ entails $\mathfrak{F}(c_1) < \mathfrak{F}(c_2)$.

($\mathfrak{F}$2) Given any sequence $\{c_n\} \subset \mathbb{C}^{\succsim 0}$, the convergence $\lim_{n\to\infty} c_n = 0$ is logically equivalent to $\lim_{n\to\infty} \mathfrak{F}(c_n) = -\infty$.

($\mathfrak{F}$3) There exists a parameter $k \in (0, 1)$ such that for every sequence $\{c_n\}$ in $\mathbb{C}^{\succsim 0}$ that converges to the origin, the limit $\lim_{n\to\infty} \Big|c_n\Big|^k \mathfrak{F}(c_n) = 0$ is maintained.

Example 3.1. *As examples of elements of $\mathcal{G}$, we have*

(1) $\mathfrak{F}_1(c) = \ln\Big|c\Big|$,

(2) $$\mathfrak{F}_2(c) = \begin{cases} \dfrac{\ln\Big|c\Big|}{\sqrt{\Big|c\Big|}}, & 0 < \Big|c\Big| \leq e^2, \\ \Big|c\Big| + \dfrac{2}{e} - e^2, & \Big|c\Big| > e^2. \end{cases}$$

(3) $\mathfrak{F}_3(c) = \ln\Big|c\Big| + \Big|c\Big|$,

(4) $\mathfrak{F}_4(c) = \dfrac{-1}{\Big|c\Big|^{\alpha}}, \quad 0 < \alpha < 1$,

(5) $$\begin{aligned}\mathfrak{F}_5(c) = \ln\Big(\frac{\mathcal{Re}\,(c)}{\cos\theta} + \frac{\mathcal{Im}\,(c)}{\sin\theta}\Big) \\ + \frac{\mathcal{Re}\,(c)}{\cos\theta} + \frac{\mathcal{Im}\,(c)}{\sin\theta},\ \theta \in (0, \tfrac{\pi}{2}).\end{aligned}$$

We also define some properties $\mathfrak{F}_{s\tau}$ and $\mathfrak{F}'_{s\tau}$ as follows:

Definition 3.2. *Let $s \geq 1$ and $\tau > 0$. We define the set $\mathcal{G}_{s,\tau}$ as a subclass of $\mathcal{G}$ consisting of functions $\mathfrak{F}$ that verify the following condition: ($\mathfrak{F}_{s\tau}$) For any $u, v, w \in \mathbb{C}^{\succsim 0}$, the inequalities $\tau + \mathfrak{F}(su) \leq \mathfrak{F}(v)$ and $\tau + \mathfrak{F}(sv) \leq \mathfrak{F}(w)$ jointly imply $\tau + \mathfrak{F}(s^2 u) \leq \mathfrak{F}(sv)$.*

Definition 3.3. *Let $s \geq 1$ and $\tau > 0$. We delineate $\mathcal{G}'_{s,\tau}$ as a subclass of $\mathcal{G}$ containing functions $\mathfrak{F}$ such that: ($\mathfrak{F}'_{s\tau}$) For every sequence $\{c_n\} \subset \mathbb{C}^{\succsim 0}$ adhering to $\tau + \mathfrak{F}(sc_n) \leq \mathfrak{F}(c_{n-1})$ for all $n \in \mathbb{N}$, the condition $\tau + \mathfrak{F}(s^n c_n) \leq \mathfrak{F}(s^{n-1} c_{n-1})$ is satisfied for all $n \in \mathbb{N}$.*

It is straightforward to verify that for any strictly increasing mapping $\mathfrak{F} : \mathbb{C}^{\succsim 0} \to \mathbb{R}$, the conditions ($\mathfrak{F}_{s\tau}$) and ($\mathfrak{F}'_{s\tau}$) are logically equivalent. Consequently, the classes $\mathcal{G}_{s,\tau}$ and $\mathcal{G}'_{s,\tau}$ coincide (the proof is completely similar to the proof in [42] for the case where the domain of $\mathfrak{F}$ is $(0, \infty)$).

Theorem 3.4. *Consider a complete CVBBMS (Ω, ρ_c) characterized by a constant coefficient $s \geq 1$, and let $\Upsilon : \Omega \to \Omega$ represent a self-mapping. Suppose there* *exist a function $\mathfrak{F} \in \mathfrak{F}_{s\tau}$, a constant $\tau > 0$, and a triangular function $\xi : \Omega \times \Omega \to [0,\infty)$ such that the following contractive constraint is imposed:*

$$\tau + \mathfrak{F}(s\rho_c(\Upsilon\ell, \Upsilon\upsilon)) \leq \mathfrak{F}(\rho_c(\ell, \upsilon)) \tag{1}$$

whenever $\xi(\ell, \upsilon) \geq 1$ and $\Upsilon\ell \neq \Upsilon\upsilon$. Furthermore, the following conditions are assumed to be satisfied:

(i) the operator Υ is ξ-admissible;

(ii) there exists at least one element $\ell_0 \in \Omega$ such that $\xi(\ell_0, \Upsilon\ell_0) \geq 1$;

(iii) the space Ω is ξ-regular or the mapping Υ is continuous.

Under these premises, the mapping Υ guarantees the existence of a fixed point. Additionally, the uniqueness of this fixed point is verified provided that $\xi(\ell, \upsilon) \geq 1$ holds for every pair of fixed points $\ell, \upsilon \in \Omega$.

Proof. Consider a sequence $\{\ell_n\}$ in Ω such that $\ell_n = \Upsilon\ell_{n-1} = \Upsilon^n\ell_0$ for all $n \in \mathbb{N}$. Then $\ell_{n+1} = \Upsilon\ell_n$ for all $n \in \mathbb{N} \cup \{0\}$. If $\ell_{n+1} = \ell_n$, for some $n \in \mathbb{N} \cup \{0\}$, then ℓ_n is a fixed point of Υ and we have nothing to prove. Suppose that $\ell_{n+1} \neq \ell_n$ for all $n \in \mathbb{N}$. Thus $\Upsilon\ell_n \neq \Upsilon\ell_{n-1}$ for all $n \in \mathbb{N}$. Also, we have $\xi(\ell_0, \ell_1) = \xi(\ell_0, \Upsilon\ell_0) \geq 1$ and since Υ is ξ-admissible, we can show by induction that $\xi(\ell_n, \ell_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Since ξ is triangular, we have also $\xi(\ell_n, \ell_m) \geq 1$ for all $n, m \in \mathbb{N} \cup \{0\}$ with $n < m$. Now, we show that $\ell_n \neq \ell_m$ for all $n, m \in \mathbb{N}$ with $n \neq m$.

Assume that $n < m$ and $\ell_n = \ell_m$, seeking a contradiction.

Then by (1), for any $k \in \mathbb{N}$,

$$\begin{aligned}\mathfrak{F}(\rho_c(\ell_k, \ell_{k+1})) &\leq \mathfrak{F}(s\rho_c(\ell_k, \ell_{k+1})) \\ &= \mathfrak{F}(s\rho_c(\Upsilon\ell_{k-1}, \Upsilon\ell_k)) \\ &\leq \mathfrak{F}(\rho_c(\ell_{k-1}, \ell_k)) - \tau \\ &< \mathfrak{F}(\rho_c(\ell_{k-1}, \ell_k)).\end{aligned}$$

Since $\mathfrak{F}$ is strictly increasing, we get $\rho_c(\ell_k, \ell_{k+1}) \precnsim \rho_c(\ell_{k-1}, \ell_k)$. Now, we have

$$\begin{aligned}\rho_c(\ell_m, \ell_{m+1}) &\precnsim \rho_c(\ell_{m-1}, \ell_m) \precnsim \cdots \\ &\precnsim \rho_c(\ell_n, \ell_{n+1}) \\ &= \rho_c(\ell_n, \Upsilon\ell_n) = \rho_c(\ell_m, \Upsilon\ell_m) \\ &= \rho_c(\ell_m, \ell_{m+1}),\end{aligned}$$

which is a contradiction. Thus $\ell_n \neq \ell_m$ for all $n, m \in \mathbb{N}$ with $n \neq m$. Now by (1),

$$\begin{aligned}\mathfrak{F}(s\rho_c(\ell_n, \ell_{n+1})) &= \mathfrak{F}(s\rho_c(\Upsilon\ell_{n-1}, \Upsilon\ell_n)) \\ &\leq \mathfrak{F}(\rho_c(\ell_{n-1}, \ell_n)) - \tau,\end{aligned} \tag{2}$$

for all $n \in \mathbb{N}$. Taking $c_n = \rho_c(\ell_n, \ell_{n+1})$, we have $\mathfrak{F}(sc_n) \leq \mathfrak{F}(c_{n-1}) - \tau$ for all $n \in \mathbb{N}$. From $(\mathfrak{F}'_{s\tau})$, we get $\tau + \mathfrak{F}(s^n c_n) \leq \mathfrak{F}(s^{n-1}c_{n-1})$ for all $n \in \mathbb{N}$. Now, we have

$$\begin{aligned}\mathfrak{F}(s^n c_n) &\leq \mathfrak{F}(s^{n-1}c_{n-1}) - \tau \\ &\leq \mathfrak{F}(s^{n-2}c_{n-2}) - 2\tau \leq \cdots \leq \mathfrak{F}(c_0) - n\tau.\end{aligned} \tag{3}$$

