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On the Approximate Solutions for Fractional Differential Equations with Caputo-Fabrizio Fractional Derivative

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Abstract

In this paper, the Sumudu Daftardar-Jafari method, abbreviated as (SDJM) is established and successfully applied to obtain approximate analytical solutions for the fractional differential equations with the Caputo-Fabrizio fractional derivative (CFFD). SDJM is expressed with a combining of the Sumudu transform and the Daftardar-Jafari method. Based on this method, the solutions are considered in the form of a series. Three examples are provided to illustrate the efficiency and ease of implementation of the proposed method. Moreover, the obtained solutions are compared with the exact solutions. We also present our results with the help of tables and figures.

Keywords: Fractional differential equations; Daftardar-Jafari method, Sumudu transform, Caputo-Fabrizio fractional derivative.
2010 MSC: 35R11, 65M70, 91G60

1. Introduction

In the last decades, fractional differential equations have attracted the attention of many scientists and have been useful in various fields such as control theory, Fluid Mechanics, physics, Hydrology, Electromagnetics, Mechanics, Finance, and other fields of science and engineering (see [1, 2, 3]).

Various fractional derivative definitions have been introduced in recent years. Some of this are Atangana–Baleanu [4], Hilfer [5], Hadamard [6], Caputo–Fabrizio [7], etc. Applications of these fractional derivatives have been investigated by many researchers in various fields of science and engineering. For instance, Bahar Acay et al. [11] investigated fractional physical models based on falling body problem, H. K. Jassim et al. [26] studied Burger's and coupled Burger's equations, Arif, M.S et al. [28] investigated a fractional-order predator-prey model, Alcantara-Lopez, et al. [29] studied a fractional growth model applied to COVID-19

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Data, K. Shah et al. [32] investigated the fractional model of dengue fever disease under Caputo-Fabrizio derivative, Zeliha Korpinar et al. [39] investigated the fractional model of fokker-planck equations with two different operators, Ved Prakash Dubey et al. [45] investigated the fractional order model of transmission dynamics of HIV/AIDS with the effect of weak CD4+T cells, the fractional model of phytoplankton-toxic Phytoplankton-Zooplankton system is analyzed and investigated by Ved Prakash Dubey et al. [46].

There are different types of powerful techniques that have been introduced and developed to obtain solutions to the differential equations that explain the phenomena of the problems of the real world in order to predict them, especially those formulated in the form of non-linear equations, which is difficult to find a solution to in closed form. For example the Daftardar-Jafari method [42], Sumudu Adomian decomposition method [42, 43], the variational iteration method [44], the Adomian decomposition method [14], the residual power series method [8, 9, 10], the Laplace transform[11, 12], the differential transform method [13, 15], the homotopy analysis transform method [18], the homotopy perturbation method [19], Laplace Variational Iteration Method [20], the Hussein–Jassim method [21], the Laplace residual power series [22], the homotopy analysis method [23, 24, 25], the Sumudu variational iteration method [26], the numerical inverse Laplace transform methods [27] and others (see [28]–[41]).

In this paper, a coupling of the Sumudu transform with the Daftardar-Jafari method is utilized to solve the fractional differential equations under the fractional operator of the Caputo-Fabrizio type. The paper has been organized as follows: In Section 2, we review the definitions of fractional calculus and the Sumudu transform. In Section 3, the analysis of the SDJM is offered and discussed. An illustrative examples that shows the effectiveness of the method is given in Section 4. Finally, a conclusion is presented in Section 5.

2. Preliminaries

In this section, we recap some useful definition of fractional calculus and sumudu transform which are required to introduce the proposed method.

Definition 1. ([7, 16]). The fractional derivative Caputo-Fabrizio fractional derivative is defined as:

$$^{CF}D_t^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t f^{'}(\tau) \exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau, \tag{2.1}$$

where $0 < \alpha \leq 1$, $a \in [-\infty, t)$, $f \in H^1(a,b)$ and $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$.

