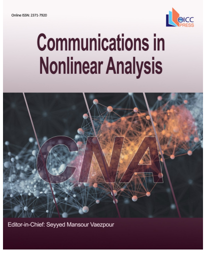


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Research Article

# Characterizations of Triple and $*$-Triple $H$-Biderivations on $C^*$-Algebras

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**Abstract:**

We introduce the concepts of triple and $*$-triple $H$-biderivations on $*$-triple systems and investigate their analytical and algebraic properties. We establish a local-to-global extension theorem by proving that every continuous bilinear map behaving locally as a $*$-triple $H$-biderivation on a unital $C^*$-algebra is a global one. In terms of structural characterization, we demonstrate that these maps vanish on commutative unital algebras, while on the non-commutative algebra $B(\mathcal{H})$, they are inner and take the form $D(x, y) = \lambda[H(x), H(y)]$.

**Keywords:** Triple $H$-biderivation; Associative $*$-triple system; Local derivation; $C^*$-algebra; Inner structure; MSC 2020: 47B47, 46L05, 16W25

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## 1. Introduction

The concept of derivations in algebraic structures has been extensively studied across various fields, playing a crucial role in diverse areas of physics and mathematics, particularly in cohomology and representation theory. Derivations serve as a bridge connecting algebra and geometry and pave the way for new algebraic constructions. Beyond the classical setting, derivations in Hom–Lie algebras [1], Lie algebras [2], and Lie triple systems [3] have also been investigated. Moreover, several generalizations have been introduced to capture more intricate symmetries, such as quasi-derivations of Lie algebras [2], generalized derivations [2], $\delta$-derivations and $(\alpha, \beta, \gamma)$-derivations [4], and Lie triple derivations [5].

Parallel to linear derivations, bilinear maps known as *biderivations* have emerged as valuable tools, with recent studies extending these concepts to generalized biderivations on various algebras [6]. A bi-additive map $\mathcal{D} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is called a biderivation if it acts as a derivation in each component. In numerous studies concerned with characterizing commutative maps, authors typically begin by determining the biderivations of the underlying algebra. The exploration of local derivations, which map an algebra into a module in a pointwise manner, was prominently initiated by Kadison [7, 8] and has since become a standard method for testing the rigidity of algebraic structures.

Despite this substantial body of work, the interplay between biderivations, triple structures, and homomorphic twists remains largely unexplored. In this paper, we introduce the concept of *triple $H$-biderivations* on associative $*$-triple systems, generalizing the classical theory by incorporating a $*$-homomorphism $H$ into the Leibniz rule for the triple product $xy^*z$. Within this framework, our primary focus is on *$*$-triple $H$-biderivations*, which are further distinguished by an additional conjugate-skew symmetry condition. Throughout the paper, we restrict attention to unital algebras in order to employ the specific properties of the identity element in our structural results.

Our main objective is to characterize these mappings on $C^*$-algebras from both analytical and algebraic perspectives. The paper is organized as follows. Section 2 recalls necessary preliminaries. Section 3 introduces triple $H$-biderivations and provides fundamental examples. Section 4 examines their analytical behavior, where we establish a local-to-global extension theorem, proving that any continuous bilinear map acting locally as a $*$-triple $H$-biderivation on a unital $C^*$-algebra is, in fact, global. Finally, Section 5 presents a complete structural characterization: we obtain a vanishing result for commutative unital $C^*$-algebras and a strong structural theorem for the non-commutative algebra $B(\mathcal{H})$. Specifically, we show that every continuous triple $H$-biderivation on $B(\mathcal{H})$ is inner and must be of the form

$$D(x, y) = \lambda[H(x), H(y)].$$

## 2. Why Associative Triple Systems?

When extending the theory of biderivations to triple systems, Lie triple systems and Jordan triple systems naturally emerge as primary candidates due to their rich algebraic properties and deep connections with geometry and physics.

Let us briefly address the challenges encountered when attempting to construct non-trivial examples of $*$-triple $H$-biderivations on these structures.

A Lie $*$-triple system is a vector space $\mathcal{L}$ equipped with a trilinear map $[\cdot,\cdot,\cdot] : \mathcal{L} \times \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ and an involution $*$, satisfying the following identities for all $u, v, x, y, z \in \mathcal{L}$

1. $[x, y, z] = -[y, x, z]$;
2. $[x, y, z] + [y, z, x] + [z, x, y] = 0$ (Jacobi identity);
3. $[u, v, [x, y, z]] = [[u, v, x], y, z] + [x, [u, v, y], z] + [x, y, [u, v, z]]$ (Fundamental identity);
4. $[x, y, z]^* = [x^*, y^*, z^*]$.