As $n \to \infty$, we find $\lim_{n\to\infty} \mathfrak{F}(s^n c_n) = -\infty$. From $(\mathfrak{F}2)$, we get $\lim_{n\to\infty} s^n c_n = 0$. Now we shall show that $\{\ell_n\}$ is a Cauchy sequence. From (3), we have

$$n\tau \leq \mathfrak{F}(c_0) - \mathfrak{F}(s^n c_n). \tag{4}$$

Multiplying $\left|s^n c_n\right|^k$ in both sides of (4) and taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \left|s^n c_n\right|^k n\tau = 0.$$

Thus

$$\lim_{n\to\infty} \left|s^n c_n\right|^k n = 0.$$

Therefore there exists $N \in \mathbb{N}$ such that

$$\left|s^n c_n\right| \leq \frac{1}{n^{\frac{1}{k}}} \quad \text{for all} \quad n \geq N.$$

Now, we have

$$\sum_{n=N}^{\infty} \left|s^n c_n\right| \leq \sum_{n=N}^{\infty} \frac{1}{n^{\frac{1}{k}}} < \infty.$$

Thus

$$\sum_{n=0}^{\infty} \left|s^n c_n\right| < \infty,$$

that is, $\sum_{n=0}^{\infty} \left|s^n c_n\right|$ is convergent. Similarly, taking $d_n = \rho_c(\ell_n, \ell_{n+2})$ and using (3.4), we can also prove $\lim_{n\to\infty} d_n = 0$. For any odd integer $p > 1$ we have, if $p = 3$, then

$$\begin{aligned}\left|\rho_c(\ell_n, \ell_{n+3})\right| &\leq s\Big(\left|\rho_c(\ell_n, \ell_{n+1})\right| + \left|\rho_c(\ell_{n+1}, \ell_{n+2})\right| \\ &\quad + \left|\rho_c(\ell_{n+2}, \ell_{n+3})\right|\Big) \\ &= \left|sc_n\right| + \left|sc_{n+1}\right| + \left|sc_{n+2}\right|.\end{aligned}$$

For $p = 5$,

$$\begin{aligned}\left|\rho_c(\ell_n, \ell_{n+5})\right| &\leq s\Big(\left|\rho_c(\ell_n, \ell_{n+1})\right| + \left|\rho_c(\ell_{n+1}, \ell_{n+2})\right| \\ &\quad + \left|\rho_c(\ell_{n+2}, \ell_{n+5})\right|\Big) \\ &= \left|sc_n\right| + \left|sc_{n+1}\right| + \left|s\rho_c(\ell_{n+2}, \ell_{n+5})\right| \\ &\leq \left|sc_n\right| + \left|sc_{n+1}\right| + \left|s^2 c_{n+2}\right| \\ &\quad + \left|s^2 c_{n+3}\right| + \left|s^2 c_{n+4}\right|.\end{aligned}$$

Continuing this process, we get

$$\begin{aligned}|d(\ell_n, \ell_{n+p})| &\leq \left|sc_n\right| + \left|sc_{n+1}\right| \\ &\quad + \left|s^2 c_{n+2}\right| + \left|s^2 c_{n+3}\right| + \cdots \\ &\quad + \left|s^{\frac{p-3}{2}} c_{n+p-5}\right| + \left|s^{\frac{p-3}{2}} c_{n+p-4}\right| \\ &\quad + \left|s^{\frac{p-1}{2}} c_{n+p-3}\right| + \left|s^{\frac{p-1}{2}} c_{n+p-2}\right| \\ &\quad + \left|s^{\frac{p-1}{2}} c_{n+p-1}\right|\end{aligned}$$

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$$
\begin{aligned}
&\leq \left|s^{\frac{2}{2}}c_n\right| + \left|s^{\frac{3}{2}}c_{n+1}\right| \\
&\quad + \left|s^{\frac{4}{2}}c_{n+2}\right| + \left|s^{\frac{5}{2}}c_{n+3}\right| + \cdots \\
&\quad + \left|s^{\frac{p-3}{2}}c_{n+p-5}\right| + \left|s^{\frac{p-2}{2}}c_{n+p-4}\right| \\
&\quad + \left|s^{\frac{p-1}{2}}c_{n+p-3}\right| + \left|s^{\frac{p}{2}}c_{n+p-2}\right| \\
&\quad + \left|s^{\frac{p+1}{2}}c_{n+p-1}\right| \\
&\leq \left|s^{2}c_n\right| + \left|s^{3}c_{n+1}\right| \\
&\quad + \left|s^{4}c_{n+2}\right| + \left|s^{5}c_{n+3}\right| + \cdots \\
&\quad + \left|s^{p-3}c_{n+p-5}\right| + \left|s^{p-2}c_{n+p-4}\right| \\
&\quad + \left|s^{p-1}c_{n+p-3}\right| + \left|s^{p}c_{n+p-2}\right| \\
&\quad + \left|s^{p+1}c_{n+p-1}\right| \\
&= \frac{1}{s^{n-2}}\Big[\\
&\qquad \left|s^{n}c_n\right| + \left|s^{n+1}c_{n+1}\right| \\
&\quad + \left|s^{n+2}c_{n+2}\right| + \left|s^{n+3}c_{n+3}\right| + \cdots \\
&\quad + \left|s^{n+p-5}c_{n+p-5}\right| \\
&\qquad + \left|s^{n+p-4}c_{n+p-4}\right| \\
&\quad + \left|s^{n+p-3}c_{n+p-3}\right| \\
&\qquad + \left|s^{n+p-2}c_{n+p-2}\right| \\
&\quad + \left|s^{n+p-1}c_{n+p-1}\right|\Big] \\
&\leq \frac{1}{s^{n-2}}\sum_{i=n}^{\infty}\left|s^{i}c_i\right|.
\end{aligned}
$$

In view of the convergence of the series $\sum_{i=0}^{\infty}\left|s^{i}c_i\right|$, the terminal term in the above inequality vanishes as $n \to \infty$.

Now let p is an even integer with $p > 2$. If $p = 4$, then

$$
\begin{aligned}
\left|\rho_c(\ell_n, \ell_{n+4})\right| &\leq s\Big(\left|\rho_c(\ell_n, \ell_{n+1})\right| + \left|\rho_c(\ell_{n+1}, \ell_{n+2})\right| \\
&\quad + \left|\rho_c(\ell_{n+2}, \ell_{n+4})\right|\Big) \\
&= \left|sc_n\right| + \left|sc_{n+1}\right| \\
&\quad + \left|s\rho_c(\ell_{n+2}, \ell_{n+4})\right|.
\end{aligned}
$$

For $p = 6$,

$$
\begin{aligned}
\left|\rho_c(\ell_n, \ell_{n+6})\right| &\leq s\Big(\left|\rho_c(\ell_n, \ell_{n+1})\right| + \left|\rho_c(\ell_{n+1}, \ell_{n+2})\right| \\
&\quad + \left|\rho_c(\ell_{n+2}, \ell_{n+6})\right|\Big) \\
&= \left|sc_n\right| + \left|sc_{n+1}\right| + \left|s\rho_c(\ell_{n+2}, \ell_{n+6})\right| \\
&\leq \left|sc_n\right| + \left|sc_{n+1}\right| \\
&\quad + \left|s^{2}c_{n+2}\right| + \left|s^{2}c_{n+3}\right| \\
&\quad + \left|s^{2}\rho_c(\ell_{n+4}, \ell_{n+6})\right|.
\end{aligned}
$$