Definition 2. [16] The Sumudu transform is define over the set of function:

$$A = \left\{ f(t) \middle| \exists M,\ \epsilon_1,\ \epsilon_2 > 0, |\ f(t)| < M e^{\frac{|t|}{\epsilon_i}}\ , if\ t \in (-1)^i \times [0, \infty) \right\},$$

where ϵ_1 , ϵ_2 may be finite or infinite and the constant u must be finite. The Sumudu transform of the function $f(t)$ is defined as:

$$\mathcal{S}\left[f\ (t)\right] = F(u) = \int_0^{\infty} f(ut) e^{-ut} dt\ , \qquad t \geq 0, \qquad \epsilon_1 \leq u \leq \epsilon_2. \tag{2.2}$$

Some properties of Sumudu transform

1. $\mathcal{S}\left[f(t) + g(t)\right] = \mathcal{S}\left[f(t)\right] + \mathcal{S}\left[g(t)\right]$,
2. $\mathcal{S}[1] = 1$,
3. $\mathcal{S}\left[t^n\right] = \Gamma(n+1) u^n, \qquad n > 0.$

Definition 3. [16] The Sumudu transform of the Caputo–Fabrizio fractional derivative is defined as:

$$\mathcal{S}\left[^{CF}D_t^{\alpha} f(t)\right] = \frac{1}{1-\alpha+\alpha u} \left(\mathcal{S}\left[f(t)\right] - f(0)\right)\ , \tag{2.3}$$

which we suppose the function $M(\alpha) = 1$.

3. Analysis of SDJM

To illustrate the basic idea of the SDJM for fractional differential equations, we consider a nonlinear nonhomogeneous fractional differential equation of the following form:

$$ {}^{CF}D_t^{\alpha} y(t) + \mathfrak{R}\left[y(t)\right] + \aleph\left[y(t)\right] = g(t), \quad 0 < \alpha \leq 1, \tag{3.1} $$

with initial condition

$$ y(0) = \lambda, \tag{3.2} $$

where $y(t)$ is an analytical function, ${}^{CF}D_t^{\alpha}$ is the Caputo–Fabrizio fractional derivative, $\mathfrak{R}$ is a linear operator, $\aleph$ is a nonlinear operator and $g(t)$ is a source term.

Taking sumudu transform to both sides of (3.1) we get

$$ \frac{1}{1-\alpha+\alpha u}\left[\mathcal{S}\left[y(t)\right] - f(0)\right] + \mathcal{S}\left[\mathfrak{R}\left[y(t)\right]\right] + \mathcal{S}\left[\aleph\left[y(t)\right]\right] = \mathcal{S}\left[g(t)\right], \tag{3.3} $$

or

$$ \mathcal{S}\left[y(t)\right] = (1-\alpha+\alpha u)\left[\mathcal{S}\left[g(t)\right] - \mathcal{S}\left[\mathfrak{R}\left[y(t)\right]\right] - \mathcal{S}\left[\aleph\left[y(t)\right]\right]\right] + \lambda, \tag{3.4} $$

Taking the inverse of Sumudu transform of (3.4), we obtain

$$ y(t) = \mathcal{S}^{-1}\left[(1-\alpha+\alpha u)\left[\mathcal{S}\left[g(t)\right] - \mathcal{S}\left[\mathfrak{R}\left[y(t)\right]\right] - \mathcal{S}\left[\aleph\left[y(t)\right]\right]\right]\right] + \lambda. \tag{3.5} $$

Now, we represent the solution as an infinite series given below

$$ y(t) = \sum_{i=0}^{\infty} y_i(t), \tag{3.6} $$

thus, by substituting Equation (3.6) into Equation (3.5), we obtain

$$ \sum_{i=0}^{\infty} y_i(t) = \mathcal{S}^{-1}\left[(1-\alpha+\alpha u)\left[\mathcal{S}\left[g(t)\right] - \mathcal{S}\left[\mathfrak{R}\left[\sum_{i=0}^{\infty} y_i(t)\right]\right] - \mathcal{S}\left[\aleph\left[\sum_{i=0}^{\infty} y_i(t)\right]\right]\right]\right] + \lambda. \tag{3.7} $$