On the other hand, a normed Jordan triple system is a normed complex vector space $(\mathcal{A}, \|\cdot\|)$ equipped with a trilinear product $\{\cdot,\cdot,\cdot\} : \mathcal{A}\times\mathcal{A}\times\mathcal{A} \to \mathcal{A}$. This product is linear in the first and third variables, conjugate-linear in the second variable, and satisfies

1. $\{x, y, z\} = \{z, y, x\}$ (Symmetry in outer variables);
2. $\{u, v, \{x, y, z\}\} = \{\{u, v, x\}, y, z\} - \{x, \{v, u, y\}, z\} + \{x, y, \{u, v, z\}\}$ (Jordan identity);
3. $\|\{x, y, z\}\| \leq \|x\|\|y\|\|z\|$.

A standard example is any $C^*$-algebra endowed with the triple product $\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a)$.

For such structures, a $*$-triple $H$-biderivation $\mathcal{D}$ is expected to satisfy a generalized Leibniz rule. For instance, in the Jordan setting

$$\begin{aligned}\mathcal{D}(\{x, y, z\}, w) = &\{\mathcal{D}(x, w), H(y), H(z)\}\\ &+ \{H(x), \mathcal{D}(y, w), H(z)\}\\ &+ \{H(x), H(y), \mathcal{D}(z, w)\}.\end{aligned}$$

However, our preliminary investigation suggests that the stringent algebraic constraints imposed by the Lie and Jordan identities (particularly the Fundamental and Jordan identities) make the existence of non-trivial $*$-triple $H$-biderivations unlikely. The complex interrelations between elements create a system of conditions that appears to force any such bilinear map to vanish. Attempts to construct examples analogous to the associative case systematically collapse to the trivial biderivation $\mathcal{D} = 0$.

While the non-existence of non-trivial examples in the general Lie or Jordan setting remains an open problem, the consistent failure to produce such examples suggests they are rare or trivial. We conjecture that for many standard Lie and Jordan triple systems, all $*$-triple $H$-biderivations are trivial.

This observation motivates a strategic shift in our perspective. To develop a fruitful theory with a rich class of examples, we direct our attention to the related but more flexible algebraic setting of the associative $*$-triple system. As demonstrated in the subsequent sections, the associative law provides sufficient structure to yield meaningful results while offering the necessary flexibility to construct a wide range of non-trivial $*$-triple $H$-biderivations.

## 3. Preliminaries and Basic Definitions

In this section, we lay the algebraic groundwork for our investigation. We begin by recalling standard notions from module theory to motivate their ternary analogues. We then formally define associative triple systems and bimodules, culminating in the precise formulation of the $*$-triple $H$-biderivation.

**Definition 3.1.** *[9, p. 51] Let $\mathcal{A}$ be an algebra and $\mathcal{X}$ be a linear space equipped with a bilinear mapping $(a, x) \mapsto a \cdot x$ from $\mathcal{A} \times \mathcal{X}$ to $\mathcal{X}$ satisfying*

$$a \cdot (b \cdot x) = (ab) \cdot x \qquad (a, b \in \mathcal{A},\ x \in \mathcal{X}).$$

*Then $\mathcal{X}$ is called a* left $\mathcal{A}$-module. *Similarly, the linear space $\mathcal{X}$ equipped with a bilinear mapping $(a, x) \mapsto x \cdot a$ satisfying*

$$(x \cdot b) \cdot a = x \cdot (ba) \qquad (a, b \in \mathcal{A},\ x \in \mathcal{X}),$$

*is called a* right $\mathcal{A}$-module. *When both mappings exist on $\mathcal{X}$ and the condition $a \cdot (x \cdot b) = (a \cdot x) \cdot b$ also holds for all $a, b \in \mathcal{A}$ and $x \in \mathcal{X}$, then the space $\mathcal{X}$ is called an* $\mathcal{A}$-bimodule.

**Definition 3.2.** *[9, p. 114] Let $\mathcal{A}$ be an algebra and $\mathcal{X}$ be an $\mathcal{A}$-bimodule. A linear mapping $\mathcal{D}\colon \mathcal{A} \to \mathcal{X}$ is called a* derivation *if for all $a, b \in \mathcal{A}$*

$$\mathcal{D}(ab) = \mathcal{D}(a) \cdot b + a \cdot \mathcal{D}(b).$$

**Definition 3.3.** *[10] A vector space $T$ over $\mathbb{C}$, together with a trilinear product*

$$[\,\cdot\,,\,\cdot\,,\,\cdot\,]\colon T \times T \times T \longrightarrow T, \quad (x, y, z) \mapsto [x, y, z],$$

