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Common Fixed Point Theorem for a Pair of Self Mappings in Dislocated Quasi b-Metric Space

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Abstract

This article will illustrate a common fixed point theorem for a pair of continuous self-mapping in the dislocated quasi-b-metric space framework. The established theorem extends and generalizes some well-known results given by Rahman et al. in [4]. Also, an example is given in support of the constructed result.

Keywords: b-metric space, dislocated quasi b-metric space, fixed point, common fixed point, self mapping
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1. Introduction

A point x in a set X is said to be a fixed point of a self-mapping T of X if $Tx = x$. The concept of fixed point theories is one of the most important results in Functional Analysis. The famous fixed point result called the Banach Contraction Principle (BCP) is generalized and improved in many directions. One usual way of studying the Banach contraction principle is to replace the metric space with certain generalized metric spaces. There have been several generalizations of the usual notion of a metric space. One such generalization is a b-metric space introduced by Czerwik [2] in connection with some problems concerning the convergence of non-measurable functions concerning their measure. The concept of b-metric space was generalized in different directions. One such generalization is a dislocated quasi-b-metric space introduced and studied by Rahman and Sarwar in 2015. Rahman and Sarwar open a new way for researchers to work in the field of metric fixed point theory. In the present article, we shall generalize the result of the common fixed point theorem in dislocated quasi-b-metric space given by Rahman et al. [4].

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2. Definitions and Preliminaries

Before proceeding with the main work, we shall define some important definitions, examples and some results related to dislocated quasi-b-metric spaces, which are used in further discussion.

Definition 2.1. [3] A point u is a fixed point of the function $f(x)$ if $f(u) = u$. In other words, $f(x)$ has a root at u iff $g(x) = x - f(x)$ has a fixed point at u.

There have been several generalizations of the usual notion of a metric space. One such generalization is a b-metric space introduced and studied by Bakhtin [1] and Czerwik [2] in 1989. They also investigated some fixed point results in b-metric spaces. Now, we will begin with the definition of b-metric space.

Definition 2.2. [1] Let X be a non-empty set and $k \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric provided that for all $x, y, z \in X$,

(i) $d(x, y) = 0$ if and only if $x = y$
(ii) $d(x, y) = d(y, x)$
(iii) $d(x, y) \leq k[d(x, z) + d(z, y)]$ (b-triangular inequality)

A pair (X, d) is called a b-metric space. The number k is called the coefficient of (X, d). The following are examples of b-metric space.

Example 2.3. [1] The set $l_p(\mathbb{R})$ where $l_p(\mathbb{R}) := \{(x_n) \subseteq \mathbb{R} : \sum |x|^p < \infty\}$ (with $0 < p < 1$) together with the function $d : l_p(\mathbb{R}) \times l_p(\mathbb{R}) \to [0, \infty)$ defined by

$$d(x, y) = \Big(\sum_{n=1}^{\infty} |x - y|^p\Big)^{\frac{1}{p}}$$

where $x = \{x_n\}, y = \{y_n\} \in l_p(\mathbb{R})$ is a b-metric with $k = 2^{\frac{1}{p}}$.

Example 2.4. [1] The space $L_p[0, 1]$ (*where* $0 < p < 1$) of all real functions $x(t)$, $t \in [0, 1]$ such that $\int_0^1 |x(t)|^p dt < \infty$ is a b-metric by taking

$$d(x, y) = \Big(\int_0^1 |x(t) - y(t)|^p dt\Big)^{\frac{1}{p}}$$

for each $x, y \in L_p[0, 1]$ is a b-metric with $k = 2^{\frac{1}{p}}$.

It is clear that the definition of b-metric is an extension of the usual metric space. Obviously, each metric space is a b-metric space with $k = 1$. However, Czerwik [2] has shown that a b-metric on X need not be a metric on X. The following example illustrates this.

Example 2.5. Let (X, d) be a metric space. Define $\rho(x, y) = [d(x, y)]^p$, where $p > 1$ is a real number. Then ρ is a b-metric with $k = 2^{p-1}$. However, if (X, d) is a metric space, then (X, ρ) is not necessarily a metric space. For example, if $X = \mathbb{R}$ is the set of real numbers and $d(x, y) = |x - y|$ is the usual Euclidean metric, then $\rho(x, y) = (x - y)^2$ is a b-metric on $\mathbb{R}$ with $k = 2$. But it is not a metric space.

