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Common Fixed Point Theorems for (ψ,ϕ) Interpolative type contraction Mappings in Symmetric Space with Application

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Abstract

This paper proves the fixed point theorems for (ψ,ϕ) interpolative type contraction mappings using E.A-property in symmetric spaces with some applications. In doing so, we extended and generalized the results in the literature. We also provided an illustrative example to support the results. Finally, we demonstrate the results by the applications to differential equations.

Keywords: Common fixed point, symmetric space, interpolative mapping, E.A-property, differential equations.
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1. Introduction

Wilson [15] introduced fixed point results in symmetric spaces (also called E-space, in the terminology of Fréchet) which did not require triangular inequality. Later, Sessa [14] introduced the concept of weakly commuting mappings and proved common fixed point results for commuting mappings in metric space. Further Jungck [6] introduced the notion of compatible mappings which is a generalization of [14] and proved some fixed point theorems in metric spaces. This concept of compatible mappings is used to show the existence of fixed points. Aamri and El Moutawakil [1] established property (EA) which generalizes the concept of compatible and non-compatible mapping under strict contractive conditions in metric spaces. Imdad *et al.* [5] proved the coincidence and fixed points in symmetric spaces under strict contractions in symmetric space. Aliouche [2] gave a common fixed point theorem for non-compatible self-mappings in symmetric spaces satisfying a contractive condition of integral type.

In addition, Geraghty [4] gave the results on contractive mappings in metric space. Khan *et al.* [10] proved the fixed point theorems by altering distances between the points on a compact metric space. Moradi and Farajzadeh [12] proved the result on a fixed point of (ψ,ϕ)-weak and generalized (ψ,ϕ)-weak contraction mappings in complete metric spaces. Moreover, Karapinar [7] revisits the Kannan-type contractions via interpolation mapping in metric space. Karapinar *et al.* [8] gave the proof of interpolative Reich-Rus-Ćirić

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type contractions on partial metric spaces. Karapinar *et al.* [9] proved the fixed point theory in the setting of $(\alpha, \beta, \psi, \phi)$-interpolative contractions in metric spaces. Mohammad *et al.* [11] proved the results on the interpolative (φ, ψ)-type z-contraction in metric spaces.

In this paper, we attempted to prove common fixed point theorems for interpolative type contraction mappings using property (E.A) and altering distance function in symmetric space with some applications. We generalize and extend the theorem due to Imdad *et al.* [5], Dutta and Karapinar *et al.* [9], Moradi and Farajzadeh [12], Mohammad *et al.* [11] and Aamri and El Moutawakil [1]. Also, we give an example and demonstrate the results with an application to differential equations.

2. Preliminaries

The following preliminaries and results will be helpful to develop the new theorem for this paper.

We give the metric on symmetric space as follows:

Definition 2.1. [15] A symmetric space is a pair $(\mathcal{X}, d)$ consisting of a non-empty set $\mathcal{X}$ together with a function $d : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ called the symmetric metric, if and only if it satisfies:

(W1) $d(\kappa, \varsigma) = 0$ if and only if $\kappa = \varsigma$;
(W2) $d(\kappa, \varsigma) = d(\varsigma, \kappa)$, for all $\kappa, \varsigma \in \mathcal{X}$.

Then the pair $(\mathcal{X}, d)$ is called a symmetric space.

Let d be a symmetric on a set $\mathcal{X}$. For $\kappa \in \mathcal{X}$ and $\epsilon > 0$, let $B(\kappa, \epsilon) = \{\varsigma \in \mathcal{X} : d(\kappa, \varsigma) < \epsilon\}$. Then, a topology τ_d on $\mathcal{X}$ is defined as follows:

(τ_1) $\mathcal{U} \in \tau_d$ if and only if for each $\kappa \in \mathcal{U}$, there exists an $\epsilon > 0$ such that $B(\kappa, \epsilon) \subset \mathcal{U}$;
(τ_2) A subset $\mathcal{S}$ of $\mathcal{X}$ is a neighbourhood of $\kappa \in \mathcal{X}$ if there exists $\mathcal{U} \in \tau_d$ such that $\kappa \in \mathcal{U} \in \mathcal{S}$;
(τ_3) A symmetric d is a semi-metric if for each $\kappa \in \mathcal{U}$ and each $\epsilon > 0$, $B(\kappa, \epsilon)$ is a neighbourhood of κ in the topology d.
(τ_4) The symmetric space is not a metric space it is difference that comes from the triangle inequality.

Next, we define the Cauchy sequence properties in symmetric space $(\mathcal{X}, d)$ from [15] and [3] as below:

(W3) For a sequence $\{\kappa_n\}$ in $\mathcal{X}$, $\kappa, \varsigma \in \mathcal{X}$ $\lim_{n\to\infty} d(\kappa_n, \kappa) = 0$ and $\lim_{n\to\infty} d(\kappa_n, \varsigma) = 0$ imply that $\kappa = \varsigma$.
(W4) For a sequences $\{\kappa_n\}$ and $\{\varsigma_n\}$ in $\mathcal{X}$, $\kappa, \varsigma \in \mathcal{X}$ $\lim_{n\to\infty} d(\kappa_n, \kappa) = 0$ and $\lim_{n\to\infty} d(\varsigma, \kappa_n) = 0$ implies that $\lim_{n\to\infty} d(\varsigma_n, \kappa) = 0$.
(W5) For a sequences $\{\kappa_n\}$ and $\{\varsigma_n\}$ in $\mathcal{X}$, $\kappa, \varsigma \in \mathcal{X}$ $\lim_{n\to\infty} d(\kappa_n, \kappa) = 0$ and $\lim_{n\to\infty} d(\varsigma, \kappa_n) = 0$ implies that $\lim_{n\to\infty} d(\varsigma_n, \kappa_n) = 0$.
(W6) For continuity of the sequence $\{\kappa_n\}$ in $\mathcal{X}$, $\kappa, \varsigma \in \mathcal{X}$ $\lim_{n\to\infty} d(\kappa_n, \kappa) = 0$ implies that $\lim_{n\to\infty} d(\kappa_n, \varsigma) = d(\kappa_n, \varsigma)$.

