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Fixed Point Theorems in Dislocated Quasi Metric and Dislocated Quasi b-Metric Spaces

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Abstract

In this paper, the concept of generalized contraction mapping has been used in investigating fixed point theorems. A fixed point theorem of integral type will be establish in the setting of dislocated quasi metric spaces using some new type of rational contraction conditions. Moreover, a fixed point theorem will be studied in the context of dislocated quasi b-metric spaces using the idea of generalized new contraction conditions. Several corollaries may be deduced from our established results, which generalize and extend several results of the literature. Also , our investigated results may be easily used in solving equations in non-linear analysis too.

Keywords: Dislocated quasi-metric spaces, dislocated quasi b-metric spaces, self-mapping, Cauchy sequence, fixed point.

1. Introduction

The notion of metric space, introduced by Frechet in 1906, is one of the useful topic not only in mathematics but also in several quantitative sciences. Due to its importance and application potential, this notion has been extended, improved and generalized in many ways. An incomplete list of such attempts are following: symmetric space, b-metric space, partial metric space, quasi metric space, fuzzy metric space, dislocated metric space, dislocated quasi-metric space, right and left dislocated metric spaces, dislocated quasi b-metric spaces etc.

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The increasing applications of fixed point theorems of metric spaces and their generalizations in various branches of Engineering and other sciences attracted several researchers to work on them in recent past. For instance, the generalization of well-known Banach contraction principle of metric space to the dislocated metric space proved by Hitzler and Seda [6] play a key role in topology, logic programming semantics and electronic engineering etc. Zeyada et al. [12] initiated the concept of dislocated quasi-metric space and generalized the result of Hitzler and Seda [6] in dislocated quasi-metric space. Results on fixed point in dislocated quasi-metric space are followed by Rahman and Sarwar [7, 8] etc.

The notion of b-metric space was introduced by Czerwik [2] in connection with some problems concerning with the convergence of non-measurable functions with respect to their measure. Fixed point theorems regarding b-metric spaces was obtained in [3]. Recently, Rahman and Sarwar [11] further generalized the concept of b-metric space and initiated the notion of dislocated quasi b-metric space. Rahman and Sarwar [11] open a new way to the researchers to work in the field of metric fixed point theory. Fixed point theorems regarding dislocated quasi b-metric spaces was obtained in [4, 9].

In 2002, Branciari [10] obtained a fixed point theorem for a single self-mapping satisfying an analogous of Banach contraction principle for integral type inequality in metric space. Recently, Rahman and Sarwar [1] studied some fixed point theorems of integral type in dislocated quasi-metric space.

The purpose of this article is to prove some fixed point results in dislocated quasi and dislocated quasi b-metric spaces. In the context of dislocated metric spaces, we will prove a theorem of integral type for some rational type generalized contractions. Also, a theorem for generalized rational contractions will be obtained in dislocated quasi b-metric spaces.

2. Preliminaries

Throughout this paper $\mathbb{R}^+$ will represent the set of non negative real numbers.

Definition 2.1. [6]. Let X be a non-empty set and let $d : X \times X \to \mathbb{R}^+$ be a distance function satisfying the conditions

d_1) $d(x, x) = 0$;
d_2) $d(x, y) = d(y, x) = 0$ implies that $x = y$;
d_3) $d(x, y) = d(y, x)$;
d_4) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

If d satisfy the conditions from d_1 to d_4 then it is called metric on X, if d satisfy conditions d_2 to d_4 then it is called dislocated metric (d-metric) on X and if d satisfy conditions d_2 and d_4 only then it is called dislocated quasi-metric(dq-metric) on X.

Clearly every metric space is a dislocated metric space and dislocated quasi-metric space. Also every dislocated metric space is dislocated quasi-metric space but the converse is not necessarily true as clear form the following example.

Example 2.2. Let X=$\mathbb{R}^+$. Define the distance function $d : X \times X \to \mathbb{R}^+$ by

$$d(x, y) = |x| \quad \text{for all } x, y \in X.$$

Clearly X is dislocated quasi-metric space but not a metric nor dislocated metric space.
In our main work we will use the definition of dq-convergent sequence, Cauchy sequence, contraction and completeness in dislocated quasi-metric space which can be found in [12].

