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# Convex Trigonometric Functions and Their Applications

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**Abstract**

In this paper, we first introduce a new kind of trigonometric functions called convex trigonometric functions. These functions are natural generalization of the ordinary trigonometric and hyperbolic functions. Then investigate their properties and their connections with the trigonometric Jacobian elliptic functions. Also we find the period of circular trigonometric function and power series of circular and hyperbolic trigonometric functions, using Maple package. Finally, we present some applications of these functions for obtaining the closed from of some special integrals and finding the solution of the differential equation of simple harmonic pendulum.

*Keywords:* trigonometric and hyperbolic functions, convex trigonometric functions, Jacobian elliptic functions, Eigenvalues, Eigenfunctions.
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## 1. Introduction

Despite the tremendous advancement of science in different areas there still remain many important questions, that mathematicians can offer only relatively unsatisfactory answers. One of the main reasons for this is the absence of suitable function forms. A long time has past since the proper function forms arising from algebra were the only sufficient and the powerful tools. The so-called elementary functions do not provide satisfactory increasing requirements. Therefore, several great mathematicians for example, Euler, Legendre, Gauss, Abel and Jacobi, have studied special functions, some of which have been of decisive importance for the advancement of other branches of sciences such as physics, mechanics and engineering.
To begin with, we quickly review the ordinary Sine and Cosine functions. Indeed, from geometrical point view, the trigonometric functions are defined by parameterizing the unit circle $x^2 + y^2 = 1$. Similarly, the

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hyperbolic functions are defined by parameterizing the hyperbola $x^2 - y^2 = 1$. One can also define the sine and cosine functions as the inverse of the following integrals, respectively

$$u = \int_0^s \frac{dx}{\sqrt{1-x^2}}, \qquad u = \int_c^1 \frac{dx}{\sqrt{1-x^2}},$$

by $\sin(u) := s$ and $\cos(u) := c$. A suitable change of variables in the defining integrals leads to the fundamental identity

$$\sin^2(u) + \cos^2(u) = 1.$$

In the same way, we define the hyperbolic functions as the inverse of the integrals and denote them

$$u = \int_0^s \frac{dx}{\sqrt{x^2+1}}, \qquad u = \int_1^c \frac{dx}{\sqrt{x^2+1}},$$

by $\sinh(u) = s$ and $\cosh(u) = c$. As in the trigonometric case, we have the following well-known identity (see [4, 5]):

$$\cosh^2(u) - \sinh^2(u) = 1.$$

In analogy with the above definitions, for any $k$ with $0 < k < l$, the Jacobian elliptic functions are defined by the following integral in which

$$\mathrm{sn}(u) = \sin(\theta), \quad \mathrm{cn}(u) = \cos(\theta), \quad \mathrm{dn}(u) = \sqrt{1 - k^2\sin^2(\theta)}.$$

These functions are connected via the relations $\mathrm{sn}^2(u) + \mathrm{cn}^2(u) = 1$ and $k^2\mathrm{sn}^2(u) + \mathrm{dn}^2(u) = 1$. It is interesting to note that the Jacobian elliptic functions are doubly periodic, namely they have one real period and another complex period (see [1, 6]).

## 2. Convex Trigonometric Functions

In this section, we define convex trigonometric functions of circular and hyperbolic type as the counterparts to the trigonometric and hyperbolic functions, respectively.

### *2.1. Convex Trigonometric Functions of Circular and Hyperbolic Type*

The counterparts to the trigonometric functions will be defined as inverses of integrals. Let

$$u = \int_0^{\tan(\theta)} \frac{dx}{\sqrt{1+x^4}}, \tag{2.1}$$

then we define the convex trigonometric functions of circular type as follows

$$\sin_4(u) = \frac{\sin(\theta)}{\sqrt[4]{\sin^4(\theta)+\cos^4(\theta)}}, \quad \cos_4(u) = \frac{\cos(\theta)}{\sqrt[4]{\sin^4(\theta)+\cos^4(\theta)}}, \quad \tan_4(u) = \tan(\theta). \tag{2.2}$$