*is called an* associative triple system *(or* associative ternary algebra*) if it satisfies the associativity condition*

$$[[x,y,z],u,v] = [x,[y,z,u],v] \\ = [x,y,[z,u,v]],$$

*for all $x,y,z,u,v \in T$.*

**Example 3.4.** *If $\mathcal{A}$ is an associative algebra with product $xy$, then defining*

$$[x,y,z] := x\,y\,z \quad (x,y,z \in \mathcal{A}),$$

*turns $\mathcal{A}$ into an associative triple system.*

**Definition 3.5.** *[11] An associative triple system $T$ is called an associative* $*$-triple system *if there exists a conjugate-linear involution on $T$, denoted by $x \mapsto x^*$, such that for all $x,y,z \in T$*

$$[x,y,z]^* = [z^*,y^*,x^*].$$

*(An involution satisfies $(x^*)^* = x$ for all $x \in T$.)*

**Definition 3.6.** *[11] Let $(T,[\ ,\ ,\ ],*)$ and $(S,[\ ,\ ,\ ]',*)$ be two $*$-triple systems. A linear map $H\colon T \to S$ is called a* $*$-triple homomorphism *if for all $x,y,z \in T$ we have*

$$H\big([x,y,z]\big) = \big[\,H(x),\ H(y),\ H(z)\big]' \\ \text{and} \quad H(x^*) = H(x)^*.$$

**Definition 3.7.** *[12] Let $T$ be an associative triple system. A vector space $X$ is called a* triple $T$-bimodule *if it is equipped with three trilinear actions*

$$\phi_1\colon X \times T \times T \to X, \\ \phi_2\colon T \times X \times T \to X, \\ \phi_3\colon T \times T \times X \to X,$$

*that are compatible with the triple product of $T$. We adopt the convenient notational abuse of writing $[a,b,c]$ for the image of $(a,b,c)$ under the appropriate map ($\phi_1, \phi_2, \phi_3$, or the product on $T$ itself). With this convention, the following mixed associativity identities must hold for all $u,v,x,y \in T$ and $m \in X$*

- $[[m,u,v],x,y] = [m,[u,v,x],y] = [m,u,[v,x,y]]$;
- $[[u,m,v],x,y] = [u,[m,v,x],y] = [u,m,[v,x,y]]$;
- $[[u,v,m],x,y] = [u,[v,m,x],y] = [u,v,[m,x,y]]$.

*For instance, the expression $[[m,u,v],x,y]$ denotes $\phi_1(\phi_1(m,u,v),x,y)$.*

**Definition 3.8.** *[12] Let $(T,[\ ,\ ,\ ],*)$ be an associative $*$-triple system and let $X$ be a $T$-bimodule. A linear map $\delta\colon T \to X$ is called a* (ternary) derivation *if for all $x,y,z \in T$*

$$\delta\big([x,y,z]\big) = \big[\delta(x),y,z\big] + \big[x,\delta(y),z\big] + \big[x,y,\delta(z)\big].$$

**Definition 3.9.** *[13] Let $(T,[\cdot,\cdot,\cdot],*)$ be an associative $*$-triple system and let $X$ be a triple $T$-bimodule with actions denoted by $[\cdot,\cdot,\cdot]'$. A bilinear map $D\colon T \times T \to X$ is called a* triple biderivation *from $T$ into $X$ if for all $x,y,z,w \in T$, the following Leibniz-type identities hold*

$$D([x,y,z],w) = [D(x,w),y,z]' + [x,D(y,w),z]' \\ + [x,y,D(z,w)]', \\ D(x,[y,z,w]) = [D(x,y),z,w]' + [y,D(x,z),w]' \\ + [y,z,D(x,w)]'.$$

*Furthermore, if $X$ is also a $*$-space (with involution denoted by $*$) and $D$ satisfies*

$$D(x^*,y^*) = -D(x,y)^* \quad \text{for all } x,y \in T,$$

*then $D$ is called a* $*$-triple biderivation.