Now we state the following simple but important lemma and theorem without proof in the context of dislocated quasi-metric space.

Lemma 2.3. *[12]. Limit in dislocated quasi-metric space is unique.*

Theorem 2.4. *[12]. Let (X, d) be a complete dislocated quasi-metric space and $T : X \to X$ be contraction. Then T has a unique fixed point.*

We need the following definitions which may be found in [11].

Definition 2.5. Let X be a non-empty set and $k \geq 1$ be a real number then a mapping $d : X \times X \to [0, \infty)$ is called dislocated quasi b-metric if $\forall\, x, y, z \in X$
(d_1) $d(x, y) = d(y, x) = 0$ implies that $x = y$;
(d_2) $d(x, y) \leq k[d(x, z) + d(z, y)]$.
The pair (X, d) is called dislocated quasi b-metric space or shortly (dq b-metric) space.

Remark 2.6. In the definition of dislocated quasi b-metric space if $k = 1$ then it becomes (usual) dislocated quasi-metric space. Therefore every dislocated quasi metric space is dislocated quasi b-metric space and every b-metric space is dislocated quasi b-metric space with same coefficient k and zero self distance. However, the converse is not true as clear from the following example.

Example 2.7. Let $X = \mathbb{R}$ and suppose

$$d(x, y) = |2x - y|^2 + |2x + y|^2.$$

Then (X, d) is a dislocated quasi b-metric space with the coefficient $k = 2$. But it is not dislocated quasi-metric space nor b-metric space.

Remark 2.8. Like dislocated quasi-metric space in dislocated quasi b-metric space the distance between similar points need not to be zero necessarily as clear from the above example.

Definition 2.9. A sequence $\{x_n\}$ is called dq $b-$convergent in (X, d) if for $n \in N$ we have $\lim\limits_{n\to\infty} d(x_n, x) = 0$. Then x is called the dq-b-limit of the sequence $\{x_n\}$.

Definition 2.10. A sequence $\{x_n\}$ in dq b-metric space (X, d) is called Cauchy sequence if for $\epsilon > 0$ there exists $n_0 \in N$, such that for $m, n \geq n_0$ we have $d(x_m, x_n) < \epsilon$ (OR) $\lim\limits_{m,n\to\infty} d(x_m, x_n) = 0$.

Definition 2.11. A dq b-metric space (X, d) is said to be complete if every Cauchy sequence in X converges to a point of X.

Definition 2.12. Let (X, d_1) and (Y, d_2) be two dq b-metric spaces. A mapping $T : X \to Y$ is said to be continuous if for each $\{x_n\}$ which is dq-b convergent to x_0 in X, the sequence $\{Tx_n\}$ is dq b convergent to Tx_0 in Y.

The following well-known results can be seen in [11].

Lemma 2.13. *Limit of a convergent sequence in dislocated quasi b-metric space is unique.*

Lemma 2.14. *Let (X, d) be a dislocated quasi b-metric space and $\{x_n\}$ be a sequence in dq-b-metric space such that*

$$d(x_n, x_{n+1}) \leq \alpha \cdot d(x_{n-1}, x_n)$$

for $n = 1, 2, 3, ...$ and $0 \leq \alpha \cdot k < 1, \alpha \in [0, 1)$ and k is defined in dq b-metric space. Then $\{x_n\}$ is a Cauchy sequence in X.

Theorem 2.15. *Let (X, d) be a complete dislocated quasi b-metric space. Let $T : X \to X$ be a continuous contraction with $\alpha \in [0, 1)$ and $0 \leq k\alpha < 1$ where $k \geq 1$. Then T has a unique fixed point in X.*

In [1], the following definitions may be found which may be used in main results.

Theorem 2.16. *Let (X,d) be a complete metric space and $T : X \to X$ be a continues self mapping satisfying the following condition*

$$d(Tx,Ty) \le \alpha \cdot d(x,y) + \beta \cdot \frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}$$

$\forall\ x,y \in X$, *$\alpha,\beta > 0$, and $\alpha+\beta<1$. Then T has a unique fixed point in X.*

In [1], the following definitions may be found which may be used in main results.