The nonlinear operator $\mathcal{N}$ can be decomposed as

$$ \aleph\left[\sum_{i=0}^{\infty} y_i(t)\right] = \aleph\left[y_0(t)\right] + \sum_{i=1}^{\infty}\left(\aleph\left[\sum_{n=0}^{i} y_n(t)\right] - \aleph\left[\sum_{n=0}^{i-1} y_n(t)\right]\right). \tag{3.8} $$

Moreover, the relation is defined with recurrence so that

$$ \begin{cases} y_0(t) = \lambda + \mathcal{S}^{-1}\left[(1-\alpha+\alpha u)\left[\mathcal{S}\left[g(t)\right]\right]\right], \\ y_1(t) = \mathcal{S}^{-1}\left[(1-\alpha+\alpha u)\left[-\mathcal{S}\left[\mathfrak{R}\left[y_0(t)\right]\right] - \mathcal{S}\left[\aleph\left[y_0(t)\right]\right]\right]\right], \\ y_{i+1}(t) = \mathcal{S}^{-1}\left[(1-\alpha+\alpha u)\left[-\mathcal{S}\left[\mathfrak{R}\left[y_i(t)\right]\right] - \mathcal{S}\left[\aleph\left[\sum_{n=0}^{i} y_n(t)\right] - \aleph\left[\sum_{n=0}^{i-1} y_n(t)\right]\right]\right]\right], \quad i = 1, 2, \ldots \end{cases} \tag{3.9} $$

Thus the approximate solution of Equation (3.1) is given by:

$$ y(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + \cdots . \tag{3.10} $$

4. Illustrative Examples

In this section, the SDJM is efficiently applied to fractional differential equations to validate its efficiency and high accuracy.

Example 1. [17] Consider the following fractional-order logistic differential equation:

$$ {}^{CF}D_t^{\alpha} y(t) = y(t)\,(1 - y(t))\,, \quad t > 0, \quad 0 < \alpha \leq 1, \tag{4.1} $$

with the initial condition

$$ y(0) = 2. \tag{4.2} $$

The exact solution for Equation (4.1) for $\alpha = 1$ is $y(t) = \frac{2}{2-e^{-t}}$ [17]. Now, in view of Equation (3.5) we have

$$ y(t) = \mathcal{S}^{-1}\left[(1 - \alpha + \alpha u)\left[\mathcal{S}\left[y(t)\right] - \mathcal{S}\left[y^2(t)\right]\right]\right] + 2. \tag{4.3} $$

From the relation (3.9) and Equation (4.3), we get

$$ y_0(t) = 2\ , \tag{4.4} $$

$$ y_1(t) = \mathcal{S}^{-1}[(1 - \alpha + \alpha u)[\mathcal{S}[y_0(t)] - \mathcal{S}[{y_0}^2(t)]]]] = -2(1 - \alpha + \alpha t), \tag{4.5} $$

$$ y_2(t) = \mathcal{S}^{-1}\left[(1 - \alpha + \alpha u)\left[-\mathcal{S}\left[y_1(t)\right] - \mathcal{S}\left[\left[y_0(t) + y_1(t)\right]^2 - \aleph\left[{y_0}^2(t)\right]\right]\right]\right] \tag{4.6} $$

$$ = 2(1-\alpha)^2(2\alpha+1) + 12\alpha^2(1-\alpha)t + 2\alpha^2(8\alpha-5)\frac{t^2}{2!} - 8\alpha^3\frac{t^3}{3!}\ , \tag{4.7} $$