Now, we recall the following examples in [3] which satisfy properties $(W4 - W6)$ of symmetric space as follows:

(1) Let $\mathcal{X} = [0, \infty)$ and let
$$d(\kappa, \varsigma) = \begin{cases} |\kappa - \varsigma|, & (\kappa \neq 0, \varsigma \neq 0), \\ \frac{1}{\kappa}, & (\kappa \neq 0). \end{cases} \tag{2.1}$$
Then the pair $(\mathcal{X}, d)$ is a symmetric space which satisfies $W4$.

(2) Let $\mathcal{X} = [0,1] \cup [2]$ and let

$$d(\kappa,\varsigma) = \begin{cases} |\kappa-\varsigma|, & (0 \leq \kappa \leq 1, 0 \leq \varsigma \leq 1), \\ |\kappa|, & (0 \leq \kappa \leq 1, \varsigma = 2), \end{cases} \tag{2.2}$$

and $d(0,2) = 1$. Then the pair $(\mathcal{X}, d)$ is a symmetric space which satisfies $W5$.

(3) Let $\mathcal{X} = \{\frac{1}{n} : n = 1, 2, \dots\} \cup \{0\}$ and let $d(0, \frac{1}{n}) = \frac{1}{n}$ (n is odd) and $d(0, \frac{1}{n}) = 1$ (n is even) and

$$d\Big(\frac{1}{n}, \frac{1}{m}\Big) = \begin{cases} \left|\frac{1}{n} - \frac{1}{m}\right|, & (m+n \text{ is even}), \\ \left|\frac{1}{n} - \frac{1}{m}\right|, & (m+n \text{ is odd and } |m-n| = 1), \\ 1, & (m+n \text{ is odd and } |m-n| > 2). \end{cases} \tag{2.3}$$

and $d(0,2) = 1$. Then the pair $(\mathcal{X}, d)$ is a symmetric space which satisfies $W6$.

We extend the following definition due to Aamri and Moutawakil [1] for property $(E.A)$ to the setting of symmetric spaces as follows:

Definition 2.2. A pair of self-mappings (S,T) of a symmetric space $(\mathcal{X}, d)$ is said to satisfy the property (E.A) if there exists at least one sequence $\{\kappa_n\}$ in $\mathcal{X}$ such that

$$\lim_{n\to\infty} T\kappa_n = \lim_{n\to\infty} S\kappa_n = \kappa^*,$$

for some $\kappa^* \in \mathcal{X}$.

Definition 2.3. A pair of self-mappings S and T of symmetric spaces $(\mathcal{X}, d)$ is said to be non-compatible if there exists a sequence $\{\kappa_n\}$ in $\mathcal{X}$ such that

$$\lim_{n\to\infty} T\kappa_n \quad = \quad \lim_{n\to\infty} S\kappa_n = \kappa^*,$$

but

$$\lim_{n\to\infty} d(ST\kappa_n, TS\kappa_n)$$

is non-zero or not exists.

Definition 2.4. [6] Let S and T be self mappings of a non-empty set $\mathcal{X}$. A point $\kappa \in \mathcal{X}$ is coincidence point of S and T if $t = S\kappa = T\kappa$. The st of coincidence point of S and T is denoted by $C(S,T)$.

The following results for interpolative Kannan contraction have been proved in [7] as follows:

Definition 2.5. [7] Let $(\mathcal{X}, d)$ be a metric space, a mapping $T : \mathcal{X} \to \mathcal{X}$ is said to be interpolative Kannan contraction mappings if

$$d(T\kappa, T\varsigma) \quad \leq \quad \eta[d(\kappa, T\kappa)]^{\delta}.[d(\varsigma, T\varsigma)]^{1-\delta}, \tag{2.4}$$

for all $\kappa, \varsigma \in \mathcal{X}$ with $\kappa \neq T\kappa$, where $\delta \in [0,1)$ and $\delta \in (0,1)$.

Theorem 2.6. *[7] Let $(\mathcal{X}, d)$ be a complete metric space and T be an interpolative Kannan-type contraction. Then T has a unique fixed point in $\mathcal{X}$.*

Karapinar *et al.* [8] gave the prove of interpolative Reich-Rus-Ćirić type contractions on partial metric spaces as follows.

Theorem 2.7. *[8] Let $(\mathcal{X}, p)$ be a complete metric space. $T : \mathcal{X} \to \mathcal{X}$ be a mapping such that*

$$p(T\kappa, T\varsigma) \leq \eta[p(\kappa, \varsigma)]^{\delta}.[p(\kappa, T\kappa)]^{\alpha}.[p(\varsigma, T\varsigma)]^{1-\alpha-\delta}, \tag{2.5}$$

for all $\kappa, \varsigma \in \mathcal{X}\backslash Fix(T)$ where $Fix(T) = \{\kappa \in \mathcal{X}, T\kappa = \kappa\}$, where $\alpha \in [0, 1)$ and $\delta \in (0, 1)$. Then T has a fixed point in $\mathcal{X}$.

Khan *et al.* [10] gave the following definition and theorem in metric spaces.

Definition 2.8. [10] A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied:

(i) $\psi(0) = 0$,
(ii) ψ is continuous and monotone non-decreasing functions.

Theorem 2.9. *[10] Let $(\mathcal{X}, d)$ be a complete metric spaces, let ψ be an altering distance function, and let $T : \mathcal{X} \to \mathcal{X}$ be a self mapping which satisfy the following inequality*

$$\psi(d(T\kappa, T\varsigma)) \leq \varpi\psi(d(\kappa, \varsigma)),$$

for all $\kappa, \varsigma \in \mathcal{X}$ and some $0 < \varpi < 1$. Then T has a unique fixed point.

Furthermore, Karapinar *et al.* [9] proved the following result using $(\alpha, \beta, \phi, \psi)$-interpolative mapping in metric spaces.