After simple calculations, one arrives at the formula

$$\sin_4^4(u) + \cos_4^4(u) = 1, \quad \tan_4(u) = \frac{\sin_4(u)}{\cos_4(u)}.$$

The above relations are the counterparts to $\sin^2(u) + \cos^2(u) = 1, \tan(u) = \frac{\sin(u)}{\cos(u)}$.
The convex trigonometric functions of hyperbolic type are defined by considering the following integral as follows

$$u = \int_0^{\tan(\theta)} \frac{dx}{\sqrt{1-x^4}} \tag{2.3}$$

as follows

$$\sinh_4(u) = \frac{\sin(\theta)}{\sqrt[4]{\cos^4(\theta) - \sin^4(\theta)}}, \quad \cosh_4(u) = \frac{\cos(\theta)}{\sqrt[4]{\cos^4(\theta) - \sin^4(\theta)}}, \quad \tanh_4(u) = \tan(\theta). \tag{2.4}$$

These functions satisfy the following identities

$$\cosh_4^4(u) - \sinh_4^4(u) = 1, \quad \tanh_4(u) = \frac{\sinh_4(u)}{\cosh_4(u)}.$$

The relation (2.1) can be rewritten as a formal power series. Applying the Newton binomial expansion,

$$(1 + x^4)^{-\frac{1}{2}} = 1 - \frac{2x^4}{4 \times 1!} + \frac{2(2+4)x^8}{4^2 \times 2!} - \frac{2(2+4)(2+2\times 4)x^{16}}{4^3 \times 3!} + \cdots$$

we get integrating the above series term by term, we obtain

$$u = t - \frac{2t^5}{5 \times 4 \times 1!} + \frac{2(2+4)t^9}{9 \times 4^2 \times 2!} - \frac{2(2+4)(2+2\times 4)t^{17}}{17 \times 4^3 \times 3!} + \cdots,$$

in which $t = \tan(\theta)$. We can find the power series for (2.2) in a similar way.

*2.2. Convex Trigonometric Functions Properties*

Considering relations (2.1)-(2.4) it can be easily seen that

$$\sin_4(0) = 0, \quad \cos_4(0) = 1, \quad \sinh_4(0) = 0, \quad \cosh_4(0) = 1.$$

We have also the following reduction formulas

$$\begin{aligned} \sin_4(-u) &= -\sin_4(u), \qquad \cos_4(-u) = \cos_4(u) \\ \sinh_4(-u) &= -\sinh_4(u), \quad \cosh_4(-u) = \cosh_4(u). \end{aligned}$$

Now we show that $\sin_4(u)$ and $\cos_4(u)$ are periodic functions. To do this, we define the period of these functions as follows

$$\pi_4 = 2\int_0^\infty \frac{dx}{\sqrt{1+x^4}} \tag{2.5}$$

It can be easily seen that the above integral is equal to $\pi^{\frac{3}{2}}/\Gamma\left(\frac{3}{4}\right)^2$. Thus, we obtain the approximating value of 3.70814... for it. In the following we show that the number $\pi_4$ plays the same role for the convex trigonometric functions as $\pi$ does for the classic trigonometric functions. The following theorem shows that the number $2\pi_4$ is a period for the functions $\sin_4(u)$ and $\cos_4(u)$.

**Theorem 2.1.** *The functions* $\sin_4(u)$ *and* $\cos_4(u)$ *satisfy the following translation formulas:*

$$\sin_4\left(u + \frac{\pi_4}{2}\right) = -\cos_4(u), \quad \cos_4\left(u + \frac{\pi_4}{2}\right) = \sin_4(u),$$

*furthermore, they are periodic functions with period* $2\pi_4$.