Thus the approximate solution of (4.1) is

$$ y_{SDJM}(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + \cdots \tag{4.8} $$

$$ = 2\left[(1-\alpha)^2(2\alpha+1) + \alpha\right] - 2\alpha\left[1 - 6\alpha(1-\alpha)\right]t + \alpha^2(8\alpha-5)t^2 - 8\alpha^3\frac{t^3}{3!} + \dots \tag{4.9} $$

In particular, setting $\alpha = 1$ in Equation (4.9), we get

$$ y_{SDJM}(t) \cong 2 - 2t + 3t^2 - \frac{13}{3}t^3 + \dots \tag{4.10} $$

t	y_{exact}	Y_{SDJM} $\alpha = 1$	Y_{SDJM} $\alpha = 0.95$	Y_{SDJM} $\alpha = 0.9$	Abs. error between y_{exact} & y_{SDJM}		
					$\alpha = 1$	$\alpha = 0.95$	$\alpha = 0.9$
0	2.000000	2.000000	1.914500	1.856000	0.000000	0.085500	0.144000
0.1	1.826213	1.828667	1.800972	1.790048	0.002454	0.025241	0.036165
0.2	1.693094	1.709333	1.727515	1.753904	0.016239	0.034421	0.060810
0.3	1.588333	1.634000	1.687270	1.741736	0.045667	0.098936	0.153403
0.4	1.504121	1.594667	1.673377	1.747712	0.090545	0.169256	0.243591
0.5	1.435267	1.583333	1.678979	1.766000	0.148067	0.243713	0.330733
0.6	1.378181	1.592000	1.697216	1.790768	0.213819	0.319035	0.412587
0.7	1.330305	1.612667	1.721229	1.816184	0.282362	0.390924	0.485879
0.8	1.289764	1.637333	1.744159	1.836416	0.347569	0.454394	0.546652
0.9	1.255154	1.658000	1.759147	1.845632	0.402846	0.503993	0.590478
1	1.225400	1.666667	1.759333	1.838000	0.441267	0.533934	0.612600

Table 1: Comparison between exact solution and approximate solution for the Equation (4.1) for different values of α.

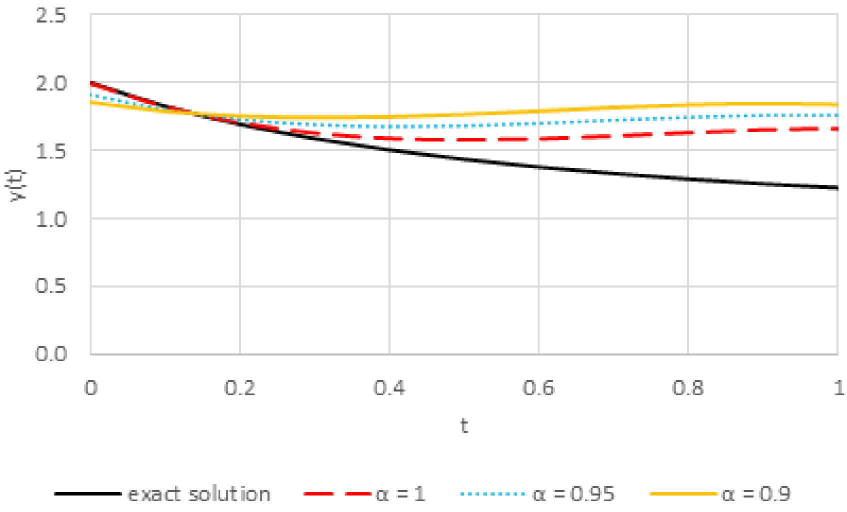

Figure 1: Plots of the exact and approximate solutions for Equation (4.1) with different values of α

Example 2. [16] Consider the following fractional-order differential equation:

$$ {}^{CF}D_t^{\alpha} y(t) + y(t) = 0, \quad y(0) = 1, \quad t > 0, \quad 0 < \alpha \leq 1. \tag{4.11} $$