Theorem 2.10. *[9] Given a mapping $T : \mathcal{X} \to \mathcal{X}$ from a complete metric spaces $(\mathcal{X}, d)$ into itself, suppose that there are two real numbers $\alpha, \beta \in [0, 1)$ such that $\alpha + \beta < 1$ and a pair of functions $\psi, \phi : [0, \infty) \to \mathbf{R}$ satisfying*

$$\psi(d(T\kappa, T\varsigma)) \leq \phi([d(\kappa, \varsigma)]^{\alpha}.[p(\kappa, T\kappa)]^{\beta}.[d(\varsigma, T\varsigma)]^{1-\alpha-\beta}),$$

for $\kappa, \varsigma \in \mathcal{X}$ with $d(T\kappa, T\varsigma) > 0$. Then T has a unique fixed point and the picard sequence $\{T^n\kappa\}_{n\in\mathbf{N}}$ converges to such a fixed point of T whenever the initial point $\kappa_0 \in \mathcal{X}$.

3. Main Results

We prove the following main results.

Theorem 3.1. *Let $(\mathcal{X}, d)$ be a complete symmetric spaces and let $S, T : \mathcal{X} \to \mathcal{X}$ be a pairs of (ψ-ϕ)-Kannan interpolative self mappings satisfying*

(i) S and T satisfy property $(E.A)$,
(ii) $S(\mathcal{X})$ and $T(\mathcal{X})$ is d-closed,
(iii) there exists $\eta, \delta \in (0, 1)$, such that

$$\psi(d(S\kappa, T\varsigma)) \leq \psi(\mathcal{M}_s(\kappa, \varsigma)) - \phi(\mathcal{M}_s(\kappa, \varsigma)), \tag{3.1}$$

where

$$\mathcal{M}_s(\kappa, \varsigma) = \eta[d(\kappa, S\kappa)]^{\delta}.[d(\varsigma, T\varsigma)]^{1-\delta},$$

and $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous and monotone non-decreasing functions with $\psi(t) = 0 = \phi(t)$ if and only if $t = 0$. Then S and T have a unique common fixed point in $\mathcal{X}$.

Proof. Using (i), there exists a sequence $\{\kappa_n\}$ in $\mathcal{X}$, and a point $t \in \mathcal{X}$ such that

$$\lim_{n\to\infty} d(S\kappa_n, t) = \lim_{n\to\infty} d(T\kappa_n, t) = 0,$$

which implies that

$$\lim_{n\to\infty} S\kappa_n = t, and \lim_{n\to\infty} T\kappa_n,$$

t$\in \mathcal{X}$. As $S(\mathcal{X})$ is d-closed, every convergent sequence of points of $\mathcal{X}$ has a limit in $\mathcal{X}$, therefore

$$\lim_{n\to\infty} S\kappa_n = \theta = S\theta = \lim_{n\to\infty} T\kappa_n,$$

for some $\theta \in \mathcal{X}$. If $S\theta = \theta$, then $T\theta$ is a common fixed point of S and T. hence our proof is done. On contrary to that , we have $S\theta \neq T\theta$, using (3.1) with $\kappa = \kappa_n$ and $\varsigma = \theta$, one gets

$$\psi(d(S\kappa_n, T\theta)) \leq \psi(\mathcal{M}_s(\kappa_n, \theta)) - \phi(\mathcal{M}_s(\kappa_n, \theta)), \tag{3.2}$$

where

$$\mathcal{M}_s(\kappa_n, \theta) = \eta[d(\kappa_n, S\kappa_n)]^{\delta}.[d(\theta, T\theta)]^{1-\delta},$$

which on letting $n \to \infty$, reduces to

$$\mathcal{M}_s(\kappa_n, \theta) = \eta[d(\theta, S\theta)]^{\delta}.[d(\theta, T\theta)]^{1-\delta}. \tag{3.3}$$

Using (3.3) in (3.2), we get

$$\begin{aligned}
\psi(d(S\theta, T\theta)) &\leq \psi(\eta[d(\theta, S\theta)]^{\delta}.[d(\theta, T\theta)]^{1-\delta}) - \phi(\eta[d(\theta, S\theta)]^{\delta}.[d(\theta, T\theta)]^{1-\delta}),\\
\psi(d(S\theta, T\theta)) &\leq \psi(0) - \phi(0),\\
\psi(d(S\theta, T\theta)) &\leq 0.
\end{aligned}$$

By non-decreasing property of ψ and ϕ, we have

$$d(S\theta, T\theta) < \psi(d(S\theta, T\theta)) \leq 0.$$

Therefore, we have

$$\begin{aligned}
d(S\theta, T\theta) &\leq 0,\\
d(S\theta, T\theta) &= 0.
\end{aligned}$$

Thus, $S\theta = T\theta$, which shows that θ is a coincidence fixed point of S and T.

Similarly, if S and T satisfy property $(E.A)$, such that

$$\lim_{n\to\infty} S\kappa_n = \lim_{n\to\infty} T\kappa_n = T\theta = t \in \mathcal{X},$$

for some $\theta \in \mathcal{X}$ as $S(\mathcal{X})$ is d-closed subset $\mathcal{X}$.