It is noted that the class of b-metric spaces is larger than the class of metric spaces. Now, we will begin with the definition of dislocated quasi-b-metric space.

Definition 2.6. [5] Let X be a non-empty set and $k \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a dislocated quasi b-metric provided that for all $x, y, z \in X$,

(i) $d(x, y) = d(y, x) = 0$, then $x = y$.
(ii) $d(x, y) \leq k[d(x, z) + d(z, y)]$.

A pair (X, d) is called a dislocated quasi b-metric space (or dqb-metric space).

In the definition of dislocated quasi b-metric space if $k = 1$ then it becomes (usual) dislocated quasi metric space. Therefore every dislocated quasi-metric space is dislocated quasi b-metric space and every b-metric space is dislocated quasi b-metric space with coefficient k and zero self distance. However, the converse is not true. The following are examples of dislocated quasi-b-metric space.

Example 2.7. [5] Let $X = \mathbb{R}^+$, for $p > 1$ and $d : X \times X \to [0, \infty)$ be defined as

$$d(x, y) = |x - y|^p + |x|^p, \text{ for all } x,\ y\ \in X.$$

Then, (X, d) is a dislocated quasi b-metric space with $k = 2^p > 1$. But it is neither b-metric nor dislocated quasi-metric.

Example 2.8. [5] Let $X = \mathbb{R}$ and suppose

$$d(x, y) = |2x - y|^2 + |2x + y|^2.$$

Then, (X, d) is a dislocated quasi b-metric space with the coefficient $k = 2$.

Definition 2.9. [7] A sequence $\{x_n\}$ in dislocated quasi b-metric space (dq b-metric space) (X, d) is said to be

(i) dqb-convergent to x if
$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0.$$
In this case, x is called the limit of $\{x_n\}$ and we write $x_n \to x$.

(ii) dqb-complete if every Cauchy sequence in it is a dq b-convergent.

(iii) Cauchy sequence if for $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $m, n \geq N$, we have $d(x_m, x_n) < \epsilon$.

Lemma 2.10. *[5] Limit of convergent sequence in a dq b-metric space is unique.*

The following lemma given by Rahman and Sarwar shall be used in the later theorems.

Lemma 2.11. *[5] Let (X, d) be a dq b-metric space and $\{x_n\}$ be a sequence in X such that*

$$d(x_n, x_{n+1}) \leq \alpha d(x_{n-1}, x_n) \ \ for\ n = 0, 1, 2, \dots \ and\ 0 \leq \alpha k < 1\ ,\ \alpha \in [0, 1),$$

and k is defined in dq b-metric space. Then $\{x_n\}$ is a Cauchy sequence.

Here, Rahman and Sarwar proved the famous Banach contraction principle in complete dislocated quasi-b-metric space.

Theorem 2.12. *[5] Let (X, d) be a complete dq b-metric space and $T : X \to X$ be a continuous contraction with $0 \leq \alpha k < 1$, $\alpha \in [0, 1)$, where $k \geq 1$ then T has a unique fixed point in X.*

Theorem 2.13. *[4] Let (X, d) be a complete dq b-metric space with coefficient $k \geq 1$ and $S, T : X \to X$ be continuous self mappings satisfying the condition*

$$\begin{aligned} d(Sx, Ty) \leq & \alpha d(x, y) + \beta \frac{d(y, Ty)[1 + d(x, Sx)]}{1 + d(x, y)} + \gamma \frac{d(x, Ty)d(y, Ty)}{k[d(x, y) + d(y, Ty)} + \mu \frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)} \\ & + \ \rho \frac{d(x, Sx)d(x, Ty)}{1 + d(x, y)} \end{aligned}$$

for all $x, y \in X$, where $\alpha, \beta, \gamma, \mu, \rho \geq 0$, with $k\alpha + \beta + \gamma + \mu + k(1 + k)\rho < 1$. Then S and T have a unique common fixed point in X.

On putting $S = T$ in the above theorem, we get the following corollary.