The exact solution for Equation (4.11) for $\alpha = 1$ is $y(t) = e^{-t}$ [16]. Now, in view of Equation (3.5) we have

$$ y(t) = \mathcal{S}^{-1}\left[(1-\alpha+\alpha u)\left[-\mathcal{S}\left[y(t)\right]\right]\right] + 1. \tag{4.12} $$

From the relation (3.9) and Equation (4.12), we get

$$ y_0(t) = 1, \tag{4.13} $$

$$ y_1(t) = \mathcal{S}^{-1}\left[(1-\alpha+\alpha u)\left[-\mathcal{S}\left[y_0(t)\right]\right]\right] = -(1-\alpha+\alpha t), \tag{4.14} $$

$$ y_2(t) = \mathcal{S}^{-1}\left[(1-\alpha+\alpha u)\left[-\mathcal{S}\left[y_1(t)\right]\right]\right] = (1-\alpha)^2 + 2\alpha(1-\alpha)t + \alpha^2\frac{t^2}{2!}\ , \tag{4.15} $$

$$ y_3(t) = \mathcal{S}^{-1}\left[(1-\alpha+\alpha u)\left[-\mathcal{S}\left[y_2(t)\right]\right]\right] = -\left((1-\alpha)^3 - 3\alpha(1-\alpha)^2 t + 3\alpha^2(1-\alpha)\frac{t^2}{2!} + \alpha^3\frac{t^3}{3!}\right), \tag{4.16} $$

Thus, the approximate solution of (4.11) is

$$ y_{SDJM}(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + \cdots \tag{4.17} $$

$$ = \alpha + (1-\alpha)^2 - (1-\alpha)^3 + \left[\alpha(1-2\alpha) + 3\alpha(1-\alpha)^2\right] t + \alpha^2(3\alpha-2)\frac{t^2}{2!} - \alpha^3\frac{t^3}{3!} + \ldots \tag{4.18} $$

In particular, setting $\alpha = 1$ in Equation (4.18), we get

$$ y_{SDJM}(t) \cong 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \ldots . \tag{4.19} $$

t	y_{exact}	Y_{SDJM} $\alpha=1$	Y_{SDJM} $\alpha=0.95$	Y_{SDJM} $\alpha=0.9$	Abs. error between y_{exact} & y_{SDJM} $\alpha=1$	$\alpha=0.95$	$\alpha=0.9$
0	1.000000	1.000000	0.952375	0.909000	0.000000	0.047625	0.091000
0.1	0.904837	0.904833	0.871280	0.842414	0.000004	0.033557	0.062424
0.2	0.818731	0.818667	0.796999	0.780768	0.000064	0.021731	0.037963
0.3	0.740818	0.740500	0.728675	0.723335	0.000318	0.012143	0.017484
0.4	0.670320	0.669333	0.665450	0.669384	0.000987	0.004870	0.000936
0.5	0.606531	0.604167	0.606466	0.618188	0.002364	0.000065	0.011657
0.6	0.548812	0.544000	0.550867	0.569016	0.004812	0.002055	0.020204
0.7	0.496585	0.487833	0.497795	0.521141	0.008752	0.001210	0.024555
0.8	0.449329	0.434667	0.446392	0.473832	0.014662	0.002937	0.024503
0.9	0.406570	0.383500	0.395802	0.426362	0.023070	0.010768	0.019792
1	0.367879	0.333333	0.345167	0.378000	0.034546	0.022713	0.010121

Table 2: Comparison between exact solution and approximate solution for the Equation (4.11) for different values of α.