Suppose ϑ is another coincidence fixed point of S and T in $\mathcal{X}$ such that $T\theta = S\vartheta$. Otherwise, we have $T\theta \neq S\vartheta$, then using (3.1) with $\kappa = \kappa_n$ and $\varsigma = \vartheta$, one gets

$$\psi(d(S\kappa_n, T\vartheta)) \leq \psi(\mathcal{M}_s(\kappa_n, \vartheta)) - \phi(\mathcal{M}_s(\kappa_n, \vartheta)), \tag{3.4}$$

where

$$\mathcal{M}_s(\kappa_n, \vartheta) = \eta[d(\kappa_n, S\kappa_n)]^{\delta}.[d(\vartheta, T\vartheta)]^{1-\delta},$$

letting $n \to \infty$, yields

$$\mathcal{M}_s(\kappa_n, \vartheta) = \eta[d(\theta, S\theta)]^{\delta}.[d(\vartheta, T\vartheta)]^{1-\delta}. \tag{3.5}$$

Applying (3.5) in (3.4), we get

$$\begin{aligned}
\psi(d(S\theta,T\vartheta)) &\leq \psi(\eta[d(\theta,S\theta)]^{\delta}.[d(\vartheta,T\vartheta)]^{1-\delta}) - \phi(\eta[d(\theta,S\theta)]^{\delta}.[d(\vartheta,T\vartheta)]^{1-\delta}),\\
\psi(d(S\theta,T\vartheta)) &\leq \psi(0)-\phi(0),\\
\psi(d(S\theta,T\vartheta)) &\leq 0.
\end{aligned}$$

By non-decreasing property of ψ and ϕ, we have

$$d(S\theta,T\vartheta) < \psi(d(S\theta,T\vartheta)) \leq 0.$$

Therefore, we have

$$\begin{aligned}
d(S\theta,T\vartheta) &\leq 0,\\
d(S\theta,T\vartheta) &= 0,\\
S\theta &= T\vartheta,
\end{aligned}$$

which is a contradiction. Hence $S\theta = S\vartheta = T\vartheta$.

Next, we prove that the common fixed point of S and T is unique. Since θ and ϑ are coincidence point of S and T such that $S \neq T$, then θ and ϑ are related by $S\theta \in T\theta$ and $S\vartheta \in T\vartheta \in \mathcal{X}$.

Let $\kappa = \theta$ and $\varsigma = \vartheta$ using (3.1), we obtains

$$\psi(d(S\theta,T\vartheta)) \leq \psi(\mathcal{M}_s(\theta,\vartheta)) - \phi(\mathcal{M}_s(\theta,\vartheta)),$$

where

$$\mathcal{M}_s(\theta,\vartheta) = \eta[d(\theta,S\theta)]^{\delta}.[d(\vartheta,T\vartheta)]^{1-\delta},$$

Consequently, we have

$$\psi(d(S\theta,T\vartheta)) \leq \psi(0)-\phi(0).$$

Since ψ and ϕ are non decreasing, we get

$$d(S\theta,T\vartheta) \leq 0.$$

Equivalent to

$$d(S\theta,T\vartheta) = 0.$$

Imply that

$$S\theta = T\vartheta,$$

which is a contradiction. Hence $\theta = \vartheta$ and θ is a unique common fixed point of S and T, denoted by $\theta \in C(S,T)$.

Now, we give an example to illustrate the use of Theorem 3.1.

Example 3.2. Let $\mathcal{X} = [0,2]$ equipped wity the symmetric $d(\kappa,\varsigma) = |\kappa-\varsigma|^2$ with $\psi(t) = \frac{3}{4}t$ and $\phi(t) = \frac{1}{4}t$, for all $\kappa,\varsigma \in \mathcal{X}$. Define $S,T : \mathcal{X} \to \mathcal{X}$ as follows:

$$S\kappa = \begin{cases} \frac{3}{2}-\kappa, & \kappa \in [0,\frac{1}{2}],\\ 2, & \kappa \in [\frac{3}{2},2]. \end{cases}$$

And

$$T\kappa = \begin{cases} \frac{\kappa}{2}, & \kappa \in [0,\frac{1}{2}],\\ 0, & \kappa \in (\frac{1}{2},1]. \end{cases}$$

With a sequence $\kappa_n = \{\frac{n}{n+1} : n \in \mathcal{N}\}$.

Since $\mathcal{X} = \{0\}\cup\{\frac{1}{2}, 1\}\cup\{\frac{3}{2}, 2\}$ is d-closed in $\mathcal{X}$. The pair (S, T) satisfy property (E.A) as for the sequence $\{\frac{n}{n+1}\} \subseteq [0, 1]$, Using Definition 2.2 and Definition 2.3, we have

$$\lim_{n\to\infty} S\kappa_n = \lim_{n\to\infty} T\kappa_n = \kappa^*,$$

Implied that

$$\begin{aligned}\lim_{n\to\infty} S\kappa_n &= \lim_{n\to\infty}\Big(\frac{3}{2} - \frac{n}{n+1}\Big) = \frac{3}{2} - 1 = \frac{1}{2},\\ \lim_{n\to\infty} T\kappa_n &= \lim_{n\to\infty}\frac{1}{2}\Big(\frac{n}{n+1}\Big) = \frac{1}{2}\lim_{n\to\infty}\frac{n}{n+1} = \frac{1}{2}\lim_{n\to\infty}\frac{1}{1} = \frac{1}{2}.\end{aligned}$$

But

$$\lim_{n\to\infty} d(ST\kappa_n, TS\kappa_n) \neq 0.$$

$$\begin{aligned} ST\kappa_n &= \frac{3}{2} - \frac{1}{2}\kappa_n = \frac{3}{2} - \frac{1}{2}\Big(\frac{n}{n+1}\Big)\\ TS\kappa_n &= \frac{3}{4} - \frac{1}{4}\kappa_n = \frac{3}{4} - \frac{1}{4}\Big(\frac{n}{n+1}\Big)\\ \lim_{n\to\infty} d\Big(\frac{3}{2} - \frac{1}{2}\Big(\frac{n}{n+1}\Big), \frac{3}{4} - \frac{1}{4}\Big(\frac{n}{n+1}\Big)\Big) &= d\Big(1, \frac{1}{2}\Big) = |1 - \frac{1}{2}|^2 = \frac{1}{4} \neq 0,\end{aligned}$$

S and T are non-compatible mappings. Thus, the E.A-property is satisfied.