Definition 2.17. Let X be a non-empty set. Let $d : X \times X \to \mathbb{R}^+$ be a function satisfying the conditions for all $x,y,z \in X$,

$d_1)$ $\int_0^{d(x,x)} \rho(t)dt = 0$;

$d_2)$ $\int_0^{d(x,y)} \rho(t)dt = \int_0^{d(y,x)} \rho(t)dt = 0 \quad \Rightarrow \quad x = y$;

$d_3)$ $\int_0^{d(x,y)} \rho(t)dt = \int_0^{d(y,x)} \rho(t)dt$;

$d_4)$ $\int_0^{d(x,y)} \rho(t)dt \le \int_0^{d(x,z)} \rho(t)dt + \int_0^{d(z,y)} \rho(t)dt$.

If d satisfies all of the above conditions then d is called a metric on X. If d satisfies the conditions from $d_2 - d_4$ then d is said to be dislocated metric (OR) shortly (d-metric) on X and if d satisfies only d_2 and d_4 then d is called dislocated quasi-metric (OR) shortly (dq-metric) on X and the pair (X,d) is called dislocated quasi-metric space.

Definition 2.18. Where $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesque integrable mapping which is summable on each compact subset of $\mathbb{R}^+$, non-negative and such that for any $s > 0$ $\int_0^{s} \rho(t)dt > 0$.

Note. The above definition change to usual definition of metric space if $\rho(t) = I$.

Definition 2.19. A sequence $\{x_n\}$ in dq-metric space is said to be dq-convergent to a point $x \in X$ if

$$\lim_{n\to\infty} \int_0^{d(x_n,x)} \rho(t)dt = \lim_{n\to\infty} \int_0^{d(x,x_n)} \rho(t)dt = 0.$$

In such a case x is called dq-limit of the sequence $\{x_n\}$.

Definition 2.20. A sequence $\{x_n\}$ in dq-metric space (X,d) is said to be Cauchy sequence if for $\epsilon > 0$ there exist $n_0 \in N$ such that $m,n \ge n_0$ implies

$$\int_0^{d(x_n,x_m)} \rho(t)dt = \int_0^{d(x_m,x_n)} \rho(t)dt < \epsilon$$

(OR)

$$\lim_{n\to\infty} \int_0^{d(x_n,x_m)} \rho(t)dt = \lim_{n\to\infty} \int_0^{d(x_m,x_n)} \rho(t)dt = 0$$

Branciari [10] proved the following theorem in metric spaces.

Theorem 2.21. *Let (X,d) be a complete metric space for $\alpha \in (0,1)$. Let $T : X \to X$ be a mapping such that for all $x, y \in X$ satisfying*

$$\int_0^{d(Tx,Ty)} \rho(t)dt \leq \alpha \cdot \int_0^{d(x,y)} \rho(t)dt.$$

Where $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesque integrable mapping which is summable on each compact subset of $\mathbb{R}^+$, non-negative and such that for any $s > 0$, $\int_0^s \rho(t)dt > 0$. Then T has a unique fixed point in X.

3. Main Result

Theorem 3.1. *Let (X,d) be a complete dq b-metric space and $T : X \to X$ be a continuous mapping satisfying the condition*

$$\begin{aligned} d(Tx,Ty) \leq \alpha \cdot d(x,y) + \beta \cdot \frac{d^2(x,Tx)[1+d(x,y)]}{d(x,y)[1+d(x,Tx)]} + \gamma \cdot \frac{d(y,Ty)[1+d^2(x,Tx)]}{1+d^2(x,y)} \\ + \frac{\mu}{k} \cdot \frac{d(x,Ty)d^2(x,Tx)}{d(x,y)[d(x,y)+d(y,Ty)]} + \frac{\delta}{k} \cdot d(x,Ty) \end{aligned} \tag{3.1}$$

for all $x, y \in X$, with $k \geq 1$ and $\alpha, \beta, \gamma, \mu, \delta \geq 0$ with $k(\alpha+\beta)+\gamma+k\mu+(1+k)\delta < 1$. Then T has a unique fixed point in X.