Continuing in this manner, we obtain

$$
\begin{aligned}
\left|d(\ell_n, \ell_{n+p})\right| &\leq \left|s^{\frac{2}{2}}c_n\right| + \left|s^{\frac{2}{2}}c_{n+1}\right| \\
&\quad + \left|s^{\frac{4}{2}}c_{n+2}\right| + \left|s^{\frac{4}{2}}c_{n+3}\right| + \cdots \\
&\quad + \left|s^{\frac{p-4}{2}}c_{n+p-6}\right| + \left|s^{\frac{p-4}{2}}c_{n+p-5}\right| \\
&\quad + \left|s^{\frac{p-2}{2}}c_{n+p-4}\right| + \left|s^{\frac{p-2}{2}}c_{n+p-3}\right| \\
&\quad + \left|s^{\frac{p-2}{2}}\rho_c(x_{n+p-2}, x_{n+p})\right| \\
&\leq \left|s^{\frac{2}{2}}c_n\right| + \left|s^{\frac{3}{2}}c_{n+1}\right| \\
&\quad + \left|s^{\frac{4}{2}}c_{n+2}\right| + \left|s^{\frac{5}{2}}c_{n+3}\right| + \cdots \\
&\quad + \left|s^{\frac{p-4}{2}}c_{n+p-6}\right| + \left|s^{\frac{p-3}{2}}c_{n+p-5}\right| \\
&\quad + \left|s^{\frac{p-2}{2}}c_{n+p-4}\right| + \left|s^{\frac{p-1}{2}}c_{n+p-3}\right| \\
&\quad + \left|s^{\frac{p-2}{2}}\rho_c(x_{n+p-2}, x_{n+p})\right| \\
&\leq \left|s^{2}c_n\right| + \left|s^{3}c_{n+1}\right| \\
&\quad + \left|s^{4}c_{n+2}\right| + \left|s^{5}c_{n+3}\right| + \cdots \\
&\quad + \left|s^{p-4}c_{n+p-6}\right| + \left|s^{p-3}c_{n+p-5}\right| \\
&\quad + \left|s^{p-2}c_{n+p-4}\right| + \left|s^{p-1}c_{n+p-3}\right| \\
&\quad + \left|s^{\frac{p-2}{2}}\rho_c(x_{n+p-2}, x_{n+p})\right| \\
&= \frac{1}{s^{n-2}}\Big[\\
&\qquad \left|s^{n}c_n\right| + \left|s^{n+1}c_{n+1}\right| \\
&\quad + \left|s^{n+2}c_{n+2}\right| + \left|s^{n+3}c_{n+3}\right| + \cdots \\
&\quad + \left|s^{n+p-6}c_{n+p-6}\right| \\
&\qquad + \left|s^{n+p-5}c_{n+p-5}\right| \\
&\quad + \left|s^{n+p-4}c_{n+p-4}\right| \\
&\qquad + \left|s^{n+p-3}c_{n+p-3}\right|\Big] \\
&\quad + s^{\frac{p-2}{2}}\left|d_{n+p-2}\right| \\
&\leq \frac{1}{s^{n-2}}\sum_{i=n}^{\infty}\left|s^{i}c_i\right| \\
&\quad + s^{\frac{p-2}{2}}\left|d_{n+p-2}\right|.
\end{aligned}
$$

Since $\sum_{i=0}^{\infty}\left|s^{i}c_i\right|$ is convergent and $\lim_{n\to\infty} d_n = 0$, the last two terms in the above inequality tend to zero as $n \longrightarrow \infty$. Thus, $\{\ell_n\}$ is a Cauchy sequence in the complete CVBBMS (Ω, ρ_c). Hence there is $\ell^* \in \Omega$ so that

$$
\lim_{n\longrightarrow\infty} \rho_c(\ell_n, \ell^*) = 0.
$$

If Ω is ξ-regular, then $\xi(\ell_n, \ell^*) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. We consider two cases:

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(case1): There exists $N \in \mathbb{N}$ so that $\Upsilon\ell_n \neq \Upsilon\ell^*$ for each $n \geq N$. Then, obviously $\ell_n \neq \ell^*$ for each $n \geq N$. Now

$$\begin{aligned}\mathfrak{F}(\rho_c(\ell_{n+1}, \Upsilon\ell^*)) &= \mathfrak{F}(\rho_c(\Upsilon\ell_n, \Upsilon\ell^*)) \\ &\leq \mathfrak{F}(\rho_c(\ell_n, \ell^*)) - \tau.\end{aligned}$$

Upon taking the limit as $n \to \infty$ in the preceding inequality, it follows that $\lim_{n\to\infty} \mathfrak{F}(\rho_c(\ell_{n+1}, \Upsilon\ell^*)) = -\infty$ and so $\lim_{n\to\infty} \rho_c(\ell_{n+1}, \Upsilon\ell^*) = 0$. Since the limit of ℓ_n is unique, thus $\Upsilon\ell^* = \ell^*$.

case(2): There exists a subsequence $\{\ell_{n_k}\}$ of $\{\ell_n\}$ so that $\Upsilon\ell_{n_k} = \Upsilon\ell^*$ for each $k \geq 1$. In this case, $\ell_{n_k+1} = \Upsilon\ell_{n_k} = \Upsilon\ell^*$ for each $k \geq 1$. Therefore $\lim_{k\to\infty} \ell_{n_k+1} = \Upsilon\ell^*$. Since the limit of ℓ_{n_k+1} is unique, thus $\Upsilon\ell^* = \ell^*$.

Now, let Υ is continuous. Then $\lim_{n\to\infty} \rho_c(\Upsilon\ell_n, \Upsilon\ell^*) = 0$. Thus $\lim_{n\to\infty} \rho_c(\ell_{n+1}, \Upsilon\ell^*) = 0$. By the uniqueness of limit for ℓ_{n+1}, we have $\Upsilon\ell^* = \ell^*$.

Moreover, suppose that $\xi(\ell, \upsilon) \geq 1$ for all fixed points ℓ, υ of Υ. If the fixed point were not unique, then there would exist another fixed point $\upsilon^* \neq \ell^*$ of Υ. Then by (1),

$$\begin{aligned}\mathfrak{F}\big(\rho(\ell^*, \upsilon^*)\big) &< \tau + \mathfrak{F}\big(\rho(\ell^*, \upsilon^*)\big) \\ &= \tau + \mathfrak{F}\big(\rho(\Upsilon\ell^*, \Upsilon\upsilon^*)\big) \\ &\leq \mathfrak{F}\big(\rho(\ell^*, \upsilon^*)\big).\end{aligned}$$

which leads to a contradiction. Consequently, the uniqueness of the fixed point is established. □

Taking $\mathfrak{F}(c) = \ln\Big|c\Big|$ in Theorem 3.4, we get the following Banach type contraction result in a CVBBMS:

Corollary 3.5. *Let (Ω, ρ_c) be a complete CVBBMS with constant $s \geq 1$ and $\Upsilon : \Omega \to \Omega$ be a self-mapping satisfying*

$$s\rho_c(\Upsilon\ell, \Upsilon\upsilon) \preceq \lambda\rho_c(\ell, \upsilon) \tag{5}$$

for all $\ell, \upsilon \in \Omega$ with $\Upsilon\ell \neq \Upsilon\upsilon$, where $\lambda \in \mathbb{C}$ with $0 \leq \Big|\lambda\Big| < 1$. Then, Υ has a unique fixed point ℓ^*.

Proof. From (5), we have

$$\Big|s\rho_c(\Upsilon\ell, \Upsilon\upsilon)\Big| \leq \Big|\lambda\Big|\Big|\rho_c(\ell, \upsilon)\Big|. \tag{6}$$

Applying $\ln$ on both sides of the above inequality, we have

$$\ln\Big|s\rho_c(\Upsilon\ell, \Upsilon\upsilon)\Big| \leq \ln\Big|\lambda\Big| + \ln\Big|\rho_c(\ell, \upsilon)\Big|.$$

Define $\xi : \Omega \times \Omega \to [0, \infty)$ by $\xi(\ell, \upsilon) = 1$ for all $\ell, \upsilon \in \Omega$. Then all of the conditions of Theorem 3.4 are satisfied for $\mathfrak{F}(c) = \ln\Big|c\Big|$ and $\tau = -\ln(\Big|\lambda\Big|)$. Thus, Υ has a unique fixed point ℓ^*. □

Remark 3.6. *Note that any CV Branciari metric space is also a CVBBMS with constant $s = 1$. In this case, any function $\mathfrak{F} \in \mathcal{G}$ belongs to $\mathcal{G}_{s,\tau}$. Consequently, $\mathcal{G} = \mathcal{G}_{s,\tau}$.*

Corollary 3.7. *In the framework of a complete CV Branciari metric space (Ω, ρ_c), let the self-mapping $\Upsilon : \Omega \to \Omega$ satisfy the following condition for some $\mathfrak{F} \in \mathcal{G}$ and $\tau > 0$:*

$$\tau + \mathfrak{F}(\rho_c(\Upsilon\ell, \Upsilon\upsilon)) \leq \mathfrak{F}(\rho_c(\ell, \upsilon)) \tag{7}$$

whenever $\Upsilon\ell \neq \Upsilon\upsilon$ and $\xi(\ell, \upsilon) \geq 1$, where $\xi : \Omega\times\Omega \to [0, \infty)$ is a triangular mapping. Assume further that:

(i) the operator Υ exhibits ξ-admissibility;

(ii) an element $\ell_0 \in \Omega$ exists such that $\xi(\ell_0, \Upsilon\ell_0) \geq 1$;

(iii) Υ is either continuous or the space Ω satisfies the ξ-regularity property.

Under these hypotheses, Υ possesses at least one fixed point. Furthermore, the uniqueness of the fixed point is guaranteed provided that $\xi(\ell, \upsilon) \geq 1$ for any pair of fixed points $\ell, \upsilon \in \Omega$.