For $\kappa, \varsigma \in [0, 2]$, we calculate the following symmetric

$$\begin{aligned} d(S\kappa, T\varsigma) &= d\Big(\frac{3}{2} - \kappa, \frac{\varsigma}{2}\Big) = \Big|\frac{3}{2} - \kappa - \frac{\varsigma}{2}\Big|^2 = \Big|\frac{3 - 2\kappa - \varsigma}{2}\Big|^2,\\ d(\kappa, S\kappa) &= d\Big(\kappa, \frac{3}{2} - \kappa\Big) = \Big|\kappa - \Big(\frac{3}{2} - \kappa\Big)\Big|^2 = \Big|\frac{4\kappa - 3}{2}\Big|^2,\\ d(\varsigma, T\varsigma) &= d\Big(\varsigma, \frac{\varsigma}{2}\Big) = \Big|\varsigma - \frac{\varsigma}{2}\Big|^2 = \Big|\frac{\varsigma}{2}\Big|^2.\end{aligned}$$

Using (3.1), we get

$$\psi(d(S\kappa, T\varsigma)) \leq \psi(\mathcal{M}_s(\kappa, \varsigma)) - \phi(\mathcal{M}_s(\kappa, \varsigma)), \tag{3.6}$$

where

$$\begin{aligned}\mathcal{M}_s(\kappa, \varsigma) &= \eta[d(\kappa, S\kappa)]^{\delta}.[d(\varsigma, T\varsigma)]^{1-\delta},\\ &= \eta\Big[\Big|\frac{4\kappa - 3}{2}\Big|^2\Big]^{\delta}.\Big[\Big|\frac{\varsigma^2}{4}\Big|\Big]^{1-\delta},\end{aligned}$$

Since $\psi(t) = \frac{3}{4}t$ and $\phi(t) = \frac{1}{4}t$, we have $\psi(t) = \psi(t) - \phi(t) = \psi(t) = \frac{1}{2}t$

$$\psi\Big(\Big|\frac{3 - 2\kappa - \varsigma}{2}\Big|^2\Big) \leq \psi\Bigg(\eta\Big[\Big|\frac{4\kappa - 3}{2}\Big|^2\Big]^{\delta}.\Big[\Big|\frac{\varsigma^2}{4}\Big|\Big]^{1-\delta}\Bigg).$$

By substituting $\eta = \frac{1}{2}, \kappa = 1, \varsigma = 1.5, \delta = \frac{1}{5}$ in the above inequality, we obtain

$$\begin{aligned}
\psi\Big(\Big|\frac{3-2-1.5}{2}\Big|^2\Big) &\leq \psi\Bigg(\eta\Big[\Big|\frac{4-3}{2}\Big|^2\Big]^{\delta}.\Big[\Big|\frac{(1.5)^2}{4}\Big|\Big]^{1-\delta}\Bigg),\\
\psi\Big(\frac{1}{16}\Big) &\leq \psi\Big(\eta\Big[0.25\Big]^{0.2}.\Big[0.5625\Big]^{0.8}\Big),\\
\psi\Big(\frac{1}{16}\Big) &\leq \psi\Big(\eta\Big[0.75783\Big].\Big[0.631099\Big]\Big),\\
\psi\Big(\frac{1}{16}\Big) &\leq \psi\Big(0.4783\eta\Big).\\
\frac{3}{4}.\frac{1}{16} &\leq \frac{1}{2}\times 0.4783\eta,\\
0.046875 &\leq 0.4783\eta.
\end{aligned}$$

Therefore, the condition in Theorem 3.1 is satisfied. Hence our proof is completed.

□

The following is our second main result, which discusses the common fixed point theorem for (ψ, ϕ) interpolative Reich-Rus-Ćirić-type using E. A property in the sense of Aamri and Moutawakil [1] which extend Theorem 2.10 by Karapinar *et al.* [9], Mohammad *et al.*[11] and Moradi and Farajzadeh [12] in the setting of symmetric spaces.

Theorem 3.3. *Let $(\mathcal{X}, d)$ be a complete symmetric spaces and let $T, S : \mathcal{X} \to \mathcal{X}$ be two weakly compatible interpolative Reich-Rus-Ćirić type contraction mappings such that*

(i) S and T are weakly non-compatible mappings,
(ii) T and S satisfy property $(E.A)$,
(iii) $S(\mathcal{X})$ and $T(\mathcal{X})$ is d-closed,
(iv) there exits $\alpha + \beta < 1$, $\eta \in (0, 1)$ such that

$$\psi(d(T\kappa, S\varsigma)) \leq \psi(\mathcal{M}_s(\kappa, \varsigma)) - \phi(\mathcal{M}_s(\kappa, \varsigma)), \tag{3.7}$$

where

$$\mathcal{M}_s(\kappa, \varsigma) = \eta([d(\kappa, \varsigma)]^{\alpha}.[d(\kappa, T\kappa)]^{\beta}.[d(\varsigma, S\varsigma)]^{1-\alpha-\beta}, \tag{3.8}$$

with $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous and monotone non-decreasing functions with $\psi(t) = 0 = \phi(t)$ if and only if $t = 0$. Then T and S has a unique common fixed point in $\mathcal{X}$.

Proof. Using (i) and Definition 2.3, for a pair (T, S) to be closed and weakly non-compatible. we have

$$\begin{aligned}
TS\kappa_n &= ST\kappa,\\
T\kappa_n &= S\kappa,\\
ST\kappa_n &= SS\kappa,\\
TS\kappa_n &= TT\kappa.
\end{aligned}$$

We show that $T\kappa$ is a common fixed point of T and S.