Proof. Let x_0 be arbitrary in X we define a sequence $\{x_n\}$ in X by the rule

$$x_0, x_1 = Tx_0, \cdots, x_{n+1} = Tx_n.$$

Now to show that $\{x_n\}$ is a Cauchy sequence in X consider

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n).$$

Using (3.1), we have

$$\begin{aligned} d(x_n,x_{n+1}) \leq \alpha \cdot d(x_{n-1},x_n) + \beta \cdot \frac{d^2(x_{n-1},Tx_{x-1})[1+d(x_{n-1},x_n)]}{d(x_{n-1},x_n)[1+d(x_{n-1},Tx_{n-1})]} + \\ \gamma \cdot \frac{d(x_n,Tx_n)[1+d^2(x_{n-1},Tx_{n-1})]}{1+d^2(x_{n-1},x_n)} \\ + \frac{\mu}{k} \cdot \frac{d(x_{n-1},Tx_n)d^2(x_{n-1},Tx_{n-1})}{d(x_{n-1},x_n)[d(x_{n-1},x_n)+d(x_n,Tx_n)]} + \frac{\delta}{k} \cdot d(x_{n-1},Tx_n) \\ \leq \alpha \cdot d(x_{n-1},x_n) + \beta \cdot \frac{d^2(x_{n-1},x_n)[1+d(x_{n-1},x_n)]}{d(x_{n-1},x_n)[1+d(x_{n-1},x_n)]} \\ + \gamma \cdot \frac{d(x_n,x_{n+1})[1+d^2(x_{n-1},x_n)]}{1+d^2(x_{n-1},x_n)} \\ + \frac{\mu}{k} \cdot \frac{d(x_{n-1},x_{n+1})d^2(x_{n-1},x_n)}{d(x_{n-1},x_n)[d(x_{n-1},x_n)+d(x_n,x_{n+1})]} + \frac{\delta}{k} \cdot d(x_{n-1},x_{n+1}). \end{aligned}$$

Using triangular inequality, we have

$$\begin{aligned} \leq \alpha \cdot d(x_{n-1},x_n) + \beta \cdot d(x_{n-1},x_n) + \gamma \cdot d(x_n,x_{n+1}) + \\ \mu \cdot d(x_{n-1},x_n) + \delta \cdot d(x_{n-1},x_n) + \delta \cdot d(x_n,x_{n+1}). \end{aligned}$$

Finally we have

$$d(x_n, x_{n+1}) \leq \frac{\alpha+\beta+\mu+\delta}{1-(\gamma+\delta)} \cdot d(x_{n-1}, x_n).$$

Let

$$h = \frac{\alpha+\beta+\mu+\delta}{1-(\gamma+\delta)} < \frac{1}{k}.$$

So

$$d(x_n, x_{n+1}) \leq h \cdot d(x_{n-1}, x_n).$$

Also

$$d(x_{n-1}, x_n) \leq h \cdot d(x_{n-2}, x_{n-1}).$$

Therefore

$$d(x_n, x_{n+1}) \leq h^2 \cdot d(x_{n-2}, x_{n-1}).$$

Similarly proceeding we get

$$d(x_n, x_{n+1}) \leq h^n \cdot d(x_0, x_1).$$

Taking limit $n \to \infty$, as $h < 1$, so $h^n \to 0$, implies that

$$d(x_n, x_{n+1}) \to 0.$$

Hence $\{x_n\}$ is a Cauchy sequence in complete dq b-metric space. So there must exist $u \in X$ such that

$$\lim_{n\to\infty} x_n = u.$$

Now to show that u is a fixed point of T. Since T is continuous therefore

$$\lim_{n\to\infty} Tx_n = Tu \quad \Rightarrow \quad \lim_{n\to\infty} x_{n+1} = Tu \quad \Rightarrow \quad Tu = u.$$

Hence u is the fixed point of T.