Corollary 2.14. *[4] Let (X, d) be a complete dq b-metric space with coefficient $k \geq 1$ and $T : X \to X$ be a continuous self mapping satisfying the condition*

$$\begin{aligned} d(Tx, Ty) \leq & \alpha d(x, y) + \beta \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \gamma \frac{d(x, Ty)d(y, Ty)}{k[d(x, y) + d(y, Ty)} + \mu \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)} \\ & + \rho \frac{d(x, Tx)d(x, Ty)}{1 + d(x, y)}. \end{aligned}$$

for all $x, y \in X$, where $\alpha, \beta, \gamma, \mu, \rho \geq 0$, with $k\alpha + \beta + \gamma + \mu + k(1 + k)\rho < 1$. Then T has a unique fixed point in X.

3. Main Result

In this section, we shall extend and generalize the result on common fixed point theorem in dislocated quasi b-metric space given by Rahman et al. [4].

Theorem 3.1. *Let (X, d) be a complete dq b-metric space with coefficient $k \geq 1$ and $S, T : X \to X$ be continuous self mappings satisfying the condition*

$$\begin{aligned} d(Sx, Ty) \leq & \alpha d(x, y) + \beta \frac{d(y, Ty)[1 + d(x, Sx)]}{1 + d(x, y)} + \gamma \frac{d(x, Ty)d(y, Ty)}{k[d(x, y) + d(y, Ty)]} + \mu \frac{d(x, Ty)d(x, Sx)}{k[d(x, y) + d(y, Ty)]} \\ & + \rho \frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)} + \lambda \frac{d(x, Sx)d(x, Ty)}{1 + d(x, y)}, \end{aligned}$$

for all $x, y \in X$, where $\alpha, \beta, \gamma, \mu, \rho, \lambda \geq 0$ with $k\alpha + \beta + \gamma + k\mu + \rho + k(1 + k)\lambda < 1$. Then S and T have a unique common fixed point in X.

Proof. Let x_0 be arbitrary in X. We define a sequence $\{x_n\}$ by

$$x_0, x_1 = Sx_0, x_3 = Sx_2, ..., x_{2n+1} = Sx_{2n} \text{ and } x_2 = Tx_1, x_4 = Tx_3, ..., x_{2n} = Tx_{2n-1}.$$

Now, we show that $\{x_n\}$ is a Cauchy sequence in X. For this, consider

$$d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1}).$$

Now, using the given condition in the theorem and the definition of the sequence defined above, we have

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) \leq & \alpha d(x_{2n}, x_{2n+1}) + \beta \frac{d(x_{2n+1}, Tx_{2n+1})[1 + d(x_{2n}, Sx_{2n})]}{1 + d(x_{2n}, x_{2n+1})} \\ & + \gamma \frac{d(x_{2n}, Tx_{2n+1})d(x_{2n+1}, Tx_{2n+1})}{k[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, Tx_{2n+1})]} + +\mu \frac{d(x_{2n}, Tx_{2n+1})d(x_{2n}, Sx_{2n})}{k[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, Tx_{2n+1})]} \\ & + \rho \frac{d(x_{2n}, Sx_{2n})d(x_{2n+1}, Tx_{2n+1})}{1 + d(x_{2n}, x_{2n+1})} + \lambda \frac{d(x_{2n}, Sx_{2n})d(x_{2n}, Tx_{2n+1})}{1 + d(x_{2n}, x_{2n+1})} \\ \leq & \alpha d(x_{2n}, x_{2n+1}) + \beta \frac{d(x_{2n+1}, x_{2n+2})[1 + d(x_{2n}, x_{2n+1})]}{1 + d(x_{2n}, x_{2n+1})} \\ & + \gamma \frac{d(x_{2n}, x_{2n+2})d(x_{2n+1}, x_{2n+2})}{k[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]} + \mu \frac{d(x_{2n}, x_{2n+2})d(x_{2n}, x_{2n+1})}{k[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]} \\ & + \rho \frac{d(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1})} + \lambda \frac{d(x_{2n}, x_{2n+1})d(x_{2n}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1})}. \end{aligned}$$

Now, using the triangular inequality, we have

$$\begin{aligned}d(x_{2n+1},x_{2n+2}) \leq &\alpha d(x_{2n},x_{2n+1}) + \beta d(x_{2n+1},x_{2n+2}) + \gamma d(x_{2n+1},x_{2n+2}) + \mu d(x_{2n},x_{2n+1})\\ &+ \rho d(x_{2n+1},x_{2n+2}) + \lambda k[d(x_{2n},x_{2n+1}) + d(x_{2n+1},x_{2n+2})].\end{aligned}$$