Let $\kappa = t$ and $\varsigma = T\kappa_n = Tt$ in (3.7), we get

$$\psi(d(Tt, STt)) \leq \psi(\mathcal{M}_s(t, Tt)) - \phi(\mathcal{M}_s(t, Tt)), \tag{3.9}$$

where

$$
\begin{aligned}
\mathcal{M}_s(t,Tt) &= \eta([d(t,Tt)]^{\alpha}.[d(t,Tt)]^{\beta}.[d(Tt,STt)]^{1-\alpha-\beta}, \\
&= \eta([d(t,Tt)]^{\alpha}.[d(t,STt)]^{\beta}.[d(Tt,STt)]^{1-\alpha-\beta}, \\
&= \eta([d(t,Tt)]^{\alpha}.[d(t,STt)]^{\beta}.[d(Tt,TTt)]^{1-\alpha-\beta}, \\
&= \eta([d(t,Tt)]^{\alpha}.[d(t,STt)]^{\beta}.[d(Tt,Tt)]^{1-\alpha-\beta}, \\
&= 0. \qquad (3.10)
\end{aligned}
$$

Using (3.10) in (3.9), we get

$$
\begin{aligned}
\psi(d(Tt,STt)) &\leq \psi(\mathcal{M}_s(t,Tt)) - \phi(\mathcal{M}_s(t,Tt)), \\
\psi(d(Tt,STt)) &\leq \psi(0) - \phi(0).
\end{aligned}
$$

By the property of ψ and ϕ, we obtain

$$
\begin{aligned}
\psi(d(Tt,TTt)) &\leq \psi(0) - \phi(0), \\
d(Tt,TTt) &\leq 0.
\end{aligned}
$$

Thus, $STt = TTt = Tt$ and Tt is a common fixed point of T and S. The proof of (ii), (iii) and (iv) follows a similar proof of Theorem 3.1. Thus our proof is completed. □

We give the following corollary for useful of Theorem 3.1.

Corollary 3.4. Let $(\mathcal{X},d)$ be a complete symmetric spaces and let $T,S:\mathcal{X}\to\mathcal{X}$ be a Chatterjea type interpolative self mapping satisfying

$$
\psi(d(T\kappa,S\varsigma)) \leq \psi(\mathcal{M}_s(\kappa,\varsigma)) - \phi(\mathcal{M}_s(\kappa,\varsigma)),
$$

where

$$
\mathcal{M}_s(\kappa,\varsigma) = \eta[d(\kappa,T\varsigma)]^{\delta}.[d(\varsigma,S\kappa)]^{1-\delta}, \qquad (3.11)
$$

with $\eta\in(0,1)$ and $\psi,\phi:[0,\infty)\to[0,\infty)$ are both continuous and monotone non-decreasing functions with $\psi(t)=0=\phi(t)$ if and only if $t=0$. Then T has a unique common fixed point in $\mathcal{X}$.

Proof. The proof of this corollary follows the similar steps of Theorem (3.1). This completes the proof. □

The following example is used to verify the use of Theorem 3.3.

Example 3.5. Let $\mathcal{X}=[0,1]$ and a symmetric d defined by $d(\kappa,\varsigma)=|\kappa-\varsigma|^2$ with $\psi(t)=\frac{3}{4}t$ and $\phi(t)=\frac{1}{4}t$, for all $\kappa,\varsigma\in\mathcal{X}$ and a mapping $T,S:\mathcal{X}\to\mathcal{X}$ defined by

$$
T\kappa = \frac{2}{\kappa+2}
$$

and

$$
S\kappa = \frac{1}{\kappa+1},
$$

for $\kappa_n=\frac{1}{n}:n>0$.

Now, we prove that T and S satisfies (3.7).

We start by showing that T and S satisfy the E.A-property. Let $\kappa_n=\frac{1}{n}:n>0$. Using Definition 2.3 such that

$$
\lim_{n\to\infty} T\kappa_n = \lim_{n\to\infty} S\kappa_n = \kappa^*,
$$

we have

$$\begin{aligned}\lim_{n\to\infty} T\kappa_n &= \lim_{n\to\infty}\frac{2}{\frac{1}{n}+2}=\lim_{n\to\infty}\frac{2}{0+2}=\lim_{n\to\infty}\frac{2}{2}=1,\\ \lim_{n\to\infty} S\kappa_n &= \lim_{n\to\infty}\frac{1}{\frac{1}{n}+1}=\lim_{n\to\infty}\frac{1}{0+1}=\lim_{n\to\infty}\frac{1}{1}=1,\end{aligned}$$

But

$$\lim_{n\to\infty} d(ST\kappa_n, TS\kappa_n)\neq 0.$$

$$\begin{aligned}ST\kappa_n &= \frac{\kappa_n+2}{\kappa_n+4}=\frac{\frac{1}{n}+2}{\frac{1}{n}+4},\\ TS\kappa_n &= \frac{2\kappa_n+3}{2\kappa_n+1}=\frac{\frac{2}{n}+2}{\frac{2}{n}+3},\\ \lim_{n\to\infty} d\Big(\frac{\frac{1}{n}+2}{\frac{1}{n}+4},\frac{\frac{2}{n}+2}{\frac{2}{n}+3}\Big) &= d\Big(\frac{1}{2},\frac{2}{3}\Big)=\frac{1}{36}\neq 0,\end{aligned}$$

for non-compatible mappings. Thus, the E.A-property is satisfied.

For $\kappa,\varsigma\in[0,1]$, we have

$$\begin{aligned}d(T\kappa,S\varsigma) &= d\Big(\frac{2}{\kappa+2},\frac{1}{\varsigma+1}\Big)=\left|\frac{2}{\kappa+2}-\frac{1}{\varsigma+1}\right|^2=\left|\frac{2(1+\varsigma)-(\kappa+2)}{(\kappa+2)(1+\varsigma)}\right|^2,\\ d(\kappa,T\kappa) &= d\Big(\kappa,\frac{2}{\kappa+2}\Big)=\left|\kappa-\frac{2}{\kappa+2}\right|^2=\Big|\frac{\kappa}{\kappa+2}\Big|^2,\\ d(\varsigma,S\varsigma) &= d\Big(\varsigma,\frac{1}{\varsigma+1}\Big)=\left|\varsigma-\frac{1}{\varsigma+1}\right|^2=\Big|\frac{\varsigma(\varsigma+1)-1}{\varsigma+1}\Big|^2,\\ d(\kappa,\varsigma) &= |\kappa-\varsigma|^2.\end{aligned}$$