*Proof.* Considering relations (2.3) and (2.5), we have

$$v = u + \frac{\pi_4}{2} = \int_0^{\tan(\theta)} \frac{dx}{\sqrt{1+x^4}} + \int_0^\infty \frac{dx}{\sqrt{1+x^4}},$$

by changing variables $x = \tan(\varphi)$, we have

$$v = \int_0^\theta \frac{\left(1+\tan^2(\varphi)\right)d\varphi}{\sqrt{1+\tan^4(\varphi)}} + \int_0^{\frac{\pi}{2}} \frac{\left(1+\tan^2(\varphi)\right)d\varphi}{\sqrt{1+\tan^4(\varphi)}},$$

now, by applying the change of variable $\psi = \varphi + \frac{\pi}{2}$ to the first integral, we obtain

$$\begin{aligned} v &= \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\theta} \frac{\big(1+\cot^2(\psi)\big)d\psi}{\sqrt{1+\cot^4(\psi)}} + \int_0^{\frac{\pi}{2}} \frac{\big(1+\tan^2(\psi)\big)d\psi}{\sqrt{1+\tan^4(\psi)}} \\ &= \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\theta} \frac{\big(1+\tan^2(\psi)\big)d\psi}{\sqrt{1+\tan^4(\psi)}} + \int_0^{\frac{\pi}{2}} \frac{\big(1+\tan^2(\psi)\big)d\psi}{\sqrt{1+\tan^4(\psi)}} \\ &= \int_0^{\theta+\frac{\pi}{2}} \frac{\big(1+\tan^2(\psi)\big)d\psi}{\sqrt{1+\tan^4(\psi)}} \\ &= \int_0^{\tan(\theta+\frac{\pi}{2})} \frac{dx}{\sqrt{1+x^4}}. \end{aligned}$$

Finally, by definition (2.2), we have

$$\sin_4\Big(u+\frac{\pi_4}{2}\Big) = \frac{\sin\big(\frac{\pi}{2}+\theta\big)}{\sqrt{\sin^4\big(\frac{\pi}{2}+\theta\big)+\cos^4\big(\frac{\pi}{2}+\theta\big)}} = -\cos_4(u).$$

A similar device works for the other function as well. So we obtain the translation formula

$$\cos_4\Big(u+\frac{\pi_4}{2}\Big) = \sin_4(u).$$

□

Thus we get the following formulas

$$\sin_4(u+\pi_4) = -\sin_4(u), \qquad \sin_4(u+2\pi_4) = \sin_4(u).$$

Hence $\sin_4(u)$ has the period equal to $2\pi_4$. The function $\cos_4(u)$ can be treated in the same way. Using Maple and plotting parameterized curve, the diagram for the convex trigonometric functions can be plotted on arbitrary interval.

## 3. The Addition Formulas

The addition formulas for the ordinary trigonometric and hyperbolic functions are a direct consequence of the fundamental property $e^{z_1+z_2} = e^{z_1}e^{z_2}$ of the exponential function. For elliptic functions the situation is more involved.

**Theorem 3.1** ([2]). *If*

$$\begin{aligned} d(u,v) = {} & \sin_4^2(u)\sin_4^2(v) + \cos_4^2(u)\cos_4^2(v) + 2\sin_4(u)\cos_4^3(u)\sin_4^3(v)\cos_4(v) \\ & + 2\sin_4^3(u)\cos_4(u)\sin_4(v)\cos_4^3(v), \end{aligned}$$

*then*

$$\begin{aligned} \cos_4^2(u+v) &= \big(\cos_4^2(u)\cos_4^2(v) - \sin_4^2(u)\sin_4^2(v)\big)^2/d(u,v), \\ \sin_4^2(u+v) &= \big(\sin_4^2(u)\cos_4^2(u) + \sin_4^2(v)\cos_4^2(v)\big)^2/d(u,v), \\ \sin_4(u+v).\cos_4(u+v) &= \big(\cos_4^2(u)\cos_4^2(v) - \sin_4^2(u)\sin_4^2(v)\big)\times \\ & \quad \times \big(\sin_4^2(u)\cos_4^2(u) + \sin_4^2(v)\cos_4^2(v)\big)/d(u,v). \end{aligned}$$