**Remark 3.10.** *Before introducing our main concept, it is crucial to clarify the choice of sign in the involution compatibility condition. While standard $*$-derivations typically satisfy the symmetric condition $\mathcal{D}(x^*) = \mathcal{D}(x)^*$, we explicitly impose the conjugate-skew condition $\mathcal{D}(x^*,y^*) = -\mathcal{D}(x,y)^*$. This choice is not arbitrary but is strictly dictated by the algebraic behavior of commutator-based biderivations. Since these canonical examples inherently satisfy this property, enforcing the standard symmetry would exclude the very class of inner maps we aim to characterize. With this essential clarification established, we now formally introduce the central concept of this work.*

**Definition 3.11.** *Let $(T,[\ ,\ ,\ ],*)$ be an associative $*$-triple system, $X$ be a triple $T$-bimodule with actions $[\cdot,\cdot,\cdot]'$, and $H\colon T \to T$ be a $*$-triple homomorphism. A bilinear map $\mathcal{D}\colon T \times T \to X$ is called a* triple $H$-biderivation *if for all $x,y,z,w \in T$, the following generalized Leibniz rules are satisfied*

$$\mathcal{D}([x,y,z],w) = [\mathcal{D}(x,w),H(y),H(z)]' \\ + [H(x),\mathcal{D}(y,w),H(z)]' \\ + [H(x),H(y),\mathcal{D}(z,w)]', \\ \mathcal{D}(x,[y,z,w]) = [\mathcal{D}(x,y),H(z),H(w)]' \\ + [H(y),\mathcal{D}(x,z),H(w)]' \\ + [H(y),H(z),\mathcal{D}(x,w)]'.$$

*Furthermore, if $\mathcal{D}$ satisfies the condition*

$$\mathcal{D}(x^*,y^*) = -\mathcal{D}(x,y)^* \quad (x,y \in T),$$

*then $\mathcal{D}$ is called a* $*$-triple $H$-biderivation.

**Remark 3.12.** *In this paper, we will focus on the important case where the target bimodule is the triple system $T$ itself (i.e., $X = T$). In this setting, the bimodule actions $[\cdot,\cdot,\cdot]'$ are simply the triple product of $T$, and the definition simplifies accordingly. We will refer to these as triple $H$-biderivations on $T$.*

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To validate our framework, we now present non-trivial examples in matrix and Clifford algebras. These constructions confirm that the associative setting admits a rich class of non-zero solutions satisfying the conjugate-skew condition which distinguishes it from the more restrictive structures discussed earlier.

**Example 3.13.** *Let $\mathcal{A} = M_2(\mathbb{C})$ with the usual matrix multiplication and the involution $x^* = \overline{x}^T$, for each $x \in \mathcal{A}$. We further endow $\mathcal{A}$ with the triple product*

$$[x, y, z] = x\,y\,z \quad (x, y, z \in \mathcal{A}).$$

*Fix the unitary permutation matrix $u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then, $u^* = u$ and $u^*u = I$. Define $H\colon M_2(\mathbb{C}) \to M_2(\mathbb{C})$ by $H(x) = u\,x\,u^*$. It is clear that $H$ is linear and*

$$H(x^*) = u\,x^*\,u^* = (u\,x\,u^*)^* = H(x)^*.$$

*Also, the triple product is just the matrix product, so*

$$\begin{aligned} H([x, y, z]) &= u\,(xyz)\,u^* \\ &= (uxu^*)(uyu^*)(uzu^*) \\ &= [H(x), H(y), H(z)]. \end{aligned}$$

*Therefore $H$ is a $*$-triple homomorphism, and $H$ is not the identity since $u$ is not a scalar multiple of the identity. Now define $\mathcal{D}\colon M_2(\mathbb{C}) \times M_2(\mathbb{C}) \to M_2(\mathbb{C})$ by*

$$\mathcal{D}(x, y) = [H(x), H(y)]_c = H(x)H(y) - H(y)H(x).$$

*(Here $[\cdot,\cdot]_c$ denotes the standard commutator). The bilinearity of $\mathcal{D}$ is clear. Also, $\mathcal{D}$ satisfies the $*$-compatibility condition because*

$$\begin{aligned} \mathcal{D}(x^*, y^*) &= H(x^*)H(y^*) - H(y^*)H(x^*) \\ &= H(x)^*H(y)^* - H(y)^*H(x)^*, \end{aligned}$$

*and*

$$\begin{aligned} -\mathcal{D}(x, y)^* &= -(H(x)H(y) - H(y)H(x))^* \\ &= -(H(y)^*H(x)^* - H(x)^*H(y)^*) \\ &= H(x)^*H(y)^* - H(y)^*H(x)^*. \end{aligned}$$

*Hence $\mathcal{D}(x^*, y^*) = -\mathcal{D}(x, y)^*$. Now we check the first Leibniz rule. For the left-hand side we have*

$$\mathcal{D}\big([x, y, z], w\big) = [H(x)H(y)H(z), H(w)]_c.$$

*The right-hand side is*

$$\begin{aligned} &[\mathcal{D}(x, w), H(y), H(z)] + [H(x), \mathcal{D}(y, w), H(z)] \\ &\quad + [H(x), H(y), \mathcal{D}(z, w)] \\ &= [H(x), H(w)]_c H(y)H(z) \\ &\quad + H(x)[H(y), H(w)]_c H(z) \\ &\quad + H(x)H(y)[H(z), H(w)]_c. \end{aligned}$$