If we put $\mathfrak{F}(c) = \ln(\Big|c\Big|) + \Big|c\Big|$ in Corollary 3.7, we derive the following result:

Corollary 3.8. *Consider a complete CV Branciari metric space (Ω, ρ_c) and a self-mapping Υ defined on Ω. Suppose that for some constant $\tau > 0$ and a triangular mapping $\xi : \Omega \times \Omega \to [0, \infty)$, the following inequality is satisfied:*

$$\frac{\Big|\rho_c(\Upsilon\ell, \Upsilon\upsilon)\Big|}{\Big|\rho_c(\ell, \upsilon)\Big|} e^{\Big|\rho_c(\Upsilon\ell,\Upsilon\upsilon)\Big| - \Big|\rho_c(\ell,\upsilon)\Big|} \leq e^{-\tau}$$

for all $\ell, \upsilon \in \Omega$ such that $\Upsilon\ell \neq \Upsilon\upsilon$ and $\xi(\ell, \upsilon) \geq 1$. Furthermore, assume that the conditions below are met:

(i) the mapping Υ is ξ-admissible;

(ii) there exists at least one element $\ell_0 \in \Omega$ satisfying $\xi(\ell_0, \Upsilon\ell_0) \geq 1$;

(iii) either the continuity of Υ is guaranteed or Ω possesses the ξ-regularity property.

Then, there exists a fixed point for Υ. In addition, this fixed point is unique provided that $\xi(\ell, \upsilon) \geq 1$ for all $\ell, \upsilon \in Fix(\Upsilon)$.

Considering $\mathfrak{F}(c) = \ln(\frac{\mathcal{R}e(c)}{\cos\theta} + \frac{\mathcal{I}m(c)}{\sin\theta})$ in Corollary 3.7, where $\theta \in (0, \frac{\pi}{2})$, we get the following result:

Corollary 3.9. *Let the pair (Ω, ρ_c) denote a complete CV Branciari metric space and let Υ be a self-mapping on Ω. Suppose that for some constant $k \in (0, 1)$, a parameter $\theta \in (0, \frac{\pi}{2})$, and a triangular mapping ξ :*

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$\Omega \times \Omega \to [0,\infty)$, *the following contractive condition is satisfied:*

$$\frac{\mathcal{R}e\,\rho_c(\Upsilon\ell,\Upsilon\upsilon)}{\cos\theta}+\frac{\mathcal{I}m\,\rho_c(\Upsilon\ell,\Upsilon\upsilon)}{\sin\theta} \le k\left(\frac{\mathcal{R}e\,\rho_c(\ell,\upsilon)}{\cos\theta}+\frac{\mathcal{I}m\,\rho_c(\ell,\upsilon)}{\sin\theta}\right)$$

for all $\ell,\upsilon\in\Omega$ *such that* $\xi(\ell,\upsilon)\ge 1$ *and* $\Upsilon\ell\neq\Upsilon\upsilon$. *Assume further that:*

(i) the operator Υ *is endowed with the property of* ξ*-admissibility;*

(ii) the set $\{\ell\in\Omega:\xi(\ell,\Upsilon\ell)\ge 1\}$ *is non-empty;*

(iii) either the continuity of Υ *holds or the space* Ω *is characterized as* ξ*-regular.*

Under these requirements, Υ *admits a fixed point. Furthermore, if the condition* $\xi(\ell,\upsilon)\ge 1$ *is satisfied for all* $\ell,\upsilon\in Fix(\Upsilon)$, *then the uniqueness of the fixed point is ensured.*

If we take $\mathfrak{F}(c)=\ln(\frac{\mathcal{R}e\,(c)}{\cos\theta}+\frac{\mathcal{I}m\,(c)}{\sin\theta})+\frac{\mathcal{R}e\,(c)}{\cos\theta}+\frac{\mathcal{I}m\,(c)}{\sin\theta}$ in Corollary 3.7, we get the following result:

Corollary 3.10. *Consider a complete CV Branciari metric space* (Ω,ρ_c) *and let* $\Upsilon:\Omega\to\Omega$ *be a self-mapping. Suppose that for a fixed* $\tau>0$, *a parameter* $\theta\in(0,\frac{\pi}{2})$, *and a triangular function* $\xi:\Omega\times\Omega\to[0,\infty)$, *the following exponential-type contraction holds:*

$$\frac{\frac{\mathcal{R}e\,\rho_c(\Upsilon\ell,\Upsilon\upsilon)}{\cos\theta}+\frac{\mathcal{I}m\,\rho_c(\Upsilon\ell,\Upsilon\upsilon)}{\sin\theta}}{\frac{\mathcal{R}e\,\rho_c(\ell,\upsilon)}{\cos\theta}+\frac{\mathcal{I}m\,\rho_c(\ell,\upsilon)}{\sin\theta}} \times\exp\left(\frac{\mathcal{R}e\,\rho_c(\Upsilon\ell,\Upsilon\upsilon)}{\cos\theta}+\frac{\mathcal{I}m\,\rho_c(\Upsilon\ell,\Upsilon\upsilon)}{\sin\theta} -\left[\frac{\mathcal{R}e\,\rho_c(\ell,\upsilon)}{\cos\theta}+\frac{\mathcal{I}m\,\rho_c(\ell,\upsilon)}{\sin\theta}\right]\right)\le e^{-\tau}$$

whenever $\xi(\ell,\upsilon)\ge 1$ *and* $\Upsilon\ell\neq\Upsilon\upsilon$. *Furthermore, assume the following:*

(i) the mapping Υ *satisfies the* ξ*-admissibility condition;*

(ii) the set of elements $\{\ell\in\Omega:\xi(\ell,\Upsilon\ell)\ge 1\}$ *is non-empty;*

(iii) the requirement of Υ *being continuous or* Ω *being* ξ*-regular is fulfilled.*

Then, Υ *possesses a fixed point. This fixed point is unique provided that* $\xi(\ell,\upsilon)\ge 1$ *for each pair of fixed points* $\ell,\upsilon\in\Omega$.

Example 3.11. *Let* $A=\{\frac{1}{n^2}\mid n=2,3,\cdots\}$, $B=\{0,1\}$, $\Omega=A\cup B$ *and define* $\rho_c:\Omega\times\Omega\to\mathbb{C}$ *by*

$$\rho_c(\ell,\upsilon)=\begin{cases}\frac{1}{\sqrt{2}n^2}(1+i); & \ell\in A,\upsilon\in B \text{ or } \ell\in B,\upsilon\in A,\\ \frac{1}{\sqrt{2}}(1+i); & \ell,\upsilon\in A,\ \ell\neq\upsilon,\\ \frac{1}{\sqrt{2}}(1+i); & \ell,\upsilon\in B,\ \ell\neq\upsilon\end{cases}$$

and $\rho_c(\ell,\ell)=0$ *for all* $\ell\in\Omega$. *Define a mapping* $\Upsilon:\Omega\to\Omega$ *by*

$$\Upsilon\ell=\begin{cases}\frac{1}{(n+1)^2}, & \ell=\frac{1}{n^2}\in A,\\ 0, & \ell\in B.\end{cases}$$

Define a function $\xi:\Omega\times\Omega\to[0,\infty)$ *by*

$$\xi(\ell,\upsilon)=\begin{cases}0; & \ell\in A,\ \upsilon\in A,\\ 1; & otherwise.\end{cases}$$

Take

$$\mathfrak{F}(c)=\begin{cases}\frac{\ln\left|c\right|}{\sqrt{\left|c\right|}}, & 0<\left|c\right|\le e^2,\\ \left|c\right|+\frac{2}{e}-e^2, & \left|c\right|>e^2.\end{cases}$$

It is easy to check that (Ω,ρ_c) *is a complete CV Branciari metric space,* Ω *is* ξ*-regular and* Υ *is* ξ*-admissible. Also, for* $\ell_0=1$, *we have* $\xi(\ell_0,\Upsilon\ell_0)=\xi(1,0)=1\ge 1$. *We want to obtain the contraction (1) in Corollary 3.7, for* $\tau=\ln 2$. *For any* $\ell,\upsilon\in\Omega$ *with* $\xi(\ell,\upsilon)\ge 1$ *and* $\Upsilon\ell\neq\Upsilon\upsilon$, *we have* $\ell\in A,\upsilon\in B$ *or* $\ell\in B,\upsilon\in A$. *Therefore, we have*

$$\rho_c(\Upsilon\ell,\Upsilon\upsilon)=\rho_c\left(\frac{1}{(n+1)^2},0\right)=\frac{1}{\sqrt{2}(n+1)^2}(1+i)$$

and

$$\rho_c(\ell,\upsilon)=\rho_c\left(\frac{1}{n^2},\upsilon\right)=\frac{1}{\sqrt{2}n^2}(1+i).$$

Also, we have $\left|\rho_c(\Upsilon\ell,\Upsilon\upsilon)\right|=\frac{1}{(n+1)^2}\le e^2$ *and* $\left|\rho_c(\ell,\upsilon)\right|=\frac{1}{n^2}\le e^2$. *Thus we shall show that*

$$\tau+\frac{\ln\left|\rho_c(\Upsilon\ell,\Upsilon\upsilon)\right|}{\sqrt{\left|\rho_c(\Upsilon\ell,\Upsilon\upsilon)\right|}}\le\frac{\ln\left|\rho_c(\ell,\upsilon)\right|}{\sqrt{\left|\rho_c(\ell,\upsilon)\right|}}. \tag{8}$$