On simplification, we have

$$(1-\beta-\gamma-\rho-k\lambda)d(x_{2n+1},x_{2n+2}) \leq (\alpha+\mu+k\lambda)d(x_{2n},x_{2n+1}).$$

This implies that

$$d(x_{2n+1},x_{2n+2}) \leq \left(\frac{\alpha+\mu+k\lambda}{1-\beta-\gamma-\rho-k\lambda}\right)d(x_{2n},x_{2n+1}).$$

Let $h = \frac{\alpha+\mu+k\lambda}{1-\beta-\gamma-\rho-k\lambda} < \frac{1}{k}$, so the above equation becomes

$$d(x_{2n+1},x_{2n+2}) \leq h\, d(x_{2n},x_{2n+1}).$$

Similarly, we get

$$d(x_{2n},x_{2n+1}) \leq h\, d(x_{2n-1},x_{2n}).$$

Continuing the same procedure, we get

$$d(x_n,x_{n+1}) \leq h\, d(x_{n-1},x_n).$$

From Lemma 2.11, we get $\{x_n\}$ is a Cauchy sequence in a complete dislocated quasi b-metric space X, so $\{x_n\}$ converges to some $u \in X$ i.e., there exists $u \in X$, such that $x_n \to u$. Also, the subsequence $\{x_{2n+1}\}$ and $\{x_{2n+2}\}$ converge to $u \in X$. Using the continuity of T, we have

$$\begin{aligned}Tu =&T \lim_{n\to\infty}(x_{2n+1})\\ =&\lim_{n\to\infty} T(x_{2n+1})\\ =&\lim_{n\to\infty}(x_{2n+2})\\ =&u.\end{aligned}$$

Similarly, using the continuity of S, we can show that $Su = u$. Therefore, u is the common fixed point of S and T.

Uniqueness: Let u, v be two common fixed points of S and T i.e., $Su = Tu = u$ and $Sv = Tv = v$. Using the given condition in the theorem and triangular inequality, we can show that

$$d(u,u) = d(v,v) = 0.$$

Now,

$$\begin{aligned}d(u,v) =&d(Su,Tv)\\ \leq&\alpha d(u,v) + \beta\frac{d(v,Tv)[1+d(u,Su)]}{1+d(u,v)} + \gamma\frac{d(u,Tv)d(v,Tv)}{k[d(u,v)+d(v,Tv)]}\\ &+ \mu\frac{d(u,Tv)d(u,Su)}{k[d(u,v)+d(v,Tv)]} + \rho\frac{d(u,Su)d(v,Tv)}{1+d(u,v)} + \lambda\frac{d(u,Su)d(u,Tv)}{1+d(u,v)}.\end{aligned}$$

Since u, v are common fixed points of S and T, we have

$$\begin{aligned}d(u,v) =&d(Su,Tv)\\ \leq&\alpha d(u,v) + \beta\frac{d(v,v)[1+d(u,u)]}{1+d(u,v)} + \gamma\frac{d(u,v)d(v,v)}{k[d(u,v)+d(v,v)]}\\ &+ \mu\frac{d(u,v)d(u,u)}{k[d(u,v)+d(v,v)]} + \rho\frac{d(u,u)d(v,v)}{1+d(u,v)} + \lambda\frac{d(u,u)d(u,v)}{1+d(u,v)}\\ \leq&\alpha d(u,v).\end{aligned}$$

Thus, $d(u,v) = 0$, as $k\alpha < 1$. Similarly, we can show that $d(v,u) = 0$. So, $d(u,v) = d(v,u) = 0$. This implies that $u = v$. Hence, u is the common unique fixed point of S and T. $\square$

On putting $S = T$, in the above theorem, we get the following corollary.