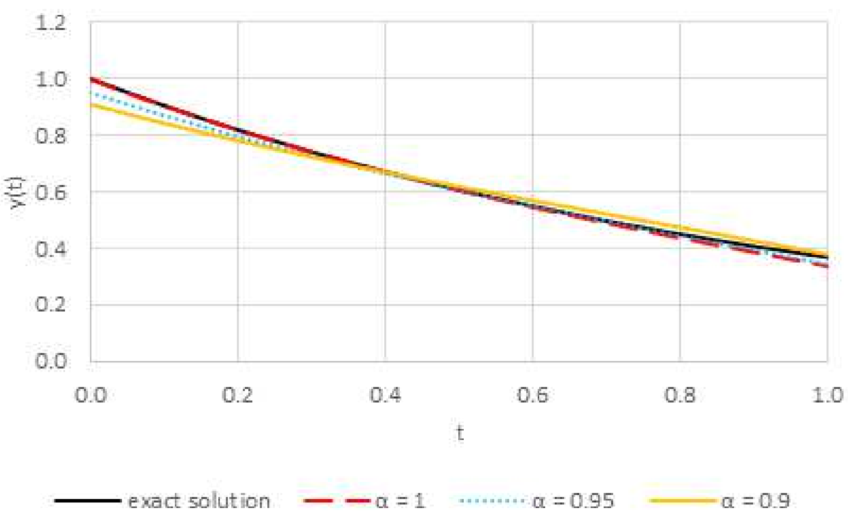

Figure 2: Plots of the exact and approximate solutions for Equation (4.11) with different values of α.

Example 3. [18] Consider the following nonlinear fractional wave equation:

$$ {}^{CF}D_t^{\alpha} y(x,t) = \frac{\partial^2 y(x,t)}{\partial x^2} + \frac{1}{2}\frac{\partial y^2(x,t)}{\partial x}, \quad t>0, \quad 0<\alpha\leq 1, \tag{4.20}$$

with the initial condition

$$ y(x,0) = x. \tag{4.21}$$

The exact solution for Equation (4.20) for $\alpha=1$ is $y(x,t)=\frac{x}{1+t}$ [18].
Now, in view of Equation (3.5) we have

$$ y(x,t) = \mathcal{S}^{-1}\left[(1-\alpha+\alpha u)\left[\mathcal{S}\left[-\frac{\partial^2 y(x,t)}{\partial x^2}\right] - \mathcal{S}\left[\frac{1}{2}\frac{\partial y^2(x,t)}{\partial x}\right]\right]\right] + x. \tag{4.22}$$

From the relation (3.9) and Equation (4.22), we get

$$ y_0(x,t) = x \ , \tag{4.23}$$

$$y_1(x,t) = \mathcal{S}^{-1}\left[(1-\alpha+\alpha u)\left[\mathcal{S}\left[-\frac{\partial^2 y_0(x,t)}{\partial x^2}\right] - \mathcal{S}\left[\frac{1}{2}\frac{\partial y_0^2(x,t)}{\partial x}\right]\right]\right] = -x(1-\alpha+\alpha t), \tag{4.24}$$

$$y_2(x,t) = \mathcal{S}^{-1}\left[(1-\alpha+\alpha u)\left[-\mathcal{S}\left[-\frac{\partial^2 y_1(x,t)}{\partial x^2}\right] - \mathcal{S}\left[\frac{1}{2}\frac{\partial\left(y_0+y_1\right)^2}{\partial x} - \frac{1}{2}\frac{\partial y_0^2}{\partial x}\right]\right]\right] \tag{4.25}$$

$$= -x\left((1-\alpha)\left(\alpha^2-1\right) + \alpha(1-\alpha)(1+3\alpha)t + \alpha^2(1-2\alpha)t^2 + 2\alpha^3\frac{t^3}{3!}\right), \tag{4.26}$$

and so on.
Thus the approximate solution of (4.20) is

$$y_{SDJM}(x,t) = y_0(x,t) + y_1(x,t) + y_2(x,t) + y_3(x,t) + \cdots \tag{4.27}$$

$$= -x\left(\left(-1+(1-\alpha)\left(1+\left(\alpha^2-1\right)\right)\right) + \alpha\left(1+(1-\alpha)(1+3\alpha)\right)t + \alpha^2(1-2\alpha)t^2 + 2\alpha^3\frac{t^3}{3!}\right). \tag{4.28}$$

