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Some Advancements in Fixed Point Results of Kannan-Type F-contractions

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Abstract

In this manuscript, a new family of contractions called Kannan-type F-contraction is introduced in the framework of a compact metric space. Sufficient conditions for the existence and uniqueness of fixed points for such contractions are investigated. A non-trivial example is constructed to support the assumptions of our obtained theorems. The ideas proposed herein extend some existing fixed point results in the corresponding literature.

Keywords: Fixed point, compact metric space, F-contraction
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1. Introduction

The first most celebrated fixed point theorem [2] in metric space appeared explicitly in Banach's thesis, where it was initially applied to obtain the existence of a solution to an integral equation. The theorem is now popularly known as Banach fixed point theorem (or the contraction mapping principle). Indeed, Banach contraction principle [2] is a reformulation of the successive approximation techniques originally used by some earlier mathematicians, namely Cauchy, Liouville, Picard, Lipschitz, and so on. The original idea of fixed point theorem due to Banach has been developed and applied in different directions. In some generalizations of the contraction mapping principle, the contractive inequality is weakened (see, for example, [1, 3, 7, 11, 23]), and in others, the supporting assumptions are modified (for example, see [8, 12, 13, 15, 20]). In an attempt to extend the Banach contraction principle, Wardowski [23] introduced a new notion of contraction, called F-contraction and proved a fixed point theorem. Shortly, Wardowski and Dung [24] initiated the concept of F-weak contraction on a metric space and obtained a generalization of F-contraction. Secelean [22] noted that condition (F_2) in Wardowski's definition of F-contraction can be replaced with an equivalent and simpler one given by $(F_2') : \inf F = -\infty$. In like manner, Piri and Kumam [17] established a

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variant of Wardowski's theorem [23] by using (F_2'). Recently, Wardowski [25] proposed the replacement of the positive constant ξ in the original definition of F-contraction with a function φ fulfilling certain conditions, and thus established a new form of contraction on metric space under the name (φ, F)-contraction. For some recent advancements in F-contractions with related results, see [10, 14, 16].
A lot of work has been done in the area of fixed point results of contraction mappings on metric spaces. Some of these results include fixed point theorems of Kannan-type contractions and F-contractions. However, a combination of these two ideas has not been sufficiently established in the framework of compact metric spaces. With this background information, this paper proposes new conditions under which Kannan-type F-contraction possesses a unique fixed point. A particular advantage of the latter spaces over the former is that it makes it easier to discuss the existence of fixed points of contraction mappings without necessarily using the Picard iterative sequence.

2. Preliminaries

In this section, some fundamental notions that will be deployed here and hereafter are highlighted. Throughout this paper, every set X is considered non-empty, $\mathbb{N}$ is the set of natural numbers, $\mathbb{R}$ represents the set of real numbers and $\mathbb{R}_+$, the set of non-negative real numbers.

Definition 2.1. [23] Let Ω be the set of all functions F: $(0,\infty) \to \mathbb{R}$ satisfying the following conditions:

$(F1)$ F is strictly increasing, i.e, for α,$\beta \in (0,\infty)$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$;

$(F2)$ for each sequence $\{a_n\}$ of positive numbers $\lim_{n\to\infty} \mathrm{F}(a_n)$=0 if and only if $\lim_{n\to\infty} \mathrm{F}(a_n) = -\infty$;

$(F3)$ there exists $k \in (0,1)$ such that $\lim_{a\to 0^+} a^k \mathrm{F}(a) = 0$.

Let (X,d) be a metric space and $T : X \longrightarrow X$ be a mapping. Then T is said to be an F-contraction if $F \in \Omega$ and there exists $\tau > 0$ such that for all $x, y \in X$,

$$d(Tx,Ty) > 0 \ \Rightarrow \tau + F(d(Tx,Ty)) \leq F(d(x,y)). \tag{2.1}$$

Edelstein [4] obtained a version of the Banach fixed point theorem as follows.

Theorem 2.2. *Let (X,d) be a compact metric space and $T : X \longrightarrow X$ be a single-valued mapping. If for all $x, y \in X$, $d(Tx,Ty) < d(x,y)$, then T has a unique fixed point in X.*

In 1968, Kannan [9] proved the following fixed point theorem.

Theorem 2.3. *Let (X,d) be a complete metric space and let $T : X \to X$ be a mapping such that there exists $K < \frac{1}{2}$ satisfying*

$$d(Tx,Ty) \leq K[d(x,Tx) + d(y,Ty)]$$

for all $x, y \in X$. Then, T has a unique fixed point $v \in X$, and for any $x \in X$, the sequence of iterates $\{T^n x\}$ converges to v and

$$d(T^{n+1}x, v) \leq K\left(\frac{K}{1-K}\right)^n d(x,Tx), \quad n = 0,1,2,\ldots$$

Kannan's fixed point theorem and some of its generalizations are discussed in [6, 18, 19]. In particular, we have the following theorem (see [18])

Theorem 2.4. *[18] Let (X,d) be a complete metric space and let $T : X \to X$ be a mapping with the following property:*

$$d(Tx,Ty) \leq Ad(x,Tx) + Bd(y,Ty) + Cd(x,y)$$

for all $x, y \in X$, where A, B, C are nonnegative numbers and satisfy $A + B + C < 1$. Then, T has a unique fixed point $v \in X$, and for any $x \in X$ the sequence of iterates $\{T^n x\}$ converges to v.