*Additionally,*

$$\cos_4^2(iz) = \cos_4^2(z), \qquad \sin_4^2(iz) = -\sin_4^2(z)$$

*and*

$$\sin_4(iz)\cos_4(iz) = i\sin_4(z)\cos_4(z).$$

## 4. A Geometric Interpretation

In analogy with the definition of trigonometric function by parameterizing the unit circle $x^2 + y^2 = 1$ (and of hyperbolic function by parameterizing the hyperbola $x^2 - y^2 = 1$), the hyperchromatic function can be interpreted as a parameterized curve $x^4 + y^4 = 1$ (or the curve $x^4 - y^4 = 1$). To this end, consider the curve

$$x^4 + y^4 = 1 \qquad (\text{or } x^4 - y^4 = 1) \tag{4.1}$$

where we have restricted ourselves to the first quadrant of the plane. The curve is parameterized by

$$x = \cos_4(u), \quad y = \sin_4(u) \qquad \big(\text{or } x = \cosh_4(u), \; y = \sinh_4(u)\big) \quad (0 \le u \le \frac{\pi_4}{2}).$$

The parameter $u$ has a very simple geometric interpretation. Let area$(S_u)$ denote the area of the sector bounded by the $x$-axis, the radius from the origin to the point $(\cos_4(u), \sin_4(u))$(or $(\cosh_4(u), \sinh_4(u))$, and the curve $x^4 + y^4 = 1$ (or $x^4 - y^4 = 1$). Now we have the well-known interpretation of $u$ as two times the area$(S_u)$. Integrating, one can obtain

$$u = \int_0^\theta \frac{dt}{\sqrt[4]{\cos_4^4(t) + \sin_4^4(t)}}, \qquad \left(\text{or} \quad u = \int_0^\theta \frac{dt}{\sqrt[4]{\cos_4^4(t) - \sin_4^4(t)}}\right).$$

Here is an inequality for the generalized sine:

**Theorem 4.1.** *Let $u \in (0, \frac{\pi_4}{2})$, we have*

$$\frac{2}{\pi_4} \le \frac{\sin_4(u)}{u} \le 1$$

*Proof.* By changing variables $t = \sin_4(u)s$, we obtain

$$u = \int_0^{\sin_4(u)} (1 - t^4)^{-1/4} dt = \sin_4(u) \int_0^1 \Big(1 - (\sin_4(u))^4 s^4\Big)^{-1/4} ds.$$

Now, according to the following inequality,

$$1 \le \int_0^1 \Big(1 - (\sin_4(u))^4 s^4\Big)^{-1/4} ds \le \frac{\pi_4}{2},$$

the proof is complete. □

## 5. Derivation and Integration Formulas

In this section, we discuss the derivation of convex trigonometric functions. Using relations (2.1) and (2.2), we have

$$\frac{d \sin_4(u)}{du} = \frac{d \sin_4(u)}{d\theta} \times \frac{1}{\frac{du}{d\theta}} = \cos_4^3(u).$$

Thus, we have the following theorem:

**Theorem 5.1.**

$$\frac{d \sin_4(u)}{du} = \cos_4^3(u), \qquad \frac{d \cos_4(u)}{du} = -\sin_4^3(u), \qquad \frac{d \tan_4(u)}{du} = \sqrt{1 + \tan_4^4(u)},$$
$$\frac{d \sinh_4(u)}{du} = \cosh_4^3(u), \qquad \frac{d \cosh_4(u)}{du} = \sinh_4^3(u), \qquad \frac{d \tanh_4(u)}{du} = \sqrt{1 - \tanh_4^4(u)}.$$

Now, we define the inverse function of $\sin_4(u)$ as follows

$$\sin_4^{-1} : [-1,1] \to \left[-\frac{\pi_4}{2}, \frac{\pi_4}{2}\right]$$
$$\sin_4^{-1}(x) = u \Leftrightarrow \sin_4(u) = x.$$