This is equal to show that

$$\left|\rho_c(\Upsilon\ell,\Upsilon\upsilon)\right|^{\frac{1}{\sqrt{\left|\rho_c(\Upsilon\ell,\Upsilon\upsilon)\right|}}}\left|\rho_c(\ell,\upsilon)\right|^{-\frac{1}{\sqrt{\left|\rho_c(\ell,\upsilon)\right|}}}\le\frac{1}{2},$$

or, equivalently

$$\left(\frac{1}{(n+1)^2}\right)^{n+1}\left(\frac{1}{n^2}\right)^{-n} \leq \frac{1}{2},$$

which is equal to

$$\frac{n^{2n}}{(n+1)^{2n+2}} \leq \frac{1}{2}.$$

This is true, since

$$\frac{n^{2n}}{(n+1)^{2n+2}} = \left(\frac{n}{n+1}\right)^{2n}\frac{1}{n+1} \leq \frac{1}{2}.$$

Thus (8) and so (7) are satisfied in Corollary 3.7. Moreover, $\xi(\ell,\upsilon) \geq 1$ for all fixed points ℓ,υ of Υ. So, by Corollary 3.7, Υ has a unique fixed point, which is 0 here. Note that

$$\begin{aligned}\sup_{n\in\mathbb{N}} \frac{\left|\rho_c(\Upsilon\ell,\Upsilon\upsilon)\right|}{\left|\rho_c(\ell,\upsilon)\right|} &= \sup_{n\in\mathbb{N}} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} \\ &= \sup_{n\in\mathbb{N}} \frac{n^2}{(n+1)^2} = 1.\end{aligned}$$

So, Υ is not a Banach type contraction and so the result of Ege [38] on CV Branciari metric spaces is not applicable in this example.

Taking $\mathfrak{F}(c) = \frac{-1}{\sqrt{|c|}}$ in Theorem 3.4, we get the following result:

Corollary 3.12. *Within the framework of a complete CV metric space (Ω,ρ_c), let $\Upsilon : \Omega \to \Omega$ act as a self-mapping. Suppose that for a positive constant $\tau > 0$ and a mapping $\xi : \Omega \times \Omega \to [0,\infty)$, the following contractive inequality is valid:*

$$\left|\rho_c(\Upsilon\ell,\Upsilon\upsilon)\right| \leq \frac{\left|\rho_c(\ell,\upsilon)\right|}{(1+\tau\sqrt{\left|\rho_c(\ell,\upsilon)\right|})^2} \tag{9}$$

for every pair $\ell,\upsilon \in \Omega$ satisfying $\Upsilon\ell \neq \Upsilon\upsilon$ and $\xi(\ell,\upsilon) \geq 1$. Furthermore, assume the following prerequisites are met:

(i) there exists at least one element $\ell_0 \in \Omega$ such that $\xi(\ell_0,\Upsilon\ell_0) \geq 1$;

(ii) the operator Υ is ξ-admissible;

(iii) the space Ω is ξ-regular or the operator Υ is continuous.

Under these conditions, Υ attains at least one fixed point $\ell^* \in \Omega$. Additionally, the uniqueness of this fixed point is secured provided that $\xi(\ell,\upsilon) \geq 1$ for any two fixed points $\ell,\upsilon \in \Omega$.

4. Application to fractional integral equations

First, we provide a brief overview of essential concepts regarding fractional calculus, following the terminology in [43, 44]. Consider a continuous mapping $f : [0,T] \to \mathbb{R}$. The fractional-order Riemann-Liouville integral (where $\alpha > 0$) is characterized as:

$$\mathfrak{J}^{\alpha}_{0+} f(w) = \frac{1}{\Gamma(\alpha)}\int_0^T (w-u)^{\alpha-1} f(u)\,du. \tag{10}$$

In the limiting case where $\alpha = 0$, we set $\mathfrak{J}^0_{0+} f(w) = f(w)$ by convention.

Furthermore, for an order γ such that $n-1 < \gamma \leq n$ and $n = -[-\gamma]$, the Riemann-Liouville fractional derivative is given by:

$${}^{RL}\mathfrak{D}^{\gamma}_{0+} f(w) = \left(\frac{d}{dw}\right)^n \mathfrak{J}^{n-\gamma}_{0+} f(w), \tag{11}$$

where the symbol $[.]$ represents the standard floor (bracket) function on $\mathbb{R}$.

The literature contains an extensive body of work exploring fractional integro-differential equations through the lens of Riemann-Liouville operators (refer to [45–52] for further details).

In our analysis, we denote the Banach space of all real-valued continuous functions on $[0,T]$ by $\nabla = C([0,T],\mathbb{R})$, which is furnished with the uniform norm:

$$\|\zeta\| = \sup_{t\in[0,T]} |\zeta(t)|.$$

Moreover, for any $r \geq 0$, we introduce the space $\nabla_r = C_r([0,T],\mathbb{R})$ as the collection of functions $u : [0,T] \to \mathbb{R}$ that satisfy the condition that $u_r(t) = t^r u(t)$ remains continuous. This space is equipped with the weighted supremum norm defined by:

$$\|u\|_r = \sup_{t\in[0,T]} \{t^r|u(t)|\}.$$

Ahmad and Nieto in [53] investigated the existence of unique solution for the following Cauchy type fractional differential equation:

$$\begin{aligned}{}^{RL}\mathfrak{D}^{\gamma}_{0+} f(w) &= \mathcal{K}(w, f(w)),\\ &\quad w \in [0,T],\ 1 < \gamma \leq 2,\end{aligned} \tag{12}$$

with the boundary conditions of fractional order

$$\begin{aligned}{}^{RL}\mathfrak{D}^{\gamma-2}_{0+} f(0^+) &= b_0\, {}^{RL}\mathfrak{D}^{\gamma-2}_{0+} f(T^-),\\ {}^{RL}\mathfrak{D}^{\gamma-1}_{0+} f(0^+) &= b_1\, {}^{RL}\mathfrak{D}^{\gamma-1}_{0+} f(T^-),\end{aligned} \tag{13}$$

where $b_0 \neq 1$, $b_1 \neq 1$ are real constants, $f \in C_{2-\gamma}([0,T],\mathbb{R})$ and $\mathcal{K} : [0,T]\times\mathbb{R} \to \mathbb{R}$ is a continuous complex-valued function.

Lemma 4.1. *[53] A function $f \in C_{2-\gamma}([0,T],\mathbb{R})$ is a solution of (12) with the boundary conditions (13) if and* *only if*

$$f(w) = \frac{1}{\Gamma(\gamma)}\int_0^w (w-u)^{\gamma-1}\mathcal{K}(u,f(u))\,du + \frac{b_1 w^{\gamma-1}}{(1-b_1)\Gamma(\gamma)}\int_0^T \mathcal{K}(u,f(u))\,du + \frac{b_0 w^{\gamma-2}}{(1-b_0)(1-b_1)\Gamma(\gamma-1)} \times \int_0^T (T-(1-b_1)u)\mathcal{K}(u,f(u))\,du. \tag{14}$$

Theorem 4.2. *[53] Assume that there exists a constant $L > 0$ such that*

$$|\mathcal{K}(t,u)-\mathcal{K}(t,v)| \le L|u-v|, \quad \forall t\in[0,T],\ u,v\in\mathbb{R}.$$

Then the problem (12) with the boundary conditions (13) has a unique solution in $C_{2-\gamma}([0,T],\mathbb{R})$ if $L < \frac{1}{\Theta}$ where

$$\Theta = T^\gamma\left[\frac{\Gamma(\gamma-1)}{\Gamma(2\gamma-1)} + \left|\frac{b_1}{(1-b_1)(\gamma-1)\Gamma(\gamma)}\right| + \left|\frac{b_0(\gamma+(\gamma-1)|1-b_1|)}{(1-b_0)(1-b_1)\Gamma(\gamma+1)}\right|\right].$$

In fact, Ahmad and Nieto in [53] used the Banach contraction principle to show the existence of a unique solution for the equation (12).

Denote by $\Omega_r = C_r([0,T],\mathbb{C})$ the set of all complex valued functions f on $[0,T]$ which $u_r(t) = t^r f(t)$ is continuous. Obviously, for any $f\in\Omega_r$, we have

$$f(t) = \mathcal{R}ef(t) + \mathcal{I}mf(t)i,$$

where $\mathcal{R}ef, \mathcal{I}mf \in \nabla_r$. Now for any $f,g\in\Omega_r$, define

$$\rho_r(f,g) = \|\mathcal{R}ef - \mathcal{R}eg\|_r + \|\mathcal{I}mf - \mathcal{I}mg\|_r i.$$

Then it is easy to check that (Ω,ρ_r) is a complete CV metric space and so is a complete CVBMS. In this paper, we investigated the existence of unique solution for the following Cauchy type fractional differential equation:

$$^{RL}\mathfrak{D}^{\gamma}_{0+} f(w) = \mathcal{K}(w,f(w)), \quad w\in[0,T],\ 1<\gamma\le 2, \tag{15}$$

with the boundary conditions of fractional order

$$\begin{aligned} {}^{RL}\mathfrak{D}^{\gamma-2}_{0+} f(0^+) &= b_0\, {}^{RL}\mathfrak{D}^{\gamma-2}_{0+} f(T^-),\\ {}^{RL}\mathfrak{D}^{\gamma-1}_{0+} f(0^+) &= b_1\, {}^{RL}\mathfrak{D}^{\gamma-1}_{0+} f(T^-), \end{aligned} \tag{16}$$

where $f\in C_{2-\gamma}([0,T],\mathbb{C})$, $b_0\ne 1, b_1\ne 1$ are constant complex numbers and $\mathcal{K} : [a,b]\times\mathbb{C}\to\mathbb{C}$ is a continuous complex-valued function and the second side function $\mathcal{K}(.,f(.))$ of the equality (15) does not satisfy necessarily the Banach contraction and satisfies a weaker contractive condition, namely F-contraction in a CVBMS.