Corollary 3.2. *Let (X,d) be a complete dq b-metric space with coefficient $k \geq 1$ and $T : X \to X$ be a continuous self mapping satisfying the condition*

$$\begin{aligned} d(Tx,Ty) \leq & \alpha d(x,y) + \beta \frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)} + \gamma \frac{d(x,Ty)d(y,Ty)}{k[d(x,y)+d(y,Ty)]} + \mu \frac{d(x,Ty)d(x,Tx)}{k[d(x,y)+d(y,Ty)]} \\ & + \rho \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)} + \lambda \frac{d(x,Tx)d(x,Ty)}{1+d(x,y)} \end{aligned}$$

for all $x,y \in X$, where $\alpha,\beta,\gamma,\mu,\rho,\lambda \geq 0$, with $k\alpha + \beta + \gamma + k\mu + \rho + k(1+k)\lambda < 1$. Then T has a unique fixed point in X.

Remark 3.3. If we put $\mu = 0$ in Theorem 3.1, then we get the Theorem 2.13.

Remark 3.4. If we put $\mu = 0$ in Corollary 3.2, then we get Corollary 2.14.

Remark 3.5. If we put $\beta = \gamma = \mu = \rho = \lambda = 0$, then we get the Banach contraction theorem in the context of dislocated quasi b-mertric space.

Example 3.6. Let $X = [0,1]$, and define

$$d(x,y) = |2x-y|^2 + |2x+y|^2$$

for all $x,y \in X$. Then, (X,d) is a complete dq b-metric space with $k = 2$. Let $S,T : X \to X$ be two continuous self-mappings defined by $Sx = 0$ and $Tx = \frac{x}{3}$. In particular, if we take

$$\alpha = \frac{1}{10}, \beta = \frac{1}{6}, \gamma = \frac{1}{12}, \rho = \frac{1}{16}, \mu = \frac{1}{30}, \lambda = \frac{1}{36}.$$

Then, all conditions of Theorem 3.1 are satisfied and $x = 0 \in [0,1]$ is the unique common fixed point of S and T.

Theorem 3.7. *Let (X,d) be a complete dq b-metric space with coefficient $k \geq 1$ and $S,T : X \to X$ be continuous self mappings satisfying condition*

$$\begin{aligned} d(Sx,Ty) \leq & \alpha d(x,y) + \beta \frac{d(y,Ty)[1+d(x,Sx)]}{1+d(x,y)} + \gamma \frac{d(x,Ty)d(y,Ty)}{k[d(x,y)+d(y,Ty)]} + \mu \frac{d(x,Ty)d(x,Sx)}{k[d(x,y)+d(y,Ty)]} \\ & + \rho \frac{d(x,Sx)d(y,Ty)}{1+d(x,y)} + \lambda \frac{d(x,Sx)d(x,Ty)}{1+d(x,y)} + \eta \frac{d(x,Ty)[1+d(x,Sx)]}{k[1+d(x,y)]} \end{aligned}$$

for all $x,y \in X$, where $\alpha,\beta,\gamma,\mu,\rho,\lambda,\eta \geq 0$, with $k\alpha + \beta + \gamma + k\mu + \rho + (1+k)\eta + k(1+k)\lambda < 1$. Then S and T have a unique common fixed point in X.

Proof. Let x_0 be arbitrary in X. We define a sequence $\{x_n\}$ by

$$x_0, x_1 = Sx_0, x_3 = Sx_2, \ldots, x_{2n+1} = Sx_{2n} \text{ and } x_2 = Tx_1, x_4 = Tx_3, \ldots, x_{2n} = Tx_{2n-1}.$$

Now, we show $\{x_n\}$ is a Cauchy sequence in X. For this, consider

$$d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1}).$$

Now, using the given condition in the theorem and the definition of the sequence defined above, we have