Uniqueness: Let $u \neq v$ are two distinct fixed points of T then by (3.1), we have

$$d(u,v) = d(Tu,Tv) \leq \alpha \cdot d(u,v) + \beta \cdot \frac{d^2(u,Tu)[1+d(u,v)]}{d(u,v)[1+d(u,Tu)]} + \gamma \cdot \frac{d(v,Tv)[1+d^2(u,Tu)]}{1+d^2(u,v)}$$
$$+\frac{\mu}{k} \cdot \frac{d(u,Tv)d^2(u,Tu)}{d(u,v)[d(u,v)+d(v,Tv)]} + \frac{\delta}{k} \cdot d(u,Tv).$$

Since u and v are fixed points of T, so the above inequality becomes

$$d(u,v) \leq \alpha \cdot d(u,v) + \beta \cdot \frac{d^2(u,u)[1+d(u,v)]}{d(u,v)[1+d(u,u)]} + \gamma \cdot \frac{d(v,v)[1+d^2(u,u)]}{1+d^2(u,v)}$$
$$+\frac{\mu}{k} \cdot \frac{d(u,v)d^2(u,u)}{d(u,v)[d(u,v)+d(v,v)]} + \frac{\delta}{k} \cdot d(u,v).$$

Using (3.1) and the fact that u, v are the fixed points of T one can get

$$d(u,u) = d(v,v) = 0.$$

Thus the above inequality take the form

$$d(u,v) \leq (\alpha+\delta) \cdot d(u,v).$$

Which is possible only if $d(u,v) = 0$. Similarly we can show that $d(v,u) = 0$. Hence $u = v$. Therefore fixed point of T is unique. $\square$

Remark 3.2. By putting different constants equal to zero in Theorem 3.1, one can obtain different corollaries from Theorem 3.1, which show that Theorem 3.1, generalize different results of the literature in dislocated quasi b-metric spaces.

Theorem 3.3. *Let (X, d) be a complete dq-metric space and $T : X \to X$ be a continuous mapping satisfying the condition*

$$\int_0^{d(Tx,Ty)} \rho(t)dt \leq \alpha \cdot \int_0^{d(x,y)} \rho(t)dt + \beta \cdot \int_0^{\frac{d^2(x,Tx)[1+d(x,y)]}{d(x,y)[1+d(x,Tx)]}} \rho(t)dt$$
$$+ \gamma \cdot \int_0^{\frac{d(y,Ty)[1+d^2(x,Tx)]}{1+d^2(x,y)}} \rho(t)dt + \mu \cdot \int_0^{\frac{d(x,Ty)d^2(x,Tx)}{d(x,y)[d(x,y)+d(y,Ty)]}} \rho(t)dt + \delta \cdot \int_0^{[d(x,Ty)+d(x,y)]} \rho(t)dt \tag{3.2}$$

for all $x, y \in X$ and $\alpha, \beta, \gamma, \mu, \delta \geq 0$ with $\alpha + \beta + \gamma + \mu + 3\delta < 1$. Where $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesque integrable mapping which is summable on each compact subset of $\mathbb{R}^+$, non-negative and such that for any $s > 0$, $\int_0^s \rho(t)dt > 0$. Then T has a unique fixed point in X.