Lemma 4.3. *A function $f\in C_{2-\gamma}([0,T],\mathbb{C})$ is a solution of (15) with the boundary conditions (16) if and only if*

$$f(w) = \frac{1}{\Gamma(\gamma)}\int_0^w (w-u)^{\gamma-1}\mathcal{K}(u,f(u))\,du + \frac{b_1 w^{\gamma-1}}{(1-b_1)\Gamma(\gamma)}\int_0^T \mathcal{K}(u,f(u))\,du + \frac{b_0 w^{\gamma-2}}{(1-b_0)(1-b_1)\Gamma(\gamma-1)} \times \int_0^T (T-(1-b_1)u)\mathcal{K}(u,f(u))\,du \tag{17}$$

Proof. The proof is completely similar to the proof of Lemma 4.1 in [53]. □

Theorem 4.4. *Let $\mathcal{K} : [0,T]\times\mathbb{C}\to\mathbb{C}$ be a continuous complex-valued function satisfying*

$$|\mathcal{R}e\mathcal{K}(t,\mathcal{Z}_1) - \mathcal{R}e\mathcal{K}(t,\mathcal{Z}_2)| \le \frac{\alpha}{\sqrt{\Theta}}\cdot\frac{\sqrt{|\mathcal{R}e\mathcal{Z}_1-\mathcal{R}e\mathcal{Z}_2|^2+|\mathcal{I}m\mathcal{Z}_1-\mathcal{I}m\mathcal{Z}_2|^2}}{\left(1+\sqrt[4]{(t^{2-\gamma}|\mathcal{R}e\mathcal{Z}_1-\mathcal{R}e\mathcal{Z}_2|)^2+(t^{2-\gamma}|\mathcal{I}m\mathcal{Z}_1-\mathcal{I}m\mathcal{Z}_2|)^2}\right)^2} \tag{18}$$

and

$$|\mathcal{I}m\mathcal{K}(t,\mathcal{Z}_1) - \mathcal{I}m\mathcal{K}(t,\mathcal{Z}_2)| \le \frac{\beta}{\sqrt{\Theta}}\cdot\frac{\sqrt{|\mathcal{R}e\mathcal{Z}_1-\mathcal{R}e\mathcal{Z}_2|^2+|\mathcal{I}m\mathcal{Z}_1-\mathcal{I}m\mathcal{Z}_2|^2}}{\left(1+\sqrt[4]{(t^{2-\gamma}|\mathcal{R}e\mathcal{Z}_1-\mathcal{R}e\mathcal{Z}_2|)^2+(t^{2-\gamma}|\mathcal{I}m\mathcal{Z}_1-\mathcal{I}m\mathcal{Z}_2|)^2}\right)^2}, \tag{19}$$

for all $t\in[0,T]$ and $\mathcal{Z}_1,\mathcal{Z}_2\in\mathbb{C}$, where α,β are positive constant real numbers satisfying $\alpha^2+\beta^2=1$ and

$$\Theta = T^\gamma\left[\frac{\Gamma(\gamma-1)}{\Gamma(2\gamma-1)} + \left|\frac{b_1}{(1-b_1)(\gamma-1)\Gamma(\gamma)}\right| + \left|\frac{b_0(\gamma+(\gamma-1)|1-b_1|)}{(1-b_0)(1-b_1)\Gamma(\gamma+1)}\right|\right].$$

Then, the problem (15) with the boundary conditions (16) has a unique solution in $\Omega_{2-\gamma} = C_{2-\gamma}([0,T],\mathbb{C})$.

Proof. Let $f,g\in C_{2-\gamma}([0,T],\mathbb{C})$. From (18), for any $u\in[0,T]$ we have

$$u^{2-\gamma}|\mathcal{R}e\mathcal{K}(u,f(u)) - \mathcal{R}e\mathcal{K}(u,g(u))| \le \frac{\alpha}{\Theta}\cdot\frac{\sqrt{(u^{2-\gamma}|\mathcal{R}ef(u)-\mathcal{R}eg(u)|)^2+(u^{2-\gamma}|\mathcal{I}mf(u)-\mathcal{I}mg(u)|)^2}}{\left(1+\sqrt[4]{(u^{2-\gamma}|\mathcal{R}ef(u)-\mathcal{R}eg(u)|)^2+(u^{2-\gamma}|\mathcal{I}mf(u)-\mathcal{I}mg(u)|)^2}\right)^2}.$$

Taking sup on both sides of the above inequality as $u\in[0,T]$, we get

$$\|\mathcal{R}e\mathcal{K}(u,f(u)) - \mathcal{R}e\mathcal{K}(u,g(u))\|_{2-\gamma} \le \frac{\alpha}{\Theta}\cdot\frac{\sqrt{(\|\mathcal{R}ef-\mathcal{R}eg\|_{2-\gamma})^2+(\|\mathcal{I}mf-\mathcal{I}mg\|_{2-\gamma})^2}}{\left(1+\sqrt[4]{(\|\mathcal{R}ef-\mathcal{R}eg\|_{2-\gamma})^2+(\|\mathcal{I}mf-\mathcal{I}mg\|_{2-\gamma})^2}\right)^2}. \tag{20}$$

Similarly, from (19), we get

$$\|\mathcal{I}m\mathcal{K}(u,f(u))-\mathcal{I}m\mathcal{K}(u,g(u))\|_{2-\gamma} \leq \frac{\beta}{\Theta}\cdot\frac{\sqrt{(\|\mathcal{R}ef-\mathcal{R}eg\|_{2-\gamma})^2+(\|\mathcal{I}mf-\mathcal{I}mg\|_{2-\gamma})^2}}{\left(1+\sqrt[4]{(\|\mathcal{R}ef-\mathcal{R}eg\|_{2-\gamma})^2+(\|\mathcal{I}mf-\mathcal{I}mg\|_{2-\gamma})^2}\right)^2}. \tag{21}$$

Define a mapping $\Upsilon : C_{2-\gamma}([0,T],\mathbb{C}) \to C_{2-\gamma}([0,T],\mathbb{C})$ with

$$\begin{aligned}\Upsilon(f)(w) &= \frac{1}{\Gamma(\gamma)}\int_0^w (w-u)^{\gamma-1}\mathcal{K}(u,f(u))\,du \\ &+ \frac{b_1 w^{\gamma-1}}{(1-b_1)\Gamma(\gamma)}\int_0^T \mathcal{K}(u,f(u))\,du \\ &+ \frac{b_0 w^{\gamma-2}}{(1-b_0)(1-b_1)\Gamma(\gamma-1)} \\ &\quad\times\int_0^T (T-(1-b_1)u)\mathcal{K}(u,f(u))\,du.\end{aligned} \tag{22}$$

By Lemma 4.3, it is sufficient to show that there is a unique fixed point for the mapping Υ. Let $f,g\in C_{2-\gamma}([0,T],\mathbb{C})$. For all $t\in[0,T]$, from (18), (20), (21) and (22), we have