*Expanding the commutators and canceling terms, this simplifies to*

$$H(x)H(y)H(z)H(w) - H(w)H(x)H(y)H(z),$$

*which is precisely $[H(x)H(y)H(z), H(w)]_c = \mathcal{D}([x, y, z], w)$. Hence the first Leibniz rule holds. The second rule is verified analogously. In addition, $\mathcal{D}$ is non-zero. For the standard matrix units $e_{12}$ and $e_{21}$, we have $H(e_{12}) = e_{21}$ and $H(e_{21}) = e_{12}$. Therefore*

$$\begin{aligned} \mathcal{D}(e_{12}, e_{21}) &= [H(e_{12}), H(e_{21})]_c \\ &= [e_{21}, e_{12}]_c = e_{22} - e_{11} \neq 0. \end{aligned}$$

*Thus, $\mathcal{D}$ is a non-trivial $*$-Triple $H$-biderivation.*

**Example 3.14.** *Let $V$ be a complex vector space of dimension $n \geq 2$ equipped with a non-degenerate symmetric bilinear form $B : V \times V \to \mathbb{C}$. The* Clifford algebra *[14], denoted by $\mathcal{A} = \mathrm{CL}(V, B)$, is the associative algebra generated by $V$ subject to the defining relation*

$$vw + wv = 2B(v, w)\mathbf{1} \quad (v, w \in V),$$

*where $\mathbf{1}$ is the identity element. This algebra is non-commutative. We endow $\mathcal{A}$ with an involution defined on the generators by $v^* = -v$ for all $v \in V$, extended to the whole algebra via $(xy)^* = y^*x^*$. As before, the triple product is given by $[x, y, z] = xyz$.*

*To construct a non-trivial homomorphism, we select a vector $u \in V$ such that $B(u, u)$ is a positive real number. Let $\alpha \in \mathbb{C}$ be purely imaginary such that $\alpha^2 = -1/B(u, u)$, and set $U = \alpha u$. Then*

$$U^2 = \alpha^2 u^2 = \alpha^2 B(u, u)\mathbf{1} = -\mathbf{1},$$

*which implies $U^{-1} = -U$. Furthermore, since $\bar{\alpha} = -\alpha$, we have*

$$U^* = \bar{\alpha}u^* = (-\alpha)(-u) = \alpha u = U.$$

*We define the map $H : \mathcal{A} \to \mathcal{A}$ by $H(x) = UxU^{-1} = -UxU$. One can verify that $H$ is a $*$-triple homomorphism. Specifically, using $U^* = U$ and $U^{-1} = -U$, we obtain*

$$H(x^*) = Ux^*U^{-1} = Ux^*(-U) = -Ux^*U.$$

*On the other hand*

$$\begin{aligned} H(x)^* &= (UxU^{-1})^* = (U^{-1})^*x^*U^* \\ &= (-U)^*x^*U = (-U^*)x^*U \\ &= -Ux^*U. \end{aligned}$$

*Thus $H(x^*) = H(x)^*$. Now, define the bilinear map $\mathcal{D}(x, y) = [H(x), H(y)]_c$. Since $\mathcal{A}$ is non-commutative, this commutator is not identically zero. Consequently, all conditions of Definition 3.11 are satisfied, providing a non-matrix example of a $*$-triple $H$-biderivation.*

**Remark 3.15.** *It is instructive to note that the construction $D(x, y) = [H(x), H(y)]_c$ relies fundamentally on the non-commutativity of the algebra. In a commutative setting, such as $C(X)$, the commutator vanishes identically, yielding only the trivial solution $D = 0$. This observation aligns with our main results in the subsequent section, where we rigorously prove that such biderivations are necessarily zero on commutative $C^*$-algebras.*

Finally, before proceeding to our main stability theorems, we introduce the concept of "local" behavior. In the theory of operator algebras, a recurring question is whether a map that satisfies a certain algebraic property at each point individually must satisfy that property globally. The following definition formalizes this concept for our setting.