$$\begin{aligned}
d(x_{2n+1}, x_{2n+2}) \leq & \alpha d(x_{2n}, x_{2n+1}) + \beta \frac{d(x_{2n+1}, Tx_{2n+1})[1 + d(x_{2n}, Sx_{2n})]}{1 + d(x_{2n}, x_{2n+1})} \\
& + \gamma \frac{d(x_{2n}, Tx_{2n+1}) d(x_{2n+1}, Tx_{2n+1})}{k[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, Tx_{2n+1})]} + \mu \frac{d(x_{2n}, Tx_{2n+1}) d(x_{2n}, Sx_{2n})}{k[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, Tx_{2n+1})]} \\
& + \rho \frac{d(x_{2n}, Sx_{2n}) d(x_{2n+1}, Tx_{2n+1})}{1 + d(x_{2n}, x_{2n+1})} + \lambda \frac{d(x_{2n}, Sx_{2n}) d(x_{2n}, Tx_{2n+1})}{1 + d(x_{2n}, x_{2n+1})} \\
& + \eta \frac{d(x_{2n}, Tx_{2n+1})[1 + d(x_{2n}, Sx_{2n})]}{k[1 + d(x_{2n}, x_{2n+1})]} \\
\leq & \alpha d(x_{2n}, x_{2n+1}) + \beta \frac{d(x_{2n+1}, x_{2n+2})[1 + d(x_{2n}, x_{2n+1})]}{1 + d(x_{2n}, x_{2n+1})} \\
& + \gamma \frac{d(x_{2n}, x_{2n+2}) d(x_{2n+1}, x_{2n+2})}{k[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]} + \mu \frac{d(x_{2n}, x_{2n+2}) d(x_{2n}, x_{2n+1})}{k[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]} \\
& + \rho \frac{d(x_{2n}, x_{2n+1}) d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1})} + \lambda \frac{d(x_{2n}, x_{2n+1}) d(x_{2n}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1})} \\
& + \eta \frac{d(x_{2n}, x_{2n+2})[1 + d(x_{2n}, x_{2n+1})]}{k[1 + d(x_{2n}, x_{2n+1})]}.
\end{aligned}$$

Now, using the triangular inequality, we have

$$\begin{aligned}
d(x_{2n+1}, x_{2n+2}) \leq & \alpha d(x_{2n}, x_{2n+1}) + \beta d(x_{2n+1}, x_{2n+2}) + \gamma d(x_{2n+1}, x_{2n+2}) + \mu d(x_{2n}, x_{2n+1}) + \rho d(x_{2n+1}, x_{2n+2}) \\
& + \lambda k[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] + \eta[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})].
\end{aligned}$$

On simplification, we have

$$(1 - \beta - \gamma - \rho - k\lambda - \eta) d(x_{2n+1}, x_{2n+2}) \leq (\alpha + \mu + \eta + k\lambda) d(x_{2n}, x_{2n+1}).$$

Then,

$$d(x_{2n+1}, x_{2n+2}) \leq \left(\frac{\alpha + \mu + \eta + k\lambda}{1 - \beta - \gamma - \rho - k\lambda - \eta} \right) d(x_{2n}, x_{2n+1}).$$

Let $h = \frac{\alpha+\mu+\eta+k\lambda}{1-\beta-\gamma-\rho-\eta-k\lambda} < \frac{1}{k}$, so the above equation becomes

$$d(x_{2n+1}, x_{2n+2}) \leq h\, d(x_{2n}, x_{2n+1}).$$

Similarly, we get

$$d(x_{2n}, x_{2n+1}) \leq h\, d(x_{2n-1}, x_{2n}).$$

Continuing the same procedure, we get

$$d(x_n, x_{n+1}) \leq h\, d(x_{n-1}, x_n).$$

From Lemma 2.11, we get $\{x_n\}$ is a Cauchy sequence in a complete dislocated quasi b-metric space X, so $\{x_n\}$ converges to some $u \in X$ i.e., there exists $u \in X$, such that $x_n \to u$. Also, the subsequence $\{x_{2n+1}\}$ and $\{x_{2n+2}\}$ converge to $u \in X$. Using the continuity of T, we have

$$\begin{aligned}
Tu = & T \lim_{n \to \infty} (x_{2n+1}) \\
= & \lim_{n \to \infty} T(x_{2n+1}) \\
= & \lim_{n \to \infty} (x_{2n+2}) \\
= & u.
\end{aligned}$$

Similarly, using the continuity of S, we can show that $Su = u$. Therefore, u is the common fixed point of S and T.