Recently, Jaroslaw [21] proved some fixed point theorems for Kannan-type mappings. One of the main results in [21] which is a version Kannan fixed point theorem is given hereunder.

Theorem 2.5. *[21] Let (X, d) be a compact metric space and let $T : X \to X$ be a continuous mapping. Suppose that*

$$d(Tx, Ty) < \frac{1}{2}[d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$, with $x \neq y$. Then, T has a unique fixed point $v \in X$ and for each $x \in X$, the sequence of iterates $\{T^n x\}$ converges to v.

The second key theorem studied by Jaroslaw [21] which is motivated by Ciric contraction is given as follows:

Theorem 2.6. *[21] Let (X, d) be a compact metric space and let $T : X \to X$ be a continuous mapping. Suppose that*

$$d(Tx, Ty) < Ad(x, Tx) + Bd(y, Ty) + Cd(x, y)$$

for all $x, y \in X$, $x \neq y$, where A, B, C are positive and satisfy $A + B + C = 1$. Then, T has a unique fixed point $v \in X$ and for each $x \in X$ the sequence of iterates $\{T^n x\}$ converges to v.

3. Main Results

In this section, fixed point results of F-Kannan-type and F-Ciric-type contractions on compact metric spaces are presented.

Theorem 3.1. *Let (X, d) be a compact metric space and let $T : X \to X$ be a mapping. If there exists $\tau > 0$ such that for all $x, y \in X$, $d(Tx, Ty) > 0$ implies*

$$\tau + [d(Tx, Ty)] < F\left[\frac{1}{2}(d(x, Tx) + d(y, Ty))\right] \tag{3.1}$$

for all $x, y \in X$. Then, T has a unique fixed point in X.

Proof. The function $g : X \to [0, +\infty)$ defined by $g(x) = d(x, Tx)$ is continuous. In view of compactness, there exists a point $v \in X$ such that $g(v) = \inf\{g(x) : x \in X\}$. If $v \neq Tv$, then

$$\tau + F[d(Tv, T^2v)] < F\left[\frac{1}{2}(d(v, Tv) + d(Tv, T^2v))\right] \tag{3.2}$$

From equation 3.2, we obtain

$$\begin{aligned} F[d(Tv, T^2v)] &< F\left[\frac{1}{2}(d(v, Tv) + d(Tv, T^2v))\right] - \tau \\ &< F\left[\frac{1}{2}(d(v, Tv) + d(Tv, T^2v))\right] \end{aligned}$$

By the monotonicity of F, we have

$$d(Tv, T^2v) < \frac{1}{2}(d(v, Tv) + d(Tv, T^2v)),$$

which yields

$$d(Tv, T^2v) < d(v, Tv). \tag{3.3}$$

Hence, $g(Tv) = d(Tv, T^2v) < d(v, Tv) = g(v)$, produces a contradiction. Therefore, $Tv = v$. Now, we will show that T has a unique fixed point. Let u, v be two distinct fixed points of T. Thus, $Tv = v \neq u = Tu$.

Hence, $d(Tv, Tu) = d(v, u) > 0$
Now,

$$\begin{aligned}\tau + F(d(v,u)) &= \tau + F(d(Tv,Tu)) \\ &< F\left[\frac{1}{2}(d(v,Tv) + d(u,Tu))\right] \\ &= F\left[\frac{1}{2}(d(v,v) + d(u,u))\right],\end{aligned}$$

implies that $F(d(v, u)) < F(0) - \tau < F(0)$. Consequently, $d(v, u) < 0$, leads to a contradiction. Thus, $u = v$. □

Theorem 3.2. *Let (X, d) be a compact metric space and let $T : X \to X$ be a mapping. If there exists $\tau > 0$ such that for all $x, y \in X$, $d(Tx, Ty) > 0$, implies*

$$\tau + F[d(Tx,Ty)] < F[Ad(x,Tx) + Bd(y,Ty) + Cd(x,y)]$$

for all $x, y \in X, Tx \neq Ty, A + B + C = 1$. Then, T has a unique fixed point.