The inverse function of $\tan_4(u)$ is defined from $(-\infty,\infty)$ to $(-\frac{\pi_4}{2}, \frac{\pi_4}{2})$. Similarly, we can define the inverse functions of $\cos_4(u), \tan_4(u), \sinh_4(u), \cosh_4(u)$ and $\tanh_4(u)$ considering their domains and their ranges. Using $x = \sinh_4(u)$, we get $\frac{dx}{du} = \cos_4^3(u)$, where this relation can be written as

$$\frac{du}{dx} = \frac{1}{\cos_4^3(u)} = \frac{1}{\sqrt[4]{\left(1-\sin_4^4(u)\right)^3}} = \frac{1}{\sqrt[4]{\left(1-x^4\right)^3}}.$$

Therefore, we get

$$\int \frac{1}{\sqrt[4]{\left(1-x^4\right)^3}} = u + c = \sin_4^{-1}(x) + c.$$

Thus, we have the following theorem:

**Theorem 5.2.**

$$\int \frac{dx}{\sqrt[4]{\left(1-x^4\right)^3}} = \sin_4^{-1}(x) + c, \qquad \int \frac{-dx}{\sqrt[4]{\left(1-x^4\right)^3}} = \cos_4^{-1}(x) + c,$$
$$\int \frac{dx}{\sqrt{1+x^4}} = \tan_4^{-1}(x) + c, \qquad \int \frac{dx}{\sqrt[4]{\left(1+x^4\right)^3}} = \sinh_4^{-1}(x) + c,$$
$$\int \frac{dx}{\sqrt[4]{\left(x^4-1\right)^3}} = \cosh_4^{-1}(x) + c, \qquad \int \frac{dx}{\sqrt{1-x^4}} = \tanh_4^{-1}(x) + c.$$

## 6. Connection Between the Convex Trigonometric Functions and Elliptic Functions

Using relation (2.1), we have

$$u = \int_0^\theta \frac{dt}{\sqrt{\cos^4(t) + \sin^4(t)}},$$

or

$$u = \int_0^\theta \frac{dt}{\sqrt{1 - \frac{1}{2}\sin^2(2t)}},$$

this relation is an elliptic integral with $k = \frac{1}{\sqrt{2}}$. Now, by changing variable $\varphi = 2t$, we get

$$2u = \int_0^{2\theta} \frac{d\varphi}{\sqrt{1 - \frac{1}{2}\sin^2(\varphi)}}, \tag{6.1}$$

considering the definition of elliptic functions, we obtain

$$\mathrm{sn}(2u) = \sin(2\theta), \qquad \mathrm{cn}(2u) = \cos(2\theta), \qquad \mathrm{dn}(2u) = \sqrt{1 - \frac{1}{2}\sin^2(2\theta)}.$$

So by relations (2.2) and (6.1) and using the following identities, we have

$$\sin^2(\theta) = \frac{1 - \sqrt{1-\sin^2(2\theta)}}{2}, \qquad \cos^2(\theta) = \frac{1 + \sqrt{1-\sin^2(2\theta)}}{2}, \qquad (0 \le \theta \le \frac{\pi}{2}),$$

also we have

$$\sin_4(u) = \frac{\sqrt{2}\mathrm{sn}(u)\mathrm{dn}(u)}{\sqrt{1+\mathrm{cn}^4(u)}}, \quad \cos_4(u) = \frac{\sqrt{2}\mathrm{cn}(u)}{\sqrt{1+\mathrm{cn}^4(u)}}, \quad \left(0 \le \theta \le 0.927037\right)$$

It is worth noting that the bound 0.927037... is obtained by computing $u$ in (6.1) for $\theta = \frac{\pi}{2}$.