Using (3.7), we get

$$\psi\left(\left|\frac{2(1+\varsigma)-(\kappa+2)}{(\kappa+2)(1+\varsigma)}\right|^2\right) \leq \psi(\mathcal{M}_s(\kappa,\varsigma))-\phi(\mathcal{M}_s(\kappa,\varsigma)),$$

where

$$\mathcal{M}_s(\kappa,\varsigma) = \eta([|\kappa-\varsigma|^2]^{\alpha}.\Big[\Big|\frac{\kappa}{\kappa+2}\Big|^2\Big]^{\beta}.\left[\Big|\frac{\varsigma(\varsigma+1)-1}{\varsigma+1}\Big|^2\right]^{1-\alpha-\beta},$$

Since $\psi(t)=\frac{3}{4}t$ and $\phi(t)=\frac{1}{4}t$, we have $\psi(t)=\frac{3}{4}t-\phi(t)=\psi(t)=\frac{1}{2}t$

$$\begin{aligned}\psi\left(\left|\frac{2(1+\varsigma)-(\kappa+2)}{(\kappa+2)(1+\varsigma)}\right|^2\right) &\leq \psi\left(\eta([|\kappa-\varsigma|^2]^{\alpha}.\Big[\Big|\frac{\kappa}{\kappa+2}\Big|^2\Big]^{\beta}.\left[\Big|\frac{\varsigma(\varsigma+1)-1}{\varsigma+1}\Big|^2\right]^{1-\alpha-\beta}\right),\\ \frac{3}{4}\left|\frac{2(1+\varsigma)-(\kappa+2)}{(\kappa+2)(1+\varsigma)}\right|^2 &\leq \frac{\eta}{2}[|\kappa-\varsigma|^2]^{\alpha}.\Big[\Big|\frac{\kappa}{\kappa+2}\Big|^2\Big]^{\beta}.\left[\Big|\frac{\varsigma(\varsigma+1)-1}{\varsigma+1}\Big|^2\right]^{1-\alpha-\beta}.\end{aligned}$$

By substituting $\eta = \frac{2}{3}, \kappa = \frac{1}{2}, \varsigma = 1, \alpha = 0.25, \beta = 0.5$, we obtain

$$\begin{aligned}
\frac{3}{4}\left|\frac{2(1+1)-(0.5+2)}{(0.5+2)(1+1)}\right|^2 &\leq \frac{\eta}{2}[|0.5-1|^2]^{0.2}.\Big[\Big|\frac{0.5}{0.5+2}\Big|^2\Big]^{0.3}.\left[\Big|\frac{1(1+1)-1}{1+1}\Big|^2\right]^{1-0.2-0.3}, \\
0.75|0.3|^2 &\leq \frac{\eta}{2}[|0.5|^2]^{0.2}.[|0.2|^2]^{0.3}.[|0.5|^2]^{0.5}, \\
0.0675 &\leq \frac{\eta}{2}[0.75785].[0.3807307].[0.7071], \\
0.0675 &\leq \frac{\eta}{2} \times 0.2040, \\
0.0675 &\leq 0.1020\eta.
\end{aligned}$$

Therefore, the condition in Theorem 3.3 are satisfied. Hence our proof is completed.

It is shown that T and S satisfy the conditions in Theorem 3.3 and hence have a unique fixed point in the setting of symmetric spaces. However, using condition (iii) the results in Karapinar *et al.* [9], Mohammad *et al.*[11] and Moradi and Farajzadeh [12] can not be obtained.

3.1. An Application for Existence Theorem for the Solution of First-Order Periodic Problem in Symmetric spaces

This section investigates the existence of the solution to the first-order periodic problem in symmetric spaces, for the demonstration of Theorem 3.1. Consider the space $C(I)$, the class of real-valued continuous functions defined on $I = [0, T]$ endowed with the symmetric d defined by

$$d(\kappa, \varsigma) = \sup_{t \in I}\Big\{|\kappa(t) - \varsigma(t)|^2, t \in I\Big\}, \forall \kappa, \varsigma \in C(I).$$

Note that $(C(I), d)$ is a complete symmetric space.

Consider the first-order periodic differential problem motivated from [13, 16].

$$\begin{cases} \kappa'(t) = f(t, \kappa(t)), \; t \in (0, T), \\ \kappa(0) = \kappa(T), \end{cases} \tag{3.12}$$

where $T > 0$, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. We consider a space $C(I)(I = [0, T])$ of continuous function defined on I.

Equation (3.12) is equivalently to

$$\begin{aligned} \kappa'(t) + \lambda\kappa(t) &= f(t, \kappa(t)) + \lambda\kappa(t), \; t \in [0, T], \\ \kappa(0) = \kappa(T). \end{aligned} \tag{3.13}$$

The above problem can be transformed into an integral equation as follows:

$$\kappa(t) = \int_0^T G(t, s)[f(s, \kappa(s) + \lambda\kappa(s))]ds, \tag{3.14}$$

where $G(t, s)$ is the Green function associated with the above integral equation is given by

$$G(t, s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}, & 0 \leq s \leq t \leq T, \\ \\ \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}, & 0 \leq t \leq s \leq T. \end{cases}$$

We prove the following theorem.

Theorem 3.6. *Let $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous and suppose that there exists $\lambda > 0$ such that for $\kappa, \varsigma \in \mathbb{R}$ with $\kappa > \varsigma$, we have*

$$|f(t,\kappa) + \lambda\kappa - [f(t,\varsigma) + \lambda\varsigma]| \leq \lambda\Big[\ln(|\kappa-\varsigma|^2+1) + \frac{|\kappa-\varsigma|^4}{2(|\kappa-\varsigma|^2+1)}\Big], \tag{3.15}$$

where

$$\ln(|\kappa-\varsigma|^2+1) + \frac{|\kappa-\varsigma|^4}{2(|\kappa-\varsigma|^2+1)} = M_s(\kappa,\varsigma) = \eta[d(\kappa,T\kappa)]^{\delta}.[d(\varsigma,T\varsigma)]^{1-\delta}.$$

Then the existence solution of (3.14) provides the existence of a unique solution of (3.12).