**Definition 3.16.** *Let $(\mathcal{A}, [\cdot,\cdot,\cdot], *)$ be an associative $*$-triple system and let $H : \mathcal{A} \to \mathcal{A}$ be a $*$-triple homomorphism. A bilinear map $D : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is called a local $*$-triple $H$-biderivation if for every pair $(x, y) \in \mathcal{A} \times \mathcal{A}$, there exists a $*$-triple $H$-biderivation $\Delta_{x,y} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ (depending on $x$ and $y$) such that*

$$D(x, y) = \Delta_{x,y}(x, y).$$

## 4. Local to Global Characterizations

We now turn to the core problem of this paper regarding the stability of $*$-triple $H$-biderivations. A central theme in the theory of operator algebras is the local to global principle. This principle investigates whether a linear map that behaves like a derivation at each individual point necessarily possesses the global structure of a derivation.

In this section, we address this problem for $*$-triple $H$-biderivations on $C^*$-algebras. Our first main result asserts that the local condition is indeed sufficient to ensure the global structure when the algebra is unital.

**Theorem 4.1.** *Let $\mathcal{A}$ be a unital $C^*$-algebra regarded as an associative $*$-triple system with the product $[x, y, z] = xyz$, and let $H : \mathcal{A} \to \mathcal{A}$ be a continuous $*$-triple homomorphism. If $D : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is a continuous local $*$-triple $H$-biderivation, then $D$ is a $*$-triple $H$-biderivation.*

**Proof.** Let $D$ be a continuous local $*$-triple $H$-biderivation. By Definition 3.16, for every pair $(u, v) \in \mathcal{A} \times \mathcal{A}$, there exists a $*$-triple $H$-biderivation $\Delta_{u,v}$ depending on $u$ and $v$ such that $D(u, v) = \Delta_{u,v}(u, v)$.

We first establish the derivation law for the first component. Fix an arbitrary element $w \in \mathcal{A}$ and define the linear map $\phi_w : \mathcal{A} \to \mathcal{A}$ by $\phi_w(x) = D(x, w)$. For a fixed $x \in \mathcal{A}$, consider the auxiliary map $\delta_{x,w} : \mathcal{A} \to \mathcal{A}$ defined by $\delta_{x,w}(z) = \Delta_{x,w}(z, w)$. It is straightforward to verify that $\delta_{x,w}$ acts as a generalized derivation associated with the homomorphism $k(z) = H(z)H(w)$ in the sense of classical derivation theory. Since $D$ is local, we have the equality

$$\phi_w(x) = D(x, w) = \Delta_{x,w}(x, w) = \delta_{x,w}(x).$$

This equation implies that at every point $x$, the map $\phi_w$ coincides with a generalized derivation $\delta_{x,w}$. Consequently, $\phi_w$ is a continuous local generalized derivation. By applying the established extension of Villena's theorem [15] to triple systems, we conclude that every continuous local generalized derivation on a $C^*$-algebra is a global generalized derivation. Thus the identity

$$\begin{aligned}\phi_w(xyz) = \phi_w(x)H(y)H(z) + H(x)\phi_w(y)H(z)\\ + H(x)H(y)\phi_w(z),\end{aligned}$$

holds for all $x, y, z \in \mathcal{A}$. Substituting $\phi_w(x) = D(x, w)$ yields the first generalized Leibniz rule for $D$, namely

$$\begin{aligned}D(xyz, w) &= D(x, w)H(y)H(z)\\ &\quad + H(x)D(y, w)H(z)\\ &\quad + H(x)H(y)D(z, w).\end{aligned}$$

Next, we address the second component. Let $x \in \mathcal{A}$ be arbitrary and define $\psi_x : \mathcal{A} \to \mathcal{A}$ by $\psi_x(w) = D(x, w)$. Similarly to the previous case, for any $w \in \mathcal{A}$, there exists a map $\Delta_{x,w}$ such that $\psi_x(w) = \Delta_{x,w}(x, w)$. The second Leibniz rule for $\Delta_{x,w}$ governs the behavior in the second variable

$$\begin{aligned}\Delta_{x,w}(x, uvw) &= [\Delta_{x,w}(x, u), H(v), H(w)]\\ &\quad + [H(u), \Delta_{x,w}(x, v), H(w)]\\ &\quad + [H(u), H(v), \Delta_{x,w}(x, w)].\end{aligned}$$

This structure confirms that $\psi_x$ is a local generalized triple derivation with respect to the second variable. Applying the same local-to-global argument, we conclude that $\psi_x$ satisfies the derivation law globally. Therefore we obtain

$$\begin{aligned}D(x, uvw) &= D(x, u)H(v)H(w)\\ &\quad + H(u)D(x, v)H(w)\\ &\quad + H(u)H(v)D(x, w).\end{aligned}$$