Proof. Let x_0 be arbitrary in X we define a sequence $\{x_n\}$ in X by the rule

$$x_0, x_1 = Tx_0, \cdots, x_{n+1} = Tx_n.$$

Now to show that $\{x_n\}$ is a Cauchy sequence in X, consider

$$\int_0^{d(x_n,x_{n+1})} \rho(t)dt = \int_0^{d(Tx_{n-1},Tx_n)} \rho(t)dt.$$

Using (3.2), we have

$$\begin{aligned}
\int_0^{d(x_n,x_{n+1})} \rho(t)dt &\leq \alpha \cdot \int_0^{d(x_{n-1},x_n)} \rho(t)dt + \beta \cdot \int_0^{\frac{d^2(x_{n-1},Tx_{x-1})[1+d(x_{n-1},x_n)]}{d(x_{n-1},x_n)[1+d(x_{n-1},Tx_{n-1})]}} \rho(t)dt \\
&+ \gamma \cdot \int_0^{\frac{d(x_n,Tx_n)[1+d^2(x_{n-1},Tx_{n-1})]}{1+d^2(x_{n-1},x_n)}} \rho(t)dt + \mu \cdot \int_0^{\frac{d(x_{n-1},Tx_n)d^2(x_{n-1},Tx_{n-1})}{d(x_{n-1},x_n)[d(x_{n-1},x_n)+d(x_n,Tx_n)]}} \rho(t)dt \\
&+ \delta \cdot \int_0^{[d(x_{n-1},Tx_n)+d(x_{n-1},x_n)]} \rho(t)dt \\
&\leq \alpha \cdot \int_0^{d(x_{n-1},x_n)} \rho(t)dt + \beta \cdot \int_0^{\frac{d^2(x_{n-1},x_n)[1+d(x_{n-1},x_n)]}{d(x_{n-1},x_n)[1+d(x_{n-1},x_n)]}} \rho(t)dt + \gamma \cdot \int_0^{\frac{d(x_n,x_{n+1})[1+d^2(x_{n-1},x_n)]}{1+d^2(x_{n-1},x_n)}} \rho(t)dt \\
&+ \mu \cdot \int_0^{\frac{d(x_{n-1},x_{n+1})d^2(x_{n-1},x_n)}{d(x_{n-1},x_n)[d(x_{n-1},x_n)+d(x_n,x_{n+1})]}} \rho(t)dt + \delta \cdot \int_0^{[d(x_{n-1},x_{n+1})+d(x_{n-1},x_n)]} \rho(t)dt \\
&\leq \alpha \cdot \int_0^{d(x_{n-1},x_n)} \rho(t)dt + \beta \cdot \int_0^{d(x_{n-1},x_n)} \rho(t)dt + \gamma \cdot \int_0^{d(x_n,x_{n+1}} \rho(t)dt \\
&+ \mu \cdot \int_0^{d(x_{n-1},x_n)} \rho(t)dt + 2\delta \cdot \int_0^{d(x_{n-1},x_n)} \rho(t)dt + \delta \cdot \int_0^{d(x_n,x_{n+1})} \rho(t)dt.
\end{aligned}$$

Finally, we have from the above equation

$$\int_0^{d(x_n,x_{n+1})} \rho(t)dt \leq \frac{\alpha+\beta+\mu+2\delta}{1-(\gamma+\delta)} \cdot \int_0^{d(x_{n-1},x_n)} \rho(t)dt.$$

Let

$$h = \frac{\alpha+\beta+\mu+2\delta}{1-(\gamma+\delta)} < 1.$$

So

$$\int_0^{d(x_n,x_{n+1})} \rho(t)dt \leq h \cdot \int_0^{d(x_{n-1},x_n)} \rho(t)dt.$$

Also

$$\int_0^{d(x_{n-1},x_n)} \leq h \cdot \int_0^{d(x_{n-2},x_{n-1})} \rho(t)dt.$$

Therefore

$$\int_0^{d(x_n,x_{n+1})} \rho(t)dt \leq h^2 \cdot \int_0^{d(x_{n-2},x_{n-1})} \rho(t)dt.$$

Similarly proceeding we get

$$\int_0^{d(x_n,x_{n+1})} \rho(t)dt \leq h^n \cdot \int_0^{d(x_0,x_1)} \rho(t)dt.$$

Taking limit $n \to \infty$, as $h < 1$ so $h^n \to 0$, implies

$$\int_0^{d(x_n,x_{n+1})} \rho(t)dt \to 0.$$

Hence $\{x_n\}$ is a Cauchy sequence in complete dq-metric space. So there must exist $u \in X$ such that

$$\lim_{n\to\infty} x_n = u.$$

Now to show that u is a fixed point of T. Since T is continuous therefore

$$\lim_{n\to\infty} Tx_n = Tu \quad \Rightarrow \quad \lim_{n\to\infty} x_{n+1} = Tu \quad \Rightarrow \quad Tu = u.$$

Hence u is the fixed point of T.