Proof. Consider $C^1([a,b],\mathcal{X})$ with the symmetric

$$d(\kappa,\varsigma) = \sup_{t\in I}|\kappa(t)-\varsigma(t)|^2.$$

The $(\mathcal{X},d)$ is a complete symmetric space.

Let $S,T : C^1([a,b]) \to C^1([a,b])$ be a mapping defined by

$$S\kappa(t) = \int_0^T G(t,s)[f(s,\kappa(s)+\lambda\kappa(s))]ds. \tag{3.16}$$

$$T\varsigma(t) = \int_0^T G(t,s)[f(s,\varsigma(s)+\lambda\varsigma(s))]ds. \tag{3.17}$$

Since our mappings S and T are continuous, let $\kappa > \varsigma$, we have

$$f(t,\kappa) + \lambda\kappa \geq f(t,\varsigma) + \lambda\varsigma$$

Using (3.16), we obtain

$$S\kappa(t) = \int_0^T G(t,s)[f(s,\kappa(s)+\lambda\kappa(s))]ds \geq \int_0^T G(t,s)[f(s,\varsigma(s)+\lambda\varsigma(s))]ds = T\varsigma(t),$$

for $t \in I$. Consider $\kappa > \varsigma$, using (3.15) and (3.16), we get

$$\begin{aligned} d(T\kappa,T\varsigma) = \sup_{t\in I}|T\kappa(t)-T\varsigma(t)|^2 &\leq \Big|\sup_{t\in I}\int_0^T G(t,s)[f(s,\kappa(s)+\lambda\kappa(s)) - f(s,\varsigma(s)-\lambda\varsigma(s))]ds\Big|^2, \\ &\leq \sup_{t\in I}\int_0^T G(t,s)\lambda\Big[\ln(|\kappa-\varsigma|^2+1) + \frac{|\kappa-\varsigma|^4}{2(|\kappa-\varsigma|^2+1)}\Big]ds, \\ &\leq \Big[\ln(|\kappa-\varsigma|^2+1) + \frac{|\kappa-\varsigma|^4}{2(|\kappa-\varsigma|^2+1)}\Big]\lambda\sup_{t\in I}\int_0^T G(t,s)ds. \end{aligned} \tag{3.18}$$

Then

$$\begin{aligned} \sup_{t\in I}\int_0^T G(t,s) &= \int_0^t G(t,s)ds + \int_t^T G(t,s)ds, \\ &= \int_0^t \frac{e^{\lambda(T+s-t)}}{(e^{\lambda T}-1)}ds + \int_t^T \frac{e^{\lambda(s-t)}}{(e^{\lambda T}-1)}ds, \\ &= \frac{1}{\lambda(e^{\lambda T}-1)}(e^{\lambda T}-1), \\ &= \frac{1}{2\lambda(e^{\lambda T}-1)}(e^{\lambda T}-1), \\ &= \frac{e^{\lambda T}-1}{\lambda(e^{\lambda T}-1)} = \frac{1}{\lambda}. \end{aligned} \tag{3.19}$$

Using (3.19) in (3.18), we get

$$\begin{aligned} d(S\kappa, T\varsigma) = |S\kappa - T\varsigma|^2 &\leq \Big[\ln(|\kappa - \varsigma|^2 + 1) + \frac{(\kappa - \varsigma)^4}{2(|\kappa - \varsigma|^2 + 1)}\Big]\lambda \times \frac{1}{\lambda}, \\ |S\kappa - T\varsigma|^2 &\leq \ln(|\kappa - \varsigma|^2 + 1) + \frac{(\kappa - \varsigma)^4}{2(|\kappa - \varsigma|^2 + 1)}, \\ d(S\kappa, T\varsigma) &\leq \ln[(d(\kappa, \varsigma) + 1)] + \frac{(d(\kappa, \varsigma))^2}{2(d(\kappa, \varsigma) + 1)}, \\ d(S\kappa, T\varsigma) &\leq M_s(\kappa, \varsigma) = \eta[d(\kappa, S\kappa)]^{\delta}.[d(\varsigma, T\varsigma)]^{1-\delta}. \end{aligned}$$

By passing the property of ψ and ϕ, we can write this as

$$\begin{aligned} \psi(d(S\kappa, T\varsigma)) &\leq \psi(M_s(\kappa, \varsigma)) = \psi(\eta[d(\kappa, S\kappa)]^{\delta}.[d(\varsigma, T\varsigma)]^{1-\delta}) - \phi(\eta[d(\kappa, S\kappa)]^{\delta}.[d(\varsigma, T\varsigma)]^{1-\delta}) \\ &< \psi(\eta[d(\kappa, S\kappa)]^{\delta}.[d(\varsigma, T\varsigma)]^{1-\delta}). \end{aligned} \tag{3.20}$$

Now S and T satisfies condition of Theorem 3.1 with $\delta < 1$. Also, the conditions imposed in Theorem 3.6 are satisfied. Hence, $\kappa \in C(I, \mathbb{R})$, that is; κ is a solution of integral equation (3.16). Thus (3.12) has a solution. □

4. Conclusion

The novelty of this study to fixed point theory is the fixed point result given in Theorem 3.1 which extends and Karapinar *et al.* [9] theorems to the setting of symmetric spaces. Theorem 3.3, which discuss the common fixed point theorem for (ψ, ϕ) interpolative Reich-Rus-Ćirić-type using E.A property in the sense of Aamri and Moutawakil [1] which extend Theorem 2.10 by Karapinar *et al.* [9], Mohammad *et al.*[11] and Moradi and Farajzadeh [12] in the setting of symmetric spaces. These theorems provide the fixed points conditions for a substantial class of self-mappings and two self-mappings on various abstract spaces. We gave two examples for verification of our proven results. Also, a corollary is given for the usefulness of Theorem 3.1. Lastly, we gave an application for Theorem 3.1 to the existence solution of the first-order order periodic problem in symmetric spaces. We proved fixed point and common fixed point theorems for interpolative mappings in symmetric space using E. A property and altering distance function make our results to be unique from the previously published articles.

Conflict of Interest

The authors have no conflict of interest regarding the publication of this article.

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