In particular, setting $\alpha = 1$ in Equation (4.28), we get

$$y_{SDJM}(x,t) \cong x\left(1 - t + t^2 - t^3 + \ldots\right). \tag{4.29}$$

t	y_{exact}	Y_{SDJM} $\alpha=1$	Y_{SDJM} $\alpha=0.95$	Y_{SDJM} $\alpha=0.9$	Abs. error between y_{exact} & y_{SDJM}		
					$\alpha=1$	$\alpha=0.95$	$\alpha=0.9$
0	1.000000	1.000000	0.954875	0.919000	0.000000	0.045125	0.081000
0.1	0.909091	0.909667	0.849424	0.801937	0.000576	0.059667	0.107154
0.2	0.833333	0.837333	0.758504	0.696376	0.004000	0.074830	0.136957
0.3	0.769231	0.781000	0.680399	0.600859	0.011769	0.088832	0.168372
0.4	0.714286	0.738667	0.613394	0.513928	0.024381	0.100891	0.200358
0.5	0.666667	0.708333	0.555776	0.434125	0.041667	0.110891	0.232542
0.6	0.625000	0.688000	0.505829	0.359992	0.063000	0.119171	0.265008
0.7	0.588235	0.675667	0.461838	0.290071	0.087431	0.126397	0.298164
0.8	0.555556	0.669333	0.422090	0.222904	0.113778	0.133466	0.332652
0.9	0.526316	0.667000	0.384868	0.157033	0.140684	0.141448	0.369283
1	0.500000	0.666667	0.348458	0.091000	0.166667	0.151542	0.409000

Table 3: Comparison between exact solution and approximate solution for the Equation (4.20) for different values of $\alpha(x=1)$.

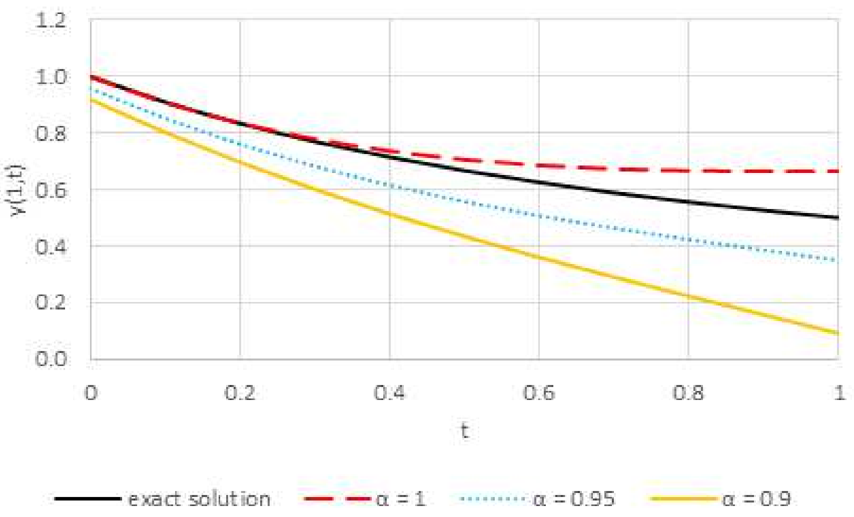

Figure 3: Plots of the exact and approximate solutions for Equation (4.20) with different values of α with fixed value $x = 1$.

5. Conclusions

In this paper, the (SDJM) has been successfully applied to obtain the analytical approximate solutions for the FDEs under CFFD. The obtained solutions were considered in the form of a series, which rapidly converges to exact solution. The examples show that the results of (SDJM) are in good agreement with the exact solution when $\alpha \to 1$. Based on the results shown, the proposed approach is a powerful technique for solving FDEs. Also, we can utilize them to obtain approximate solutions to other nonlinear problems in the sciences and engineering.

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Conflicts of Interest:

The author declare that there is no conflict of interest regarding the publication of this paper.

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