## 7. Power Series of Convex Trigonometric Functions

As we know, calculating the value of trigonometric function at a point can be done by formal power series. So we intend to find the corresponding power series for convex trigonometric functions. To begin, we observe that the functions $x(u) = \sin_4(u)$ and $y(u) = \cos_4(u)$ satisfy in the following system of differential equations:

$$\begin{cases} x'(u) = y^3(u), \\ y'(u) = -x^3(u), \end{cases} \qquad \big(x(0) = 0,\ y(0) = 1\big)$$

Now, using the differential equation package of Maple software we can obtain the following power series for $\sin_4(u)$ and $\cos_4(u)$:

```
>Order:=25;
>sys:=diff(x(u),u)=-y(u)^3,diff(y(u),u)=x(u)^3;
>ans:=dsolve(sys union x(0)=1,y(0)=0,x(u),y(u), type=series);
```

$$\sin_4(u) = u - \frac{3}{20}u^5 + \frac{19}{480}u^9 - \frac{469}{41600}u^{13} + \frac{189611}{56576000}u^{17} - \cdots,$$
$$\cos_4(u) = 1 - \frac{1}{4}u^4 + \frac{9}{160}u^8 - \frac{149}{9600}u^{12} + \frac{15147}{3328000}u^{16} - \cdots,$$

by dividing $\sin_4(u)$ to $\cos_4(u)$, we obtain the following power series for $\tan_4(u)$,

$$\tan_4(u) = u + \frac{1}{10}u^5 + \frac{1}{120}u^9 + \frac{11}{15600}u^{13} + \frac{211}{3536000}u^{17} + \cdots.$$

In the same way, one can obtain

$$\sinh_4(u) = u + \frac{3}{20}u^5 + \frac{19}{480}u^9 + \frac{469}{41600}u^{13} + \frac{189611}{56576000}u^{17} + \cdots,$$
$$\cosh_4(u) = 1 + \frac{1}{4}u^4 + \frac{9}{160}u^8 + \frac{149}{9600}u^{12} + \frac{15147}{3328000}u^{16} + \cdots,$$
$$\tanh_4(u) = u - \frac{1}{10}u^5 + \frac{1}{120}u^9 - \frac{11}{15600}u^{13} + \frac{211}{3536000}u^{17} - \cdots.$$

Now, we extend our functions to the complex plane. For example, define

$$\sin_4(z) = z - \frac{3}{20}z^5 + \frac{19}{480}z^9 - \frac{469}{41600}z^{13} + \frac{189611}{56576000}z^{17} - \cdots$$

for $z = x + iy$. The series converges only for $|z| < \pi_4/2$. Hence analytic continuation must be used for values outside the disc of convergence.

## 8. Applications

Here we present two important applications of the convex trigonometric functions in calculating some indefinite integrals and solving the differential equation of simple harmonic pendulum.

### *8.1. Calculating Some Integral Using Convex Trigonometric Functions*

Here we investigate two integrals, using the convex trigonometric functions.

**Example 8.1.** Evaluate the value of integral $I = \int \cos_4^2(u)du$.

We have

$$I=\int\frac{d\sin_4(u)}{\sqrt[4]{1-\sin_4^4(u)}}.$$

Now, by changing variables $x=\sin_4(u)$, we get

$$I=\int\frac{dx}{\sqrt[4]{1-x^4}}.$$

Using Maple (or table of integrals), we obtain

$$\int\frac{dt}{1+t^4}=\frac{1}{4\sqrt{2}}\ln\Big(\frac{1+t\sqrt{2}+t^2}{1-t\sqrt{2}+t^2}\Big)+\frac{1}{2\sqrt{2}}\tan^{-1}\Big(\frac{1-\sqrt{2}t}{1-t^2}\Big)+c$$

now, letting $t=\tan_4(u)$, calculating the $\tan_4(u)$ with respect to $\sin_4(u)$ and putting $x=\sin_4(u)$, we obtain

$$t=\tan_4(u)=\frac{\sin_4(u)}{\cos_4(u)}=\frac{x}{\sqrt[4]{1-x^4}},$$

so, we have

$$\int\frac{dt}{1+t^4}=\int\frac{dx}{\sqrt[4]{1-x^4}}.$$

Thus, considering, $x=\sin_4(u)$, we get

$$\int\cos_4^2(u)du=\frac{1}{2\sqrt{2}}\ln\Big[1+2\sqrt{2}x\sqrt[4]{(1-x^4)^3}+4x^2\sqrt{1-x^4}+2\sqrt{2}x^3\sqrt[4]{1-x^4}\Big]+$$
$$+\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}x\sqrt[4]{1-x^4}}{\sqrt{1-x^4}-x^2}\right)+c.$$