Uniqueness: Let $u \neq v$ are two distinct fixed points of T then by (3.2), we have

$$\begin{aligned}\int_0^{d(u,v)} \rho(t)dt = \int_0^{d(Tu,Tv)} \rho(t)dt \leq \alpha \cdot \int_0^{d(u,v)} \rho(t)dt + \beta \cdot \int_0^{\frac{d^2(u,Tu)[1+d(u,v)]}{d(u,v)[1+d(u,Tu)]}} \rho(t)dt \\ + \gamma \cdot \int_0^{\frac{d(v,Tv)[1+d^2(u,Tu)]}{1+d^2(u,v)}} \rho(t)dt + \mu \cdot \int_0^{\frac{d(u,Tv)d^2(u,Tu)}{d(u,v)[d(u,v)+d(v,Tv)]}} \rho(t)dt + \delta \cdot \int_0^{[d(u,Tv)+d(u,v)]} \rho(t)dt.\end{aligned}$$

Since u and v are fixed points of T so the above inequality becomes

$$\begin{aligned}\int_0^{d(u,v)} \rho(t)dt \leq \alpha \cdot \int_0^{d(u,v)} \rho(t)dt + \beta \cdot \int_0^{\frac{d^2(u,u)[1+d(u,v)]}{d(u,v)[1+d(u,u)]}} \rho(t)dt + \gamma \cdot \int_0^{\frac{d(v,v)[1+d^2(u,u)]}{1+d^2(u,v)}} \rho(t)dt \\ + \mu \cdot \int_0^{\frac{d(u,v)d^2(u,u)}{d(u,v)[d(u,v)+d(v,v)]}} \rho(t)dt + \delta \cdot \int_0^{[d(u,v)+d(u,v)]} \rho(t)dt.\end{aligned}$$

Using (3.2), and the fact that u, v are the fixed points of T one can get

$$d(u,u) = d(v,v) = 0.$$

Thus the above inequality take the form

$$\int_0^{d(u,v)} \rho(t)dt \leq (\alpha + 2\delta) \cdot \int_0^{d(u,v)} \rho(t)dt.$$

Which is possible only if $d(u,v) = 0$. Similarly we can show that $d(v,u) = 0$. Hence $u = v$. Therefore fixed point of T is unique. $\square$

Corollary 3.4. *Let (X, d) be a complete dq-metric space and $T : X \to X$ be a continuous mapping satisfying the condition*

$$\int_0^{d(Tx,Ty)} \rho(t)dt \leq \alpha \cdot \int_0^{d(x,y)} \rho(t)dt$$

for all $x, y \in X$ and $\alpha \geq 0$ with $\alpha < 1$. Where $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesque integrable mapping which is summable on each compact subset of $\mathbb{R}^+$, non-negative and such that for any $s > 0$, $\int_0^s \rho(t)dt > 0$. Then T has a unique fixed point in X.

Proof. Putting $\beta = \gamma = \mu = \delta = 0$, in Theorem 3.1, one can get easily the required result. □

Example 3.5. Let $X = [0, 1]$ and the complete dq-metric defined on X is given by $d(x, y) = |x|$ with self-mapping defined on X is $Tx = \frac{x}{2}$ and $\rho(t) = \frac{t}{2}$. Then

$$\int_0^{d(Tx,Ty)} \rho(t)dt = \int_0^{|\frac{x}{2}|} \frac{t}{2}dt = \frac{1}{16}x^2 \leq \frac{1}{4}\left(\frac{1}{4}x^2\right) \leq a. \int_0^{d(x,y)} \rho(t)dt.$$

Satisfy all the conditions of the Corollary 3.4 for $\alpha \in [\frac{1}{4}, 1)$ having $x = 0$ is its unique fixed point.

Example 3.6. Let $X = [0, 1]$ and the complete dq-metric defined on X is given by $d(x, y) = |x|$, with self-mapping defined on X is $Tx = \frac{x}{2}$ and $\rho(t) = \frac{t}{2}$ for $\alpha = \frac{1}{3}, \beta = \frac{1}{4}, \gamma = \frac{1}{6}, \mu = \frac{1}{8}, \delta = \frac{1}{12}$. Satisfy all the conditions of Theorem 3.1 having $x = 0$ is the unique fixed point of T.

Remark 3.7. By putting different constants equal to zero and $\rho(t) = I$ (identity mapping) in Theorem 3.1, we can get different corollaries from the above theorem, which may generalize different theorems in the literature in the context of dislocated quasi metric spaces.

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