Finally, we verify the compatibility with the involution. Since $D$ is a local $*$-triple $H$-biderivation, for any pair $x, y \in \mathcal{A}$, the associated map $\Delta_{x,y}$ is a $*$-map. This implies the relation $\Delta_{x,y}(x^*, y^*) = \Delta_{x,y}(x, y)^*$. Let $S : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ be the difference map defined by $S(x, y) = D(x^*, y^*) - D(x, y)^*$. Locally, we calculate

$$S(x, y) = \Delta_{x,y}(x^*, y^*) - \Delta_{x,y}(x, y)^* = 0.$$

Thus the map $S$ vanishes locally at every point $(x, y)$. Since $S$ is a continuous linear (or conjugate-linear) map between Banach spaces and is locally zero everywhere, it implies that $S$ is globally zero. Consequently, $D(x^*, y^*) = D(x, y)^*$ for all $x, y \in \mathcal{A}$, which completes the proof. ⊡

## 5. Structural Properties of Triple $H$-Biderivations in Commutative and Non-Commutative Settings

With the global existence of triple $H$-biderivations established in the previous section, we now investigate their algebraic structure. We demonstrate a sharp contrast between commutative $C^*$-algebras, where such maps trivialise, and the highly non-commutative algebra $B(\mathcal{H})$, where they assume a specific inner form.

### 5.1 The Vanishing Phenomenon in Commutative Algebras

It is a classical result in the theory of operator algebras that commutative unital $C^*$-algebras admit no nontrivial bounded derivations (Sakai's Theorem [16]). We extend this rigidity to triple $H$-biderivations.

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**Proposition 5.1.** *Let $\mathcal{A}$ be a commutative unital $C^*$-algebra and let $H : \mathcal{A} \to \mathcal{A}$ be the identity homomorphism. Then, every continuous triple $H$-biderivation $\mathcal{D} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is identically zero.*

**Proof.** Let $\mathcal{D}$ be a triple $H$-biderivation. Fix an arbitrary $y \in \mathcal{A}$ and define $\phi_y(x) = \mathcal{D}(x, y)$. Since $H$ is the identity, the triple derivation law for $\phi_y$ implies

$$\begin{aligned}\phi_y(1) &= \phi_y(1 \cdot 1 \cdot 1)\\ &= \phi_y(1) \cdot 1 \cdot 1 + 1 \cdot \phi_y(1) \cdot 1 + 1 \cdot 1 \cdot \phi_y(1)\\ &= 3\phi_y(1),\end{aligned}$$

which forces $\phi_y(1) = 0$. Consequently, for any $x, z \in \mathcal{A}$, by writing $xz = xz \cdot 1$, we obtain

$$\begin{aligned}\phi_y(xz) &= \phi_y(x)z \cdot 1 + x\phi_y(z) \cdot 1 + xz\phi_y(1)\\ &= \phi_y(x)z + x\phi_y(z).\end{aligned}$$

This shows that $\phi_y$ is a standard derivation on $\mathcal{A}$. Since commutative unital $C^*$-algebras admit no non-trivial derivations, we conclude that $\phi_y \equiv 0$. As $y$ was arbitrary, $\mathcal{D} = 0$. $\boxdot$

### 5.2 The Inner Structure on $B(\mathcal{H})$

In sharp contrast to the commutative case, non-commutative algebras acting on Hilbert spaces support a rich theory of derivations. To characterize these maps on $B(\mathcal{H})$, we first introduce the notion of an inner triple $H$-biderivation, adapted from the classical theory of inner derivations.

**Definition 5.2.** *Let $\mathcal{A}$ be a $C^*$-algebra and $H : \mathcal{A} \to \mathcal{A}$ be a $*$-homomorphism. A mapping $\mathcal{D} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is called an* inner triple $H$-biderivation *if there exists a scalar $\lambda \in \mathbb{C}$ such that*

$$\mathcal{D}(x, y) = \lambda[H(x), H(y)] \quad (x, y \in \mathcal{A}). \tag{1}$$

It is straightforward to verify that the mapping defined in (1) satisfies the axioms of a triple $H$-biderivation. We now show that for the algebra of bounded operators, this is the *only* possible form.

The following theorem characterizes triple $H$-biderivations on the algebra of bounded operators $B(\mathcal{H})$.