Proof. The function $g : X \to [0, +\infty)$ defined by $g(x) = d(x, Tx)$ is continuous. In view of compactness, there exists a point $v \in X$ such that $g(v) = \inf\{g(x) : x \in X\}$. If $v \neq Tv$, then

$$\tau + F[d(Tv,T^2v)] < F[Ad(v,Tv) + Bd(Tv,T^2v) + Cd(v,Tv)]. \tag{3.4}$$

From equation 3.4, we obtain

$$\begin{aligned}F[d(Tv,T^2v)] &< F[Ad(v,Tv) + Bd(Tv,T^2v) + Cd(v,Tv)] - \tau \\ &< F[Ad(v,Tv) + Bd(Tv,T^2v) + Cd(v,Tv)].\end{aligned}$$

By the monotonicity of F, we have

$$\begin{aligned}d(Tv,T^2v) &< Ad(v,Tv) + Bd(Tv,T^2v) + Cd(v,Tv) \\ &= (A+C)d(v,Tv) + Bd(Tv,T^2v),\end{aligned}$$

from which it follows that

$$(1-B)d(Tv,T^2v) < (A+C)d(v,Tv).$$

This yields

$$\begin{aligned}d(Tv,T^2v) &< \frac{A+C}{1-B}d(v,Tv) = d(v,Tv) \\ &< d(v,Tv).\end{aligned}$$

Hence, $g(Tv) = d(Tv, T^2v) < d(v, Tv) = g(v)$, a contradiction. Therefore, $Tv = v$ Now, we will show that T has a unique fixed point. Let u, v be two distinct fixed points of T. Thus, $Tv = v \neq u = Tu$. Hence, $d(Tv, Tu) = d(v, u) > 0$.
Now,

$$\begin{aligned}\tau + F(d(v,u)) &= \tau + F(d(Tv,Tu)) \\ &< F[Ad(v,Tv) + Bd(u,Tu) + Cd(v,u)] \\ &= F[Cd(v,u)],\end{aligned}$$

implies

$$\begin{aligned} F(d(v,u)) &< F[Cd(v,u)] - \tau \\ &< F[Cd(v,u)] \\ &< F[d(v,u)], \end{aligned}$$

which is a contradiction. Hence, $u = v$. □

Corollary 3.3. *[21] Let (X, d) be a compact metric space and let $T : X \to X$ be a continuous mapping. If there exists $\alpha \in (0, 1)$ such that for all $x, y \in X$.*

$$d(Tx, Ty) < \alpha(d(x, Tx) + d(y, Ty)) \tag{3.5}$$

Then, T has a unique fixed point.

Proof. Define the mapping $F : (0, \infty) \to (0, \infty)$ by $F(t) = \ln(t)$, for all $t > 0$ and $\alpha = \frac{1}{2e^{\tau}}$, for some $\tau > 0$, then Corollary 3.3 coincides with 3.1. Hence, Theorem 3.1 can be applied to find a unique $v \in X$ such that $Tv = v$. □

Corollary 3.4. *Let (X, d) be a compact metric space and let $T : X \to X$ be a continuous mapping. If there exists $\alpha \in (0, 1)$ such that for all $x, y \in X$.*

$$d(Tx, Ty) < Ad(x, Tx) + Bd(y, Ty) + Cd(x, y), \tag{3.6}$$

then T has a unique fixed point.

Proof. The idea of the Proof is the same as that of Corollary 3.3. □

Corollary 3.5. *(see [5, Theorem 2.2]) Let (X, d) be a compact metric space and let $T : X \to X$ be a mapping. Suppose that*

$$d(Tx, Ty) < \frac{1}{2}[d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$, with $x \neq y$. Then, T has a unique fixed point $v \in X$.

Proof. Put $A = B = \frac{1}{2}, C = 0$ in Corollary 3.4. □

Corollary 3.6. *Let (X, d) be a compact metric space and let $T : X \to X$ be a mapping. Suppose that*

$$d(Tx, Ty) < \frac{1}{3}[d(x, Tx) + d(y, Ty) + d(x, y)]$$

for all $x, y \in X$, with $x \neq y$. Then, T has a unique fixed point $v \in X$.

Proof. Put $A = B = C = \frac{1}{3}$, in Corollary 3.4. □

Corollary 3.7. *[4] Let (X, d) be a compact metric space and let $T : X \to X$ be a mapping. Suppose that $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$, with $x \neq y$. Then, T has a unique fixed point $v \in X$.*

Proof. Put $A = B = 0, C = 1$ in Corollary 3.4. □

4. Conclusion

The study of fixed point results of Kannan-type F-contractions on compact spaces is a new and promising area of research. The study has the potential to bring up new and improved fixed point theorems and algorithms for solving nonlinear equations. The main contribution of this paper is the introduction of the concept of Kannan-type F-contraction. This new type of contraction has been seen to be more general than some existing contractions, such as Kannan contractions, Edelstein contraction, F-contractions and many others in the corresponding literature. Another contribution of this article is the use of a compact metric space. Compact metric spaces are a very important class of metric spaces, and many fixed point theorems have been proven in this setting. one of the applicable properties of compact metric space in this work is that it allowed us to establish the existence and uniqueness of fixed point theorems of Kannan-type F-contractions without using the Picard iterative sequence.

Conflict of Interest

The authors have no conflict of interest regarding the publication of this article.

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