$$\begin{aligned}&w^{2-\gamma}|\mathcal{R}e\Upsilon(f)(w)-\mathcal{R}e\Upsilon(g)(w)| \\ &\leq \frac{w^{2-\gamma}}{\Gamma(\gamma)}\int_0^w (w-u)^{\gamma-1} \\ &\quad\times|\mathcal{R}e\mathcal{K}(u,f(u))-\mathcal{R}e\mathcal{K}(u,g(u))|\,du \\ &\quad+\left|\frac{b_1 w}{(1-b_1)\Gamma(\gamma)}\right| \\ &\quad\times\int_0^T |\mathcal{R}e\mathcal{K}(u,f(u))-\mathcal{R}e\mathcal{K}(u,g(u))|\,du \\ &\quad+\left|\frac{b_0}{(1-b_0)(1-b_1)\Gamma(\gamma-1)}\right|\int_0^T |T-(1-b_1)u| \\ &\quad\times|\mathcal{R}e\mathcal{K}(u,f(u))-\mathcal{R}e\mathcal{K}(u,g(u))|\,du \\ &\leq\Bigg[\frac{w^{2-\gamma}}{\Gamma(\gamma)}\int_0^w (w-u)^{\gamma-1}u^{\gamma-2}\,du \\ &\quad+\left|\frac{b_1 w}{(1-b_1)\Gamma(\gamma)}\right|\int_0^T u^{\gamma-2}\,du \\ &\quad+\left|\frac{b_0}{(1-b_0)(1-b_1)\Gamma(\gamma-1)}\right| \\ &\quad\times\int_0^T |T-(1-b_1)u|\,u^{\gamma-2}\,du\Bigg] \\ &\quad\times\|\mathcal{R}e\mathcal{K}(u,f(u))-\mathcal{R}e\mathcal{K}(u,g(u))\|_{2-\gamma} \\ &\leq T^\gamma\left[\frac{\Gamma(\gamma-1)}{\Gamma(2\gamma-1)}+\left|\frac{b_1}{(1-b_1)(\gamma-1)\Gamma(\gamma)}\right| + \left|\frac{b_0(\gamma+(\gamma-1)|1-b_1|)}{(1-b_0)(1-b_1)\Gamma(\gamma+1)}\right|\right] \\ &\quad\times\|\mathcal{R}e\mathcal{K}(u,f(u))-\mathcal{R}e\mathcal{K}(u,g(u))\|_{2-\gamma} \\ &\leq T^\gamma\left[\frac{\Gamma(\gamma-1)}{\Gamma(2\gamma-1)}+\left|\frac{b_1}{(1-b_1)(\gamma-1)\Gamma(\gamma)}\right| + \left|\frac{b_0(\gamma+(\gamma-1)|1-b_1|)}{(1-b_0)(1-b_1)\Gamma(\gamma+1)}\right|\right] \\ &\quad\times\frac{\alpha}{\Theta} \\ &\quad\cdot\frac{\sqrt{\|\mathcal{R}ef-\mathcal{R}eg\|^2_{2-\gamma}+\|\mathcal{I}mf-\mathcal{I}mg\|^2_{2-\gamma}}}{\left(1+\sqrt[4]{\|\mathcal{R}ef-\mathcal{R}eg\|^2_{2-\gamma}+\|\mathcal{I}mf-\mathcal{I}mg\|^2_{2-\gamma}}\right)^2} \\ &=\alpha\cdot \\ &\quad\cdot\frac{\sqrt{\|\mathcal{R}ef-\mathcal{R}eg\|^2_{2-\gamma}+\|\mathcal{I}mf-\mathcal{I}mg\|^2_{2-\gamma}}}{\left(1+\sqrt[4]{\|\mathcal{R}ef-\mathcal{R}eg\|^2_{2-\gamma}+\|\mathcal{I}mf-\mathcal{I}mg\|^2_{2-\gamma}}\right)^2}.\end{aligned}$$

Similarly, from (18), (20), (21) and (22),

$$\begin{aligned}&w^{2-\gamma}|\mathcal{I}m\Upsilon(f)(t)-\mathcal{I}m\Upsilon(g)(t)| \\ &\leq\beta\cdot\frac{\sqrt{\|\mathcal{R}ef-\mathcal{I}mg\|^2_{2-\gamma}+\|\mathcal{R}ef-\mathcal{I}mg\|^2_{2-\gamma}}}{\left(1+\sqrt[4]{\|\mathcal{R}ef-\mathcal{R}eg\|^2_{2-\gamma}+\|\mathcal{I}mf-\mathcal{I}mg\|^2_{2-\gamma}}\right)^2}.\end{aligned}$$

Taking sup on $t\in[0,T]$, in the above inequalities, and squaring them, we get

$$\begin{aligned}&\|\mathcal{R}e\Upsilon(f)-\mathcal{R}e\Upsilon(g)\|^2_{2-\gamma} \\ &\leq\alpha^2\cdot\frac{\|\mathcal{R}ef-\mathcal{R}eg\|^2_{2-\gamma}+\|\mathcal{I}mf-\mathcal{I}mg\|^2_{2-\gamma}}{\left(1+\sqrt[4]{\|\mathcal{R}ef-\mathcal{R}eg\|^2_{2-\gamma}+\|\mathcal{I}mf-\mathcal{I}mg\|^2_{2-\gamma}}\right)^4}\end{aligned}$$

and

$$\begin{aligned}&\|\mathcal{I}m\Upsilon(f)-\mathcal{I}m\Upsilon(g)\|^2_{2-\gamma} \\ &\leq\beta^2\cdot\frac{\|\mathcal{R}ef-\mathcal{R}eg\|^2_{2-\gamma}+\|\mathcal{I}mf-\mathcal{I}mg\|^2_{2-\gamma}}{\left(1+\sqrt[4]{\|\mathcal{R}ef-\mathcal{R}eg\|^2_{2-\gamma}+\|\mathcal{I}mf-\mathcal{I}mg\|^2_{2-\gamma}}\right)^4}.\end{aligned}$$

Adding the above two inequalities, we have

$$\begin{aligned}&\|\mathcal{R}e\Upsilon(f)-\mathcal{R}e\Upsilon(g)\|^2_{2-\gamma} \\ &\quad+\|\mathcal{I}m\Upsilon(f)-\mathcal{I}m\Upsilon(g)\|^2_{2-\gamma} \\ &\leq\frac{\|\mathcal{R}ef-\mathcal{R}eg\|^2_{2-\gamma}+\|\mathcal{I}mf-\mathcal{I}mg\|^2_{2-\gamma}}{\left(1+\sqrt[4]{\|\mathcal{R}ef-\mathcal{R}eg\|^2_{2-\gamma}+\|\mathcal{I}mf-\mathcal{I}mg\|^2_{2-\gamma}}\right)^4}.\end{aligned}$$

Taking $\sqrt[4]{\cdot}$, we have

$$\begin{aligned}&\sqrt[4]{\|\mathcal{R}e\Upsilon(f)-\mathcal{R}e\Upsilon(g)\|^2_{2-\gamma}+\|\mathcal{I}m\Upsilon(f)-\mathcal{I}m\Upsilon(g)\|^2_{2-\gamma}} \\ &\leq\frac{\sqrt[4]{\|\mathcal{R}ef-\mathcal{R}eg\|^2_{2-\gamma}+\|\mathcal{I}mf-\mathcal{I}mg\|^2_{2-\gamma}}}{1+\sqrt[4]{\|\mathcal{R}ef-\mathcal{R}eg\|^2_{2-\gamma}+\|\mathcal{I}mf-\mathcal{I}mg\|^2_{2-\gamma}}}.\end{aligned}$$

Therefore

$$\sqrt{\left|\rho_{2-\gamma}(\Upsilon(f),\Upsilon(g))\right|}\leq\frac{\sqrt{\left|\rho_{2-\gamma}(f,g)\right|}}{1+\sqrt{\left|\rho_{2-\gamma}(f,g)\right|}}.$$

Squaring two sides of the above inequality, we have

$$\left|\rho_{2-\gamma}(\Upsilon(f),\Upsilon(g))\right|\leq\frac{\left|\rho_{2-\gamma}(f,g)\right|}{(1+\sqrt{\left|\rho_{2-\gamma}(f,g)\right|})^2}.$$

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Taking $\tau = 1$, we have

$$\left|\rho_{2-\gamma}(\Upsilon(f), \Upsilon(g))\right| \leq \frac{\left|\rho_{2-\gamma}(f,g)\right|}{(1+\tau\sqrt{\left|\rho_{2-\gamma}(f,g)\right|})^2}$$

which is (9), for all $f, g \in \Omega_{2-\gamma}$. Define $\xi : \Omega_{2-\gamma} \times \Omega_{2-\gamma} \to [0,\infty)$ with $\xi(f,g) = 1$ for all $f, g \in \Omega_{2-\gamma}$. Now all of the conditions of Corollary 3.12 are satisfied. Thus, Υ has a unique fixed point in $\Omega_{2-\gamma} = C_{2-\gamma}([0,T],\mathbb{C})$. So, the fractional differential equation (15) with the boundary conditions (16) has a unique solution in $C_{2-\gamma}([0,T],\mathbb{C})$. □

Example 4.5. *We focus our analysis on the following fractional differential equation:*

$$\begin{aligned} {}^{RL}\mathfrak{D}_{0^+}^{\frac{3}{2}} f(w) = &\left(\frac{\sqrt{\pi}}{8\sqrt{3}(18+\pi)} e^{-t} \frac{(\mathcal{R}ef(t))^2 + (\mathcal{I}mf(t))^2}{1+t(\mathcal{R}ef(t))^2+(\mathcal{I}mf(t))^2}\right) \\ &+ \left(\frac{\sqrt{2\pi}}{8\sqrt{3}(18+\pi)}\sin t \frac{(\mathcal{R}ef(t))^2+(\mathcal{I}mf(t))^2}{1+t(\mathcal{R}ef(t))^2+(\mathcal{I}mf(t))^2}\right) i; \\ &t\in[0,1],\ f\in C_{\frac{1}{2}}([0,1],\mathbb{C})\cap B_{\frac{3-\sqrt{5}}{4}}(0), \end{aligned} \tag{23}$$

where

$$B_{\frac{3-\sqrt{5}}{4}}(0) = \left\{f\in C([0,1],\mathbb{C}) : \|f\| \leq B_{\frac{3-\sqrt{5}}{4}}(0)\right\}$$