Uniqueness: Let u, v be two common fixed points of S and T i.e., $Su = Tu = u$ and $Sv = Tv = v$. Using given condition in the theorem and triangular inequality, we can show that

$$d(u,u) = d(v,v) = 0.$$

Now,

$$\begin{aligned} d(u,v) =& d(Su,Tv)\\ \leq& \alpha d(u,v) + \beta\frac{d(v,Tv)[1+d(u,Su)]}{1+d(u,v)} + \gamma\frac{d(u,Tv)d(v,Tv)}{k[d(u,v)+d(v,Tv)]} + \mu\frac{d(u,Tv)d(u,Su)}{k[d(u,v)+d(v,Tv)]}\\ &+ \rho\frac{d(u,Su)d(v,Tv)}{1+d(u,v)} + \lambda\frac{d(u,Su)d(u,Tv)}{1+d(u,v)} + \eta\frac{d(u,Tv)[1+d(u,Su)]}{k[1+d(u,v)]}. \end{aligned}$$

Since u, v are common fixed points of S and T, we have

$$\begin{aligned} d(u,v) =& d(Su,Tv)\\ \leq& \alpha d(u,v) + \beta\frac{d(v,v)[1+d(u,u)]}{1+d(u,v)} + \gamma\frac{d(u,v)d(v,v)}{k[d(u,v)+d(v,v)]} + \mu\frac{d(u,v)d(u,u)}{k[d(u,v)+d(v,v)]}\\ &+ \rho\frac{d(u,u)d(v,v)}{1+d(u,v)} + \lambda\frac{d(u,u)d(u,v)}{1+d(u,v)} + \eta\frac{d(u,v)[1+d(u,u)]}{k[1+d(u,v)]}\\ \leq& \alpha d(u,v) + \frac{\eta}{k}d(u,v)\\ \leq& \Big(\alpha + \frac{\eta}{k}\Big)d(u,v). \end{aligned}$$

Thus, $d(u,v) = 0$, as $\Big(\alpha + \frac{\eta}{k}\Big) < 1$. Similarly, we can show that $d(v,u) = 0$. So, $d(u,v) = d(v,u) = 0$. This implies $u = v$. Hence, u is the common unique fixed point of S and T. □

On putting $S = T$, in the above theorem, we get the following corollary.

Corollary 3.8. *Let (X, d) be a complete dq b-metric space with coefficient $k \geq 1$ and $T : X \to X$ be a continuous self mapping satisfying the condition*

$$\begin{aligned} d(Tx,Ty) \leq& \alpha d(x,y) + \beta\,\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)} + \gamma\frac{d(x,Ty)d(y,Ty)}{k[d(x,y)+d(y,Ty)]} + \mu\frac{d(x,Ty)d(x,Tx)}{k[d(x,y)+d(y,Ty)]}\\ &+ \rho\frac{d(x,Tx)d(y,Ty)}{1+d(x,y)} + \lambda\frac{d(x,Tx)d(x,Ty)}{1+d(x,y)} + \eta\frac{d(x,Ty)[1+d(x,Tx)]}{k[1+d(x,y)]} \end{aligned}$$

for all $x, y \in X$, where $\alpha, \beta, \gamma, \mu, \rho, \lambda, \eta \geq 0$, with $k\alpha + \beta + \gamma + k\mu + \rho + (1+k)\eta + k(1+k)\lambda < 1$. Then T has a unique fixed point in X.

Remark 3.9. If we put $\eta = 0$ in Theorem 3.7, then we get the Theorem 3.1.

Remark 3.10. If we put $\eta = 0$ in Corollary 3.8, then we get Corollary 3.2.

Remark 3.11. If we put $\beta = \gamma = \mu = \rho = \lambda = \eta = 0$, in Theorem 3.7, then we get the Banach contraction theorem in the context of dislocated quasi b-mertric space.

Example 3.12. Let $X = [0, 1]$, and define

$$d(x,y) = |2x - y|^2 + |2x + y|^2,$$

for all $x, y \in X$. Then, (X, d) is a complete dq b-metric space with $k = 2$. Let $S, T : X \to X$ be two continuous self-mappings defined by $Sx = 0$ and $Tx = \frac{x}{3}$. In particular, if we take

$$\alpha = \frac{1}{10}, \beta = \frac{1}{6}, \gamma = \frac{1}{12}, \rho = \frac{1}{16}, \mu = \frac{1}{30}, \lambda = \frac{1}{36}, \eta = \frac{1}{18}.$$

Then, all the conditions of theorem 3.7 are satisfied and $x = 0 \in [0, 1]$ is the unique common fixed point of S and T.

4. Conclusion

In this paper, we have generalized and extended some existing common fixed point theorems in dislocated quasi-b-metric space given by Rahman and et. al.

Conflict of Interest

The authors have no conflict of interest regarding the publication of this article.

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