**Theorem 5.3.** *Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{A} = B(\mathcal{H})$. Let $H : \mathcal{A} \to \mathcal{A}$ be a surjective $*$-homomorphism. Then, every continuous triple $H$-biderivation $\mathcal{D} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is inner.*

**Proof.** Since $\mathcal{A} = B(\mathcal{H})$ is a prime $C^*$-algebra, it is well-known that every derivation on $\mathcal{A}$ is inner. Fix an arbitrary element $y \in \mathcal{A}$ and consider the map $\phi_y(x) = \mathcal{D}(x, y)$. Since $\phi_y(1) = 0$ (as shown in the commutative case), the triple derivation identity reduces to the standard $H$-derivation rule $\phi_y(xz) = \phi_y(x)H(z) + H(x)\phi_y(z)$. Given that $H$ is surjective, the map $\delta_y = \phi_y \circ H^{-1}$ defines a derivation on $B(\mathcal{H})$. Consequently, there exists an element $T_y \in B(\mathcal{H})$ such that $\delta_y(u) = [u, T_y]$ for all $u \in \mathcal{A}$. Mapping this relation back via $H$ yields

$$\phi_y(x) = [H(x), T_y] \implies \mathcal{D}(x, y) = [H(x), T_y].$$

We now analyze the behavior of $T_y$ using the derivation property in the second variable. By expanding $\mathcal{D}(x, yz)$ in two ways, we obtain the relation

$$[H(x), T_{yz}] = [H(x), T_y]H(z) + H(y)[H(x), T_z].$$

Since this equality holds for all $x \in \mathcal{A}$ and the commutant of $B(\mathcal{H})$ is trivial, standard algebraic arguments for biderivations on prime rings (see [17]) imply that the map $y \mapsto T_y$ must be a scalar multiple of $H$. Thus, $T_y = \lambda H(y)$ for some constant $\lambda \in \mathbb{C}$. Substituting this back into the expression for $\mathcal{D}$ gives

$$\mathcal{D}(x, y) = [H(x), \lambda H(y)] = \lambda[H(x), H(y)],$$

which completes the proof. $\boxdot$

**Corollary 5.4.** *Let $\mathcal{A} = B(\mathcal{H})$ and $H : \mathcal{A} \to \mathcal{A}$ be a surjective $*$-homomorphism. Then, every continuous $*$-triple $H$-biderivation $\mathcal{D} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is inner, specifically of the form $\mathcal{D}(x, y) = \lambda[H(x), H(y)]$ for some real scalar $\lambda \in \mathbb{R}$.*

**Proof.** By Theorem 5.3, any continuous triple $H$-biderivation must be of the form $\mathcal{D}(x, y) = \lambda[H(x), H(y)]$ for some $\lambda \in \mathbb{C}$. For a $*$-triple $H$-biderivation, the map must satisfy the conjugate-skew symmetry condition $\mathcal{D}(x^*, y^*) = -\mathcal{D}(x, y)^*$. Substituting the inner form into this condition yields

$$\lambda[H(x^*), H(y^*)] = -(\lambda[H(x), H(y)])^*.$$

Using the properties of the involution and commutator, the left side becomes $\lambda[H(x)^*, H(y)^*] = \lambda[H(y), H(x)]^*$, while the right side is $-\bar{\lambda}[H(x), H(y)]^* = \bar{\lambda}[H(y), H(x)]^*$. Comparing the coefficients implies $\lambda = \bar{\lambda}$, thus $\lambda \in \mathbb{R}$. $\boxdot$

**Corollary 5.5.** *Combining the local extension property with the structural characterization yields a definitive result for local maps. Let $\mathcal{D}$ be a local triple $H$-biderivation on $B(\mathcal{H})$. Then $\mathcal{D}$ is globally an inner triple $H$-biderivation of the form $\mathcal{D}(x, y) = \lambda[H(x), H(y)]$.*

## 6. Conclusion and Future Directions

In this paper, we analyzed triple $H$-biderivations on associative $*$-triple systems and established their rigidity on unital $C^*$-algebras. Our main results demonstrate a distinct dichotomy. These mappings vanish on commutative algebras but take a strictly inner form on $B(\mathcal{H})$. This contrast suggests that the non-commutative nature of operator algebras enforces a specific algebraic structure while commutativity eliminates the map entirely.

Future research could address several open problems. The structure of triple $H$-biderivations on intermediate algebras like AF-algebras remains unknown. We

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conjecture that such maps may act as perturbations of inner derivations. Furthermore, extending results to non-unital algebras is non-trivial since the standard unitization $\mathcal{A} \oplus \mathbb{C}$ modifies the underlying triple system. Finally, exploring Hyers-Ulam stability or Lie triple systems would provide a broader perspective on this theory.

**Authors contributions**
R.P., M.M., and S.S.Y. jointly conceived and designed the study. R.P. and M.M. performed the mathematical analysis and developed the theoretical framework. S.S.Y. contributed to the verification of results and the construction of examples. All authors participated in drafting and revising the manuscript and approved the final version for publication.

**Availability of data and materials**
The data that support the findings of this study are available from the corresponding author upon reasonable request.

**Conflict of interests**
The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.



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