**Example 8.2.** Evaluate the value of integral $I=\int\frac{d\theta}{\cos(\theta)}$.

We have

$$I=2\int\frac{d\frac{\theta}{2}}{\sqrt{\cos^2(\frac{\theta}{2})-\sin^2(\frac{\theta}{2})}}$$

if we multiply the term in radical by $\cos^2(\frac{\theta}{2})+\sin^2(\frac{\theta}{2})=1$, then we obtain

$$I=2\int\frac{d\frac{\theta}{2}}{\sqrt{\cos^4(\frac{\theta}{2})-\sin^4(\frac{\theta}{2})}}=2\int\frac{d\tan\big(\frac{\theta}{2}\big)}{\sqrt{1-\tan^2(\frac{\theta}{2})}}$$

now, considering Theorem 5.1, we get

$$I=2\tanh_4^{-1}\Big(\tan\Big(\frac{\theta}{2}\Big)\Big)+c.$$

## 9. The Pendulum Equation

Consider the simple harmonic pendulum consisting of a mass $m$ and a solid rod (we neglect the weight) with length $\ell$. We assume that the rod is hanging on from the origin. The differential equation of the motion of mass $m$ is as follows.

$$\frac{d^2\theta}{dt^2}+\frac{g}{\ell}\sin(\theta)=0,\qquad\Big(\theta(t_0)=\frac{\pi}{2},\ \theta'(t_0)=0\Big).$$

Multiplying both sides by $\theta'$ and integrating, we obtain

$$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 - \frac{g}{\ell}\cos(\theta) = c_1.$$

Applying the initial condition $\theta'(t_0) = 0$, we conclude $c_1 = 0$. Then

$$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 - \frac{g}{\ell}\cos(\theta) = 0, \qquad \left(\text{or} \quad dt = \sqrt{\frac{\ell}{2g}}\frac{d\theta}{\sqrt{\cos(\theta)}}\right). \tag{9.1}$$

Now, by integrating from (9.1) and putting $\theta(t_0) = \frac{\pi}{2}$, we obtain

$$\sqrt{\frac{2g}{\ell}}t + c_2 = 2\tanh_4^{-1}\left(\tan\left(\frac{\theta}{2}\right)\right) = \tanh_4^{-1}(1) - \frac{\pi}{2}\sqrt{\frac{2g}{\ell}},$$

where

$$\tanh_4^{-1}(1) = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\cos(\theta)}}.$$

Finally, we calculate $\theta$ versus $t$ as follows

$$\theta = 2\tan^{-1}\left[\tanh_4\left(\frac{1}{2}\sqrt{\frac{2g}{\ell}}t - \frac{\pi}{4}\sqrt{\frac{2g}{\ell}} + \tanh_4^{-1}(1)\right)\right].$$

## 10. Eigenvalues and Eigenfunctions of a Dirichlet Problem

Consider the Dirichlet problem with the following boundary values:

$$\begin{cases} \left((u')^3\right)' + \lambda u^3 = 0, \\ u(0) = u(\pi_4) = 0, \end{cases} \qquad x \in (0, \pi_4)$$

In article [3], it is shown that the eigenvalues of this problem are as follows:

$$\lambda_1 = 1, \ \lambda_2 = 2^4, \ \lambda_3 = 3^4, \cdots, \lambda_n = n^4, \cdots,$$

and their corresponding eigenfunctions are:

$$u_1(x) = \sin_4(x), \ u_2(x) = \sin_4(2x), \cdots, u_n(x) = \sin_4(nx), \cdots.$$

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