($\|f\| = \sup\{\left|f(t)\right| : t\in[0,1]\}$) with the boundary conditions of fractional order

$$\begin{aligned} {}^{RL}\mathfrak{D}_{0^+}^{-\frac{1}{2}} f(0^+) &= \tfrac{3}{2}\ {}^{RL}\mathfrak{D}_{0^+}^{-\frac{1}{2}} f(1^-), \\ {}^{RL}\mathfrak{D}_{0^+}^{\frac{1}{2}} f(0^+) &= \tfrac{1}{2}\ {}^{RL}\mathfrak{D}_{0^+}^{\frac{1}{2}} f(1^-). \end{aligned} \tag{24}$$

Here we have

$$\gamma = \tfrac{3}{2},\ T=1,\ \alpha=\sqrt{\tfrac{1}{3}},\ \beta=\sqrt{\tfrac{2}{3}},\ b_0=\tfrac{3}{2},\ b_1=\tfrac{1}{2}.$$

A direct computing shows that $\Theta = \frac{2(18+\pi)}{\sqrt{\pi}}$. Here

$$\begin{aligned} \mathcal{K}(t,\mathcal{Z}) = &\left(\frac{\sqrt{\pi}}{2\sqrt{3}(18+\pi)}e^{-t} \frac{(\mathcal{R}e\mathcal{Z})^2+(\mathcal{I}m\mathcal{Z})^2}{1+t[(\mathcal{R}e\mathcal{Z})^2+(\mathcal{I}m\mathcal{Z})^2]}\right) \\ &+ \left(\frac{\sqrt{2\pi}}{2\sqrt{3}(18+\pi)}\sin t \frac{(\mathcal{R}e\mathcal{Z})^2+(\mathcal{I}m\mathcal{Z})^2}{1+t[(\mathcal{R}e\mathcal{Z})^2+(\mathcal{I}m\mathcal{Z})^2]}\right) i. \end{aligned}$$

First note that for any $\mathcal{Z}_1, \mathcal{Z}_2 \in B_{\frac{3-\sqrt{5}}{4}}(0)$ and for any $t\in[0,4]$ we have

$$\begin{aligned} &\left|\frac{|\mathcal{Z}_1|^2}{1+t|\mathcal{Z}_1|^2} - \frac{|\mathcal{Z}_2|^2}{1+t|\mathcal{Z}_2|^2}\right| \\ &= \frac{(|\mathcal{Z}_1|-|\mathcal{Z}_2|)(|\mathcal{Z}_1|+|\mathcal{Z}_2|)}{(1+t|\mathcal{Z}_1|^2)(1+t|\mathcal{Z}_2|^2)} \\ &\leq |\mathcal{Z}_1-\mathcal{Z}_2|\frac{3-\sqrt{5}}{2} \\ &= \frac{|\mathcal{Z}_1-\mathcal{Z}_2|}{\frac{3+\sqrt{5}}{2}} = \frac{|\mathcal{Z}_1-\mathcal{Z}_2|}{(1+\sqrt{\frac{3-\sqrt{5}}{2}})^2} \\ &\leq \frac{|\mathcal{Z}_1-\mathcal{Z}_2|}{(1+\sqrt{|\mathcal{Z}_1|+|\mathcal{Z}_2|})^2} \\ &\leq \frac{|\mathcal{Z}_1-\mathcal{Z}_2|}{(1+\sqrt{\sqrt{t}|\mathcal{Z}_1-\mathcal{Z}_2|})^2}. \end{aligned}$$

Now, for any $t\in[0,1]$, we have

$$\begin{aligned} &|\mathcal{R}e\mathcal{K}(t,\mathcal{Z}_1) - \mathcal{R}e\mathcal{K}(t,\mathcal{Z}_2)| \\ &= \left|\frac{\sqrt{\pi}}{2\sqrt{3}(18+\pi)}e^{-t} \times \frac{(\mathcal{R}e\mathcal{Z}_1)^2+(\mathcal{I}m\mathcal{Z}_1)^2}{1+t[(\mathcal{R}e\mathcal{Z}_1)^2+(\mathcal{I}m\mathcal{Z}_1)^2]} - \frac{\sqrt{\pi}}{2\sqrt{3}(18+\pi)}e^{-t} \times \frac{(\mathcal{R}e\mathcal{Z}_2)^2+(\mathcal{I}m\mathcal{Z}_2)^2}{1+t[(\mathcal{R}e\mathcal{Z}_2)^2+(\mathcal{I}m\mathcal{Z}_2)^2]}\right| \\ &\leq \frac{\sqrt{\pi}}{2\sqrt{3}(18+\pi)}\left|\frac{(\mathcal{R}e\mathcal{Z}_1)^2+(\mathcal{I}m\mathcal{Z}_1)^2}{1+t[(\mathcal{R}e\mathcal{Z}_1)^2+(\mathcal{I}m\mathcal{Z}_1)^2]} - \frac{(\mathcal{R}e\mathcal{Z}_2)^2+(\mathcal{I}m\mathcal{Z}_2)^2}{1+t[(\mathcal{R}e\mathcal{Z}_2)^2+(\mathcal{I}m\mathcal{Z}_2)^2]}\right| \\ &\leq \frac{\sqrt{\pi}}{2\sqrt{3}(18+\pi)} \cdot \frac{\sqrt{|\mathcal{R}e\mathcal{Z}_1-\mathcal{R}e\mathcal{Z}_2|^2+|\mathcal{I}m\mathcal{Z}_1-\mathcal{I}m\mathcal{Z}_2|^2}}{\left(1+\sqrt[4]{t[|\mathcal{R}e\mathcal{Z}_1-\mathcal{R}e\mathcal{Z}_2|^2+|\mathcal{I}m\mathcal{Z}_1-\mathcal{I}m\mathcal{Z}_2|^2]}\right)^2} \\ &= \frac{\alpha}{\Theta}\cdot\frac{\sqrt{|\mathcal{R}e\mathcal{Z}_1-\mathcal{R}e\mathcal{Z}_2|^2+|\mathcal{I}m\mathcal{Z}_1-\mathcal{I}m\mathcal{Z}_2|^2}}{\left(1+\sqrt[4]{t[|\mathcal{R}e\mathcal{Z}_1-\mathcal{R}e\mathcal{Z}_2|^2+|\mathcal{I}m\mathcal{Z}_1-\mathcal{I}m\mathcal{Z}_2|^2]}\right)^2}. \end{aligned}$$

Similarly,

$$\begin{aligned} &|\mathcal{I}m\mathcal{K}(t,\mathcal{Z}_1) - \mathcal{I}m\mathcal{K}(t,\mathcal{Z}_2)| \\ &\leq \frac{\beta}{\Theta}\cdot\frac{\sqrt{|\mathcal{R}e\mathcal{Z}_1-\mathcal{R}e\mathcal{Z}_2|^2+|\mathcal{I}m\mathcal{Z}_1-\mathcal{I}m\mathcal{Z}_2|^2}}{\left(1+\sqrt[4]{t[|\mathcal{R}e\mathcal{Z}_1-\mathcal{R}e\mathcal{Z}_2|^2+|\mathcal{I}m\mathcal{Z}_1-\mathcal{I}m\mathcal{Z}_2|^2]}\right)^2}. \end{aligned}$$

Also, we have $\alpha^2+\beta^2 = \frac{1}{3}+\frac{2}{3} = 1$. We see that all of the conditions of Theorem 4.4 are satisfied and so the fractional differential equation (23) with the boundary conditions (24) has a unique solution in $C_{\frac{1}{2}}([0,1],\mathbb{C}) \cap B_{\frac{3-\sqrt{5}}{4}}(0)$.

5. Conclusions

In this study, we established several fixed point results for Wardowski-type admissible contractions within the framework of CVBBMS. To demonstrate the validity of our theoretical findings, a concrete example was provided. Furthermore, the practical significance of these results was highlighted through an application to the existence of solutions for fractional differential equations. The criteria developed herein provide robust sufficient conditions for the existence of fixed points for such contractive operators in complex-valued structures.

Moving forward, several promising directions for future inquiry are identified. One potential extension involves investigating common fixed points for pairs of mappings that satisfy Wardowski-type contractive conditions in CVBBMS. Additionally, exploring the existence of coupled fixed points for bivariate Wardowski-type admissible contractions presents an interesting challenge. Lastly, extending these concepts to multi-valued operators and examining their implications for fractional differential inclusions in CVBBMS could provide significant insights into the field.

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Authors contributions

M.R.S., J.H.A. and B.M. contributed to the conception and design of the study. M.R.S. and J.H.A. performed the mathematical analysis and derivation of results. B.M. supervised the research, contributed to the formulation of the main fixed point results and the application to fractional differential equations, and reviewed the manuscript. All authors participated in drafting and revising the manuscript, and approved the final version for publication.

Availability of data and materials